ON THE ABSOLUTE SUMMABILITY FACTORS

By Gen-ichiro SUNOUCHI

(Comm. by T. Kawata)

1. In the present note the author proves the following theorem which is an answer of the problem raised by M.T.Cheng [2].

Let $\varphi(t)$ be even, periodic, integral and

(1) $$\varphi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt.$$

If we denote by $\varphi_\alpha(t)$ the α -th mean of $\varphi(t)$, then we have the following theorem.

Theorem 1. If $\varphi_\alpha(t)$ is bounded variation in $(0, \pi)$, then $\{\log(n+1)\}^{-(1+\varepsilon)}$ $(\varepsilon>0)$ are the $|C.\alpha|$-summability factors of the Fourier series of $\varphi(t)$ at $t=0$.

Cheng proved this theorem for $0 \leqq \alpha \leqq 1$, and said that the case $\alpha > 1$ remains open. But this theorem is a easy consequence of

Theorem 2. Denote by σ_n^α the $(C.\alpha)$-mean of the series $\sum a_n$. If

$$\sum_{n=1}^{N} |\sigma_n^\alpha - \sigma_{n-1}^\alpha| = O(\log N),$$

then $\{\log(n+1)\}^{-(1+\varepsilon)}$ are the $|C.\alpha|$-summability factors of the series $\sum a_n$.

Denote by $\sigma_n^\alpha(0)$ the $(C.\alpha)$-mean of the Fourier series of (1) at $t=0$. Then from Bosanque's theorem [1], if $\varphi_\alpha(t)$ is bounded variation in $(0, \pi)$,

$$\sum_{n=1}^{N} |\sigma_n^\alpha(0) - \sigma_{n-1}^\alpha(0)| = O(\log N).$$

From this fact if Theorem 2 is proved, Theorem 1 is evident.

2. Concerning Theorem 2, we shall raise the problem :

if $\sum_{n=1}^{N} |\sigma_n^\alpha - \sigma_{n-1}^\alpha| = O(\log N)$ and

$$\sum_{n=1}^{\infty} a_n / \log(n+1)$$ is $|C|$-summable for some order, then whether $\{\log(n+1)\}^{-1}$ are the $|C.\alpha|$-summability factors or not. In the ordinary Cesaro summability case, this problem has been answered affirmatively by A.Zygmund [5], (cf. G.Sunouchi [4] and L.Jesmanowicz [3]). But in the $|C.\alpha|$ case, we cannot drop $\varepsilon\ (>0)$. For $\alpha=0$, there is a function of bounded variation where

$$\sum |a_n| / \log(n+1) = \infty.$$

3. We proceed the proof of Theorem 2. Put $\mu_n = (\log n)^{1+\varepsilon}$, then

(1) $$\Delta^j \frac{1}{\mu_n} = O\left\{\frac{1}{n^j (\log n)^{2+\varepsilon}}\right\},$$

for $j = 1, 2, \ldots$

From Kogbetliantz's formula, we have

(2) $$\sigma_n^\alpha - \sigma_{n-1}^\alpha$$

$$= \frac{1}{nA_n^\alpha} \sum_{\nu=1}^{n} \nu A_{n-\nu}^{\alpha-1} a_\nu = \frac{1}{nA_n^\alpha} t_n^{\alpha-1},$$

say. Further put

(3) $$\tau_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^\alpha \frac{a_\nu}{\mu_\nu}$$

$$= \frac{1}{A_n^\alpha \mu_n} \sum_{\nu=0}^{n} A_{n-\nu}^\alpha a_\nu + \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^\alpha a_\nu \left(\frac{1}{\mu_\nu} - \frac{1}{\mu_n}\right)$$

$$= u_n^\alpha + v_n^\alpha,$$

then

$$\tau_n^\alpha - \tau_{n-1}^\alpha = u_n^\alpha - u_{n-1}^\alpha + v_n^\alpha - v_{n-1}^\alpha.$$

The last term

(4) $v_n^\alpha - v_{n-1}^\alpha$

$$= \frac{1}{nA_n^\alpha}\sum_{\nu=1}^{n} \nu A_{n-\nu}^{\alpha-1} a_\nu \left(\frac{1}{\mu_\nu}-\frac{1}{\mu_n}\right)$$

$$= \frac{1}{nA_n}\sum_{\nu=1}^{n} \nu\, a_\nu\, \varepsilon_\nu ,$$

where

(5) $$\varepsilon_\nu = A_{n-\nu}^{\alpha-1}\left(\frac{1}{\mu_\nu}-\frac{1}{\mu_n}\right).$$

putting $k=[\alpha]+1$, and applying k-times Abel's transformation,

(6) $$\sum_{\nu=1}^{n} \nu\, a_\nu\, \varepsilon_\nu = \sum_{\nu=0}^{n} t_\nu^{k-1}\Delta^k \varepsilon_\nu$$

where

(7) $$\Delta^k \varepsilon_\nu$$

$$= \sum_{j=1}^{k-1}\binom{k}{j}\Delta^j\frac{1}{\mu_\nu}\,\Delta^{k-j}A_{n-\nu-j}^{\alpha-1}$$

$$+ \left(\frac{1}{\mu_\nu}-\frac{1}{\mu_n}\right)\Delta^k A_{n-\nu}^{\alpha-1} + \Delta^k\frac{1}{\mu_\nu}\cdot A_{n-\nu-k}^{\alpha-1}$$

For $1\leqq j\leqq k-1$, we get

(8) $$\sum_{\nu=0}^{n} t_\nu^{k-1}\Delta^j\frac{1}{\mu_\nu}\,\Delta^{k-j}A_{n-\nu-j}^{\alpha-1}$$

$$= \sum_{\nu=0}^{n-j} t_\nu^{k-1}\Delta^j\frac{1}{\mu_\nu}\,A_{n-j-\nu}^{\alpha-1-k+j} .$$

Substituting the formula (1) and (2), this is smaller than

$$\sum_{\nu=0}^{n-j}\nu A_\nu^k\,|\sigma_\nu^\alpha-\sigma_{\nu-1}^\alpha|\; O\left\{\frac{1}{\nu^j(\log\nu)^{2+\varepsilon}}\right\}A_{n-j-\nu}^{\alpha-k-1+j}$$

$$(\log\nu = 1 \quad \text{for} \quad \nu=0)$$

$$= \sum_{\nu=0}^{n-j} O\left\{A_\nu^{k+1-j}|\sigma_\nu^k-\sigma_{\nu-1}^k|\,(\log\nu)^{-(2+\varepsilon)}A_{n-\nu-j}^{\alpha-k-1+j}\right\}$$

$$= \sum_{\nu=0}^{[n/2]} + \sum_{\nu=[n/2]+1}^{n-j} = I_n + J_n ,$$

say. Since

$$\sum_{n=1}^{N}|\sigma_n^k-\sigma_{n-1}^k| = O(\log N) ,$$

(9) I_n

$$= O\left\{n^{\alpha-k-1+j}\sum_{\nu=0}^{[n/2]}\nu^{k+1-j}(\log\nu)^{-(2+\varepsilon)}|\sigma_\nu^k-\sigma_{\nu-1}^k|\right\}$$

$$= O\left\{n^{\alpha-k-1+j}\,n^{k+1-j}(\log n)^{-(2+\varepsilon)}\sum_{\nu=1}^{[n/2]}|\sigma_\nu^k-\sigma_{\nu-1}^k|\right\}$$

$$= O\left\{n^\alpha(\log n)^{-(2+\varepsilon)}(\log n)\right\}$$

$$= O\left\{n^\alpha(\log n)^{-(1+\varepsilon)}\right\}.$$

From (4) and (6), we get

(10) $$\sum_{n=1}^{\infty}\frac{1}{nA_n^\alpha}\,n^\alpha(\log n)^{-(1+\varepsilon)}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n(\log n)^{1+\varepsilon}} < \infty .$$

Concerning J_n, if $j\geqq 2$,

(11) J_n

$$= O\left\{n^{k+1-j}(\log n)^{-(2+\varepsilon)}\sum_{\nu=[n/2]+1}^{n-j}|\sigma_\nu^k-\sigma_{\nu-1}^k|A_{n-\nu-j}^{\alpha-k-1+j}\right\}$$

$$= O\left\{n^{k+1-j}(\log n)^{-(2+\varepsilon)}n^{\alpha-k-1+j}\sum_{\nu=[n/2]+1}^{n-j}|\sigma_\nu^k-\sigma_{\nu-1}^k|\right\}$$

$$= O\left\{n^\alpha(\log n)^{-(2+\varepsilon)}(\log n)\right\} = O\left\{n^\alpha(\log n)^{-(1+\varepsilon)}\right\}.$$

This terms are analogous to I_n.
For $j=1$,

(12) J_n

$$= \sum_{\nu=[n/2]+1}^{n-1}\nu^{k+1}|\sigma_\nu^k-\sigma_{\nu-1}^k|\cdot O\left\{\frac{1}{\nu(\log\nu)^{2+\varepsilon}}\right\}A_{n-\nu-1}^{\alpha-k}$$

$$= O\left\{\sum_{\nu=[n/2]+1}^{n}\nu^k(\log\nu)^{-(2+\varepsilon)}|\sigma_\nu^k-\sigma_{\nu-1}^k|A_{n-\nu}^{\alpha-k}\right\}.$$

(13) $$\sum_{n=1}^{\infty}\frac{|J_n|}{nA_n^\alpha} = \sum_{n=1}^{\infty}\frac{|J_n|}{n^{\alpha+1}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^{\alpha+1}}\sum_{\nu=[n/2]+1}^{n}\nu^k(\log\nu)^{-(2+\varepsilon)}|\sigma_\nu^k-\sigma_{\nu-1}^k|\,A_{n-\nu}^{\alpha-k}$$

$$= \sum_{n=1}^{\infty}n^{k-\alpha-1}(\log n)^{-(2+\varepsilon)}\sum_{\nu=[n/2]+1}^{n}|\sigma_\nu^k-\sigma_{\nu-1}^k|\,A_{n-\nu}^{\alpha-k}$$

$$= \sum_{\nu=1}^{\infty}|\sigma_\nu^k-\sigma_{\nu-1}^k|\sum_{n=\nu}^{2\nu}n^{k-\alpha-1}(\log n)^{-(2+\varepsilon)}A_{n-\nu}^{\alpha-k}$$

$$= \sum_{\nu=1}^{\infty} |\sigma_\nu^k - \sigma_{\nu-1}^k| (\log \nu)^{-(2+\varepsilon)}$$

$$= \lim_{N\to\infty} \Big[\sum_{\nu=1}^{N} \Big\{ \sum_{\nu=1}^{n} |\sigma_\nu^k - \sigma_{\nu-1}^k| \Big\} \Big\{ \frac{1}{n(\log n)^{3+\varepsilon}} \Big\} + \Big\{ \sum_{\nu=1}^{N} |\sigma_\nu^k - \sigma_{\nu-1}^k| \Big\} \frac{1}{(\log N)^{2+\varepsilon}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n(\log n)^{2+\varepsilon}} < \infty .$$

From the second term of the right side of (7), we have

$$(14) \quad \sum_{\nu=0}^{n} t_\nu^{k-1} \Big(\frac{1}{\mu_\nu} - \frac{1}{\mu_n} \Big) \Delta^k A_{n-\nu}^{\alpha-1}$$

$$= \sum_{\nu=0}^{n} t_\nu^{k-1} \Big(\frac{1}{\mu_\nu} - \frac{1}{\mu_n} \Big) A_{n-\nu}^{\alpha-k-1}$$

$$= \sum_{\nu=0}^{[n/2]} + \sum_{\nu=[n/2]+1}^{n} = K_n + L_n ,$$

say. On the other hand, from the mean value theorem

$$\frac{1}{\mu_\nu} - \frac{1}{\mu_n} = \sum_{\mu=\nu}^{n-1} \Delta (\log \mu)^{-(1+\varepsilon)}$$

$$= \sum_{\mu=\nu}^{n-1} \frac{1}{(\mu+\vartheta_\mu)\log(\mu+\vartheta_\mu)^{2+\varepsilon}} , \quad (0 < \vartheta_\mu < 1)$$

$$= O\Big(\frac{n-\nu}{\nu(\log \nu)^{2+\varepsilon}} \Big) .$$

Thus

$$(15) \quad K_n$$

$$= O\Big\{ \sum_{\nu=0}^{[n/2]} \nu^{k+1} |\sigma_\nu^k - \sigma_{\nu-1}^k| \frac{n-\nu}{\nu(\log \nu)^{2+\varepsilon}} A_{n-\nu}^{\alpha-k-1} \Big\}$$

$$= O\Big\{ n^k (\log n)^{-(2+\varepsilon)} n^{\alpha-k} \sum_{\nu=0}^{[n/2]} |\sigma_\nu^k - \sigma_{\nu-1}^k| \Big\}$$

$$= O\{ n^\alpha (\log n)^{-(2+\varepsilon)} (\log n) \}$$

$$= O\{ n^\alpha (\log n)^{-(1+\varepsilon)} \} .$$

Concerning to L_n, we have

$$(16) \quad L_n$$

$$= O\Big[\sum_{\nu=[n/2]+1}^{n} \nu^{k+1} |\sigma_\nu^k - \sigma_{\nu-1}^k| \Big\{ \frac{n-\nu}{\nu(\log \nu)^{2+\varepsilon}} \Big\} A_{n-\nu}^{\alpha-k-1} \Big]$$

$$= O\Big\{ n^k (\log n)^{-(2+\varepsilon)} \sum_{\nu=[n/2]+1}^{n} |\sigma_\nu^k - \sigma_{\nu-1}^k| A_{n-\nu}^{\alpha-k} .$$

Analogously proceeding to (13), we get

$$\sum_{n=1}^{\infty} \frac{L_n}{n^{\alpha+1}}$$

$$= \sum_{n=1}^{\infty} \frac{n^k}{n^{\alpha+1}(\log n)^{2+\varepsilon}} \sum_{\nu=[n/2]+1}^{n} |\sigma_\nu^k - \sigma_{\nu-1}^k| A_{n-\nu}^{\alpha-k} < \infty .$$

On the other hand, from (3),

$$(17) \quad u_n^\alpha - u_{n-1}^\alpha$$

$$= \frac{\sigma_n^\alpha}{\mu_n} - \frac{\sigma_{n-1}^\alpha}{\mu_{n-1}} = \frac{\mu_{n-1}\sigma_n^\alpha - \mu_n \sigma_{n-1}^\alpha}{\mu_n \mu_{n-1}}$$

$$= \frac{\mu_{n-1}(\sigma_n^\alpha - \sigma_{n-1}^\alpha) - \sigma_{n-1}^\alpha(\mu_n - \mu_{n-1})}{\mu_n \mu_{n-1}}$$

$$= O\Big(\frac{|\sigma_n^\alpha - \sigma_{n-1}^\alpha|}{\mu_n} \Big) + O\Big(\frac{\sigma_{n-1}^\alpha(\mu_n - \mu_{n-1})}{\mu_n \mu_{n-1}} \Big)$$

$$= M_n + N_n ;$$

say.

$$(18) \quad \sum_{n=1}^{\infty} |M_n| = \sum_{n=1}^{\infty} \frac{|\sigma_n^\alpha - \sigma_{n-1}^\alpha|}{(\log n)^{1+\varepsilon}}$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\sum_{\nu=1}^{n} |\sigma_\nu^\alpha - \sigma_{\nu-1}^\alpha|}{n(\log n)^{2+\varepsilon}} + \lim_{N\to\infty} \sum_{\nu=1}^{N} |\sigma_\nu^\alpha - \sigma_{\nu-1}^\alpha| / (\log N)^{1+\varepsilon}$$

$$= O\Big\{ \sum_{n=1}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}} \Big\} + \lim_{N\to\infty} \frac{\log N}{(\log N)^{1+\varepsilon}} < \infty .$$

Since

$$\sum_{n=1}^{N} |\sigma_n^\alpha - \sigma_{n-1}^\alpha| = O(\log N) ,$$

we have

$$\sigma_n^\alpha = O(\log N) ,$$

and

$$(19) \quad \sum_{n=1}^{\infty} |N_n|$$

$$= \sum_{n=1}^{\infty} \frac{\sigma_{n-1}^\alpha}{n(\log n)^{2+\varepsilon}} = \sum_{n=1}^{\infty} \frac{O(\log n)}{n(\log n)^{2+\varepsilon}} < \infty .$$

Summing up these results and from the formula (3) and (7), we can conclude that

$$\sum_{n=1}^{\infty} | \tau_n^{\alpha} - \tau_{n-1}^{\alpha} | < \infty$$

is equivalent to

$$\sum_{n=1}^{\infty} | w_n^{\alpha} | < \infty$$

where

$$w_n^{\alpha} = \frac{1}{n A_n^{\alpha}} \sum_{\nu=1}^{n-k} t_\nu^{k-1} \Delta^k \frac{1}{\mu_\nu} A_{n-\nu-k}^{\alpha-1} .$$

If $\alpha \geq 1$

(20) w_n^{α}

$$= \frac{1}{n^{\alpha+1}} \sum_{\nu=1}^{n-k} A_{n-\nu-k}^{\alpha-1} t_\nu^{k-1} \Delta^k \frac{1}{\mu_\nu}$$

$$= \frac{1}{n^{\alpha+1}} \sum_{\nu=1}^{n-k} \nu A_\nu^k | \sigma_\nu^k - \sigma_{\nu-1}^k | \frac{1}{\nu^k (\log \nu)^{2+\varepsilon}} A_{n-\nu-k}^{\alpha-1}$$

$$= \frac{n \cdot n^{\alpha-1}}{n^{\alpha+1} (\log n)^{2+\varepsilon}} \sum_{\nu=1}^{n} | \sigma_\nu^k - \sigma_{\nu-1}^k | = O \left\{ \frac{1}{n (\log n)^{1+\varepsilon}} \right\} .$$

Therefore

$$\sum_{n=1}^{\infty} | w_n^{\alpha} | < \infty \quad ,$$

and the theorem is proved. If $\alpha < 1$, then

(21) $\sum_{n=1}^{\infty} | w_n^{\alpha} |$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \sum_{\nu=1}^{n-k} A_{n-\nu-k}^{\alpha-1} t_\nu^{k-1} \Delta^k \frac{1}{\mu_\nu}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} \left(\sum_{\nu=1}^{[n/2]} + \sum_{\nu=[n/2]+1}^{n-k} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} (P_n + Q_n) ,$$

say. Similarly to the case $\alpha \geq 1$,

$$\sum_{n=1}^{\infty} \frac{| P_n |}{n^{\alpha+1}} < \infty \quad ,$$

and

$$\sum_{n=1}^{\infty} \frac{| Q_n |}{n^{\alpha+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{\alpha} (\log n)^{1+\varepsilon}} \sum_{\nu=[n/2]+1}^{n-k} | \sigma_\nu^k - \sigma_{\nu-1}^k | A_{n-\nu-k}^{\alpha-1}$$

$$= \sum_{\nu=1}^{\infty} | \sigma_\nu^k - \sigma_{\nu-1}^k | \sum_{n=\nu-k}^{2\nu} n^{-\alpha} (\log n)^{-(1+\varepsilon)} A_{n-\nu-k}^{\alpha-1}$$

$$= \sum_{\nu=1}^{\infty} | \sigma_\nu^k - \sigma_{\nu-1}^k | \cdot O \left\{ \frac{1}{(\log \nu)^{1+\varepsilon}} \right\} < \infty ,$$

by applying Abel's transformation. Thus we can prove the theorem completely.

REFERENCES

1 L.S.Bosanquet, The absolute Cesaro summability of a Fourier series, Proc. London Math. Soc., 41(1936), 517-528.

2 M.T.Cheng, Summability factors of Fourier series at a given point, Duke Math. Journ., 15 (1948), 24-36.

3 L.Jesmanowicz, On the Cesàro means, Studia Math., 12 (1951), 145-158.

4 G.Sunouchi, On a theorem of Hardy-Littlewood, Kōdai Math. Seminar Rep., (1951), 52-54.

5 A.Zygmund, Uber einige Sätze aus der Theorie der divergenten Reihen, Bull. L'Academie Polonaise, 1927, 309-331.

Mathematical Institute,
Tōhoku University, Sendai.

(*) Received June 7, 1954.