A NOTE ON STRONGLY ERGODIC SEMI-GROUP OF OPERATORS

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Let $\{T(\xi);\ 0<\xi<\infty\}$ be a semi-group of operators satisfying the following assumptions:

(i) For each ξ, $0<\xi<\infty$, $T(\xi)$ is a bounded linear operator from a complex Banach space X into itself and

$$(1)\qquad T(\xi+\eta)=T(\xi)T(\eta).$$

(ii) $T(\xi)$ is strongly measurable on $(0,\infty)$.

(iii) $\displaystyle\int_0^1 \|T(\xi)x\|\,d\xi<\infty$ for each $x\in X$.

If $T(\xi)$ satisfies the condition

(iv) $\displaystyle\lim_{\lambda\to\infty}\lambda\int_0^\infty e^{-\lambda\xi}T(\xi)x\,d\xi=x$ for each $x\in X$,

then $T(\xi)$ is said to be of class $(0,A)$. If, instead of (iv), $T(\xi)$ satisfies the stronger condition

(v) $\displaystyle\lim_{\xi\to 0}\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}T(\tau)x\,d\tau=x$ for each $x\in X$,

then $T(\xi)$ is said to be of class $(0,C_\alpha)$. If (iii) is replaced by the stronger condition

(iii') $\displaystyle\int_0^1\|T(\xi)\|\,d\xi<\infty$,

then these classes become $(1,A)$ and $(1,C_\alpha)$, respectively.

It follows from (i) and (ii) that $T(\xi)$ is strongly continuous for $\xi>0$ and $\omega_0\equiv\lim_{\xi\to\infty}\log\|T(\xi)\|/\xi<\infty$.

We shall now define $R(\lambda;A)$, for each $\lambda>\omega_0$, by

$$(2)\qquad R(\lambda;A)x=\int_0^\infty e^{-\lambda\xi}T(\xi)x\,d\xi \quad\text{for each } x\in X.$$

It is clear that this integral converges absolutely for $\lambda>\omega_0$. When $T(\xi)$ is a semi-group of class $(0,A)$, the following properties are well known [3]:

(a) There exists the complete infinitesimal generator A.

(b) The domain $D(A)$ of A is dense in X and $R(\lambda;A)$ is the resolvent of A.

(c) $\lim_{\xi\to 0}T(\xi)x=x$ for $x\in D(A)$.

2. Theorem 1. Let α be a positive number. A necessary and sufficient condition that a semi-group of class $(0,A)$ is of class $(0,C_\alpha)$, is that there exists a real number $\omega\geqq 0$ such that

$$(3)\qquad \sup_{\lambda>0,\,k\geqq 0}\left\|\frac{\alpha\cdot k!}{\Gamma(k+\alpha+1)}\sum_{i=0}^{k}\frac{\Gamma(k+\alpha-i)}{(k-i)!}\left[R(\lambda+\omega;A)\right]^{i+1}\right\| = M<\infty.$$

In case of $\alpha=1$ this theorem is due to R.S.Phillips [3] and furthermore in case where α is a positive integer, the theorem has been proved by the present author [1].

Proof. ω is a fixed non-negative number such that $\omega>\omega_0$. Then $e^{-\omega\xi}\|T(\xi)\|$ is bounded at $\xi=\infty$. We get

$$\frac{\lambda^{k+\alpha+1}}{\Gamma(k+\alpha+1)}\int_0^\infty e^{-\lambda\xi}\xi^{k+\alpha}\left[\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}e^{-\omega\tau}T(\tau)x\,d\tau\right]d\xi$$

$$=\frac{\lambda^{k+\alpha+1}}{\Gamma(k+\alpha+1)}\int_0^\infty e^{-(\omega+\lambda)\tau}T(\tau)x\,d\tau\left[\int_\tau^\infty\alpha e^{-\lambda(\xi-\tau)}\xi^{k}(\xi-\tau)^{\alpha-1}d\xi\right]$$

$$=\frac{\alpha\lambda^{k+\alpha+1}}{\Gamma(k+\alpha+1)}\int_0^\infty e^{-(\omega+\lambda)\tau}T(\tau)x\,d\tau\left[\int_0^\infty e^{-\lambda\xi}(\xi+\tau)^k\xi^{\alpha-1}d\xi\right]$$

$$=\frac{\alpha\lambda^{k+\alpha+1}}{\Gamma(k+\alpha+1)}\sum_{i=0}^{k}\binom{k}{i}\frac{\Gamma(k+\alpha-i)}{\lambda^{k+\alpha-i}}\int_0^\infty e^{-(\omega+\lambda)\tau}\tau^iT(\tau)x\,d\tau.$$

Since $\displaystyle\int_0^\infty e^{-(\lambda+\omega)\tau}\tau^iT(\tau)x\,d\tau=\frac{i!}{\lambda^{i+1}}\left[\lambda R(\lambda+\omega;A)\right]^{i+1}x$, we obtain

$$(4)\qquad \frac{\lambda^{k+\alpha+1}}{\Gamma(k+\alpha+1)}\int_0^\infty e^{-\lambda\xi}\xi^{k+\alpha}\left[\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}e^{-\omega\tau}T(\tau)x\,d\tau\right]d\xi$$

$$=\frac{\alpha\cdot k!}{\Gamma(k+\alpha+1)}\sum_{i=0}^{k}\frac{\Gamma(k+\alpha-i)}{(k-i)!}\left[\lambda R(\lambda+\omega;A)\right]^{i+1}x.$$

If $T(\xi)$ is a semi-group of class $(0,C_\alpha)$, then $e^{-\omega\xi}T(\xi)$ is of class $(0,C_\alpha)$ and $e^{-\omega\xi}\|T(\xi)\|$ is bounded at $\xi=\infty$. Thus there exists a positive number M such that

$$\sup_{\xi>0}\left\|\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}e^{-\omega\tau}T(\tau)x\,d\tau\right\| \leqq M\|x\| \quad \text{for all } x\in X.$$

Therefore we get the relation (3) from (4).

On the other hand, using the theorem that $f(\xi)$ is a bounded continuous vector valued function from $(0,\infty)$ in X and $k/\lambda\to\eta$ $(\lambda=\lambda(k)\to\infty,\ k\to\infty)$ then

$$\frac{\lambda^{k+\alpha+1}}{\Gamma(k+\alpha+1)}\int_0^\infty e^{-\lambda\xi}\xi^{k+\alpha}f(\xi)\,d\xi\longrightarrow f(\eta),$$

we obtain from (4)

$$\lim_{k/\lambda\to\eta}\frac{\alpha\cdot k!}{\Gamma(k+\alpha+1)}\sum_{i=0}^{k}\frac{\Gamma(k+\alpha-i)}{(k-i)!}\left[\lambda R(\lambda+\omega;A)\right]^{i+1}x$$
$$=\alpha\,\eta^{-\alpha}\int_0^\eta(\eta-\tau)^{\alpha-1}e^{-\omega\tau}T(\tau)x\,d\tau$$

for $x\in D(A)$. Therefore we get by (3)

$$\sup_{\xi>0}\left\|\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}e^{-\omega\tau}T(\tau)x\,d\tau\right\| \leqq M\|x\|$$

for $x\in D(A)$. Since $D(A)$ is dense in X and $\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}e^{-\omega\tau}T(\tau)x\,d\tau$ is a bounded linear operator for each $\xi>0$, the above relation is true for all $x\in X$. We have $\lim_{\xi\to0}T(\xi)x=x$ for $x\in D(A)$ and a fortiori

$$\lim_{\xi\to0}\alpha\xi^{-\alpha}\int_0^\xi(\xi-\tau)^{\alpha-1}e^{-\omega\tau}T(\tau)x\,d\tau = x$$

for $x\in D(A)$. Thus the above relation is true for all $x\in X$ by the Banach-Steinhaus theorem. Hence $e^{-\omega\xi}T(\xi)$ is a semi-group of class $(0,C_\alpha)$, so that $T(\xi)$ is of class $(0,C_\alpha)$.

The present author [2] has already given a necessary and sufficient condition that a closed linear operator A becomes the complete infinitesimal generator of a semi-group of class $(0,A)$. Thus we can obtain from Theorem 1 the following:

Theorem 2. Let α be a positive number. A necessary and sufficient condition that a closed linear operator A is the complete infinitesimal generator of a semi-group $\{T(\xi);0<\xi<\infty\}$ of class $(0,C_\alpha)$, is that

(i') $D(A)$ is a dense linear subset in X,

(ii') there exists a real number $\omega\geqq0$ such that the spectrum of A is located in $\mathfrak{R}(\lambda)$ (the real part of λ) $<\omega$ and

$$\sup_{\lambda>0,\,k\geqq0}\left\|\frac{\alpha\,k!}{\Gamma(k+\alpha+1)}\sum_{i=0}^{k}\frac{\Gamma(k+\alpha-i)}{(k-i)!}\left[\lambda R(\lambda+\omega;A)\right]^{i+1}\right\|<\infty,$$

where $R(\lambda;A)$ is the resolvent of A,

(iii') there exists a non-negative function $f(\xi,x)$ defined on the product space $(0,\infty)\times X$ having the following properties:

(a') for each $x\in X$, $f(\xi,x)$ is continuous for $\xi>0$, integrable on $[0,1]$ and $e^{-\omega\xi}f(\xi,x)$ is bounded at $\xi=\infty$,

(b') $\|R^{(k)}(\lambda+\omega;A)x\|\leqq\int_0^\infty e^{-(\lambda+\omega)}\cdot\xi^k f(\xi,x)\,d\xi$

for each $x\in X$, all real $\lambda>0$ and all integers $k\geqq0$, where $R^{(k)}(\lambda;A)$ denotes the k-th derivative of $R(\lambda;A)$ with respect to λ.

We note that, in the above theorem, if $0<\alpha\leqq1$, then "the complete infinitesimal generator" may be replaced by "the infinitesimal generator".

3. We shall give a semi-group which is of class $(0,A)$ but not of class $(1,A)$. The following example is a modification of that by R.S. Phillips [3]. Let X consist of all

sequence pairs x ;n 1,2,

such that and n

with norm x n

The operator T( )x x

n 1,2, is defined by

$$\chi_n'=\exp[-(n+in^2)\xi]\,(\chi_n\cos n\xi-\eta_n\sin n\xi),$$
$$\exp[-(n+in^2)\xi]\,(\chi_n\sin n\xi+\eta_n\cos n\xi).$$

It is easy to show that $\{T(\xi);0<\xi<\infty\}$ is a semi-group of bounded linear operator and that $T(\xi)$ is strongly continuous for $\xi>0$. Since

$$\|T(\xi)x\|=\sum_{n=1}^\infty|\chi_n'|+\sum_{n=1}^\infty n|\eta_n'|$$
$$\leqq\sum_{n=1}^\infty|\chi_n|+2\sum_{n=1}^\infty n|\eta_n|+\sum_{n=1}^\infty n e^{-n\xi}|\chi_n|,$$

we get

$$(5)\quad \int_0^1\|T(\xi)x\|\,d\xi\leqq\sum_{n=1}^\infty|\chi_n| + 2\sum_{n=1}^\infty n|\eta_n|+\sum_{n=1}^\infty n|\chi_n|\int_0^\infty e^{-n\xi}d\xi=2\|x\|$$

for all $x\in X$.

However, for $x^{(n)}=\{(\delta_{in},0);\ i=1,2,\cdots\}$ where $\delta_{ij}=0$ for $i\neq j$, $\delta_{ii}=1$, we have for all n

$$\|T(\xi)x^{(n)}\|=\exp(-n\xi)\,|\cos n\xi|+ne^{-n\xi}|\sin n\xi|\geqq ne^{-n\xi}|\sin n\xi|$$

and $\|x^n\| = 1$. Therefore, for any small $\xi > 0$, we get

$$\|T(\xi)\| \geqq \sup_n ne^{-n\xi}|\sin n\xi| \geqq \sup_{[\frac{1}{\xi}] \geqq n \geqq [\frac{1}{2\xi}]+1} ne^{-n\xi}|\sin n\xi| \geqq \frac{1}{\xi}(2e)^{-1}\sin 1/2.$$

Hence $T(\xi)$ is not of class $(1,A)$. On the other hand, we get $\|T(\xi)\| \leqq 2$ for all $\xi \geqq 1$. Then

$$R(\lambda;A)x = \int_0^\infty e^{-\lambda\xi}T(\xi)x\,d\xi = \{(\chi_n(\lambda), \eta_n(\lambda));\ n = 1,2, \cdots\}$$

is defined for all $\lambda > 0$ and for all $x \in X$. Then we have $\chi_n(\lambda) = \chi_n\alpha_n(\lambda) - \eta_n\beta_n(\lambda)$ and $\eta_n(\lambda) = \chi_n\beta_n(\lambda) + \eta_n\alpha_n(\lambda)$, where

$$\alpha_n(\lambda) = \int_0^\infty \exp[-(\lambda+n+in^2)\xi]\cos n\xi\, d\xi = \frac{\lambda+n+in^2}{(\lambda+n+in)^2+n^2}$$

$$\beta_n(\lambda) = \int_0^\infty \exp[-(\lambda+n+in^2)\xi]\sin n\xi\, d\xi = \frac{n}{(\lambda+n+in^2)^2+n^2}$$

Since $|\alpha_n(\lambda)| \leqq \lambda^{-1}$, $|\beta_n(\lambda)| \leqq \lambda^{-1}$ and $|n\beta_n(\lambda)| \leqq \lambda^{-1}$ for all $\lambda > 0$, we get

$$\|R(\lambda;A)x\| = \sum_{n=1}^\infty |\chi_n(\lambda)| + \sum_{n=1}^\infty n|\eta_n(\lambda)| \leqq \frac{2}{\lambda}\|x\|.$$

Hence $\|\lambda R(\lambda;A)\| \leqq 2$ for all $\lambda > 0$. Furthermore, for any ultimately zero vector $x = \{(\chi_1, \eta_1), \cdots, (\chi_n, \eta_n), (0, 0), (0, 0), \cdots\}$, we obtain

$$\|\lambda R(\lambda;A)x - x\| = \sum_{k=1}^n |\lambda\chi_k(\lambda) - \chi_k| + \sum_{k=1}^\infty k|\lambda\eta_k(\lambda) - \eta_k| \to 0$$

as $\lambda \to \infty$. Since the set of the ultimately zero vectors is dense in X, it follows from the Banach-Steinhaus theorem that

$$\lim_{\lambda\to\infty} \lambda R(\lambda;A)x = x \quad \text{for all } x \in X. \tag{6}$$

Thus we obtain from (5) and (6) that $T(\xi)$ is of class $(0,A)$.

References

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