THE TOPOLOGY OF SUBHARMONIC FUNCTIONS

By Mitsuru OZAWA

1. Introduction.

Let G be a ν-ply connected planar Jordan region whose boundary is denoted by B, ν being assumed to be a finite positive integer. Let $U = U(x, y)$ be a single-valued function pseudo-harmonic in G and continuous on B. If $U(x, y)$ has a finite number of points of relative extremum on B, then a relation

$$m - n = 2 - \nu$$

holds, where m is the number of boundary points affording relative minima to U and n is the sum of the orders of the saddle points of U on $\bar{G}$.

This theorem is a starting point of the theory introduced by M.Morse-M.H. Heins [1], in which the more general results under general assumptions has stated, but their methods highly depend upon the group-theoretic ones. Soon after Morse [1] has proved this relation by the most elementary method being able to consider as an extension of a previous paper of Morse-Van Schaack [1], in which they have concerned the so-called non-degenerate case only.

The object of the present paper is to extend the above mentioned relation to the subharmonic functions. We shall principally be interested to obtain the result. Therefore we shall begin with somewhat stronger assumptions than we need actually. In our case the so-called critical sets are not always the isolated ones (of course, not always the non-degenerate ones), and moreover they may consist of a critical line "en bloc". Difficulties will occur in this aspect. Thus we shall assume the stronger assumptions, some of which involve the essential parts of Morse's paper.

2. The basic assumptions.

Let G and B be the same as in the Morse's paper. Let $u = u(x, y)$ be a function defined on G, single-valued and subharmonic in G, and continuous on $\bar{G}$, and not reducing to a constant in any compact subregion of $\bar{G}$, where $\bar{G} = G + B$, where the subharmonicity of u means that the mean values of u in any small disc are always not less than the value at the center.

For the sake of simplicity, we shall confine ourselves to the case where u does not reduce to a constant on any subinterval of the boundary B unless the contrary is explicitly mentioned.

Definition 1. If a point (x, y) satisfies $u(x, y) = c$, then we call (x, y) lies on the level c. $\underline{U}(c)$ means the region below c, that is, the collection of all the points on $\bar{G}$ satisfying $u(x, y) < c$. Similarly, $\hat{U}(c)$, $\underline{U}(c)$, $\bar{U}(c)$ mean the region above c, the set below c, the set above c, respectively, which are defined by the collection of all the points satisfying $u(x, y) > c$, $u(x, y) \leq c$, $u(x, y) \geq c$ on $\bar{G}$, respectively.

Definition 2. Branching order of level at P with respect to a neighborhood $N(P)$. Let P be an arbitrary point of $\bar{G}$ and $N_\varepsilon(P)$ be a connected component of the set common to a fixed ε-neighborhood of P and to $\bar{G}$. Let $A_{N_\varepsilon}(P)$ be a set of the points each of which can be arcwisely connected to P along a continuous arc lying on the level $u(P)$ in $N_\varepsilon(P)$. That the level $u(P)$ at P has the finite branching order $b_{N_\varepsilon}(P)$ with respect to the neighborhood $N_\varepsilon(P)$ means that the point-set $A_{N_\varepsilon}(P) - P$ has a finite number of connected components. If it is not the case, we put $b_{N_\varepsilon}(P) = \infty$.

Theorem 1. At each point P of $\bar{G}$, $\lim_{\varepsilon \to 0} b_{N_\varepsilon}(P)$ exists and is either a finite non-negative integer or an infinite number.

Proof. A component of $A_{N_{\varepsilon'}}(P) - P$ is also arcwisely connectible in $N_\varepsilon(P)$ as a member of $A_{N_\varepsilon}(P) - P$ for $\varepsilon > \varepsilon' > 0$. Thus the monotoneity holds, i.e., $b_{N_\varepsilon}(P) \leq b_{N_{\varepsilon'}}(P)$ for $\varepsilon > \varepsilon' > 0$ and hence the desired result.

Definition 3. $\lim_{\varepsilon \to 0} b_{N_\varepsilon}(P) = b(P)$ is called the relating branching order of level at P.

Corollary 1. If $b(P) \neq \infty$, then there is a small positive number ε such that $b_{N_\varepsilon}(P) = b(P)$.

Each connected component of $N_\varepsilon(P) \cap \underline{U}(c)$ or of $N_\varepsilon(P) \cap \overline{U}(c)$ for a sufficiently small positive number ε is called the sector below $u(P)$ or above $u(P)$ relating to ε-neighborhood at P, where $c = u(P)$.

Definition 4. Canonical neighborhood of an interior point P. If there is a neighborhood $N(P)$ of P, satisfying the following conditions, then $N(P)$ is called the canonical neighborhood of P:

i) Let T map homeomorphically $\overline{N(P)}$ to the unit disc $T\overline{N(P)}$ $(|z| \leq 1,\ z = x+iy = re^{i\theta})$ such that $T(P) = 0$ and the boundary curve $(\overline{N(P)} - N(P))$ of $\overline{N(P)}$, the closure of $N(P)$, corresponds to the peripherie $(|z| = 1)$.

ii) If $b(P) \neq 0$, then each connected component of $A_N(P) - P$ corresponds to a radius

$$\theta = \frac{2\pi}{b(P)} \cdot i, \quad i = 0, 1, \cdots, b(P) - 1.$$

iii) There is no point lying on the same level at P except the set $A_N(P)$ on $\overline{N(P)}$.

iv) There are a finite number of extremum points of u on the boundary of $N(P)$, where we consider u as the function defined on that boundary.

Canonical neighborhood of a boundary point P.

If there is a neighborhood $N(P)$ of P, satisfying the following conditions, then $N(P)$ is also called the canonical neighborhood of P:

i) Let T map homeomorphically $\overline{N(P)}$ to the right half of the unit disc $T\overline{N(P)}$ such that $T(P) = 0$ and $B \cap \overline{N(P)}$ corresponds to the diameter $(x = 0,\ -1 \leq y \leq 1)$ and $(\overline{N(P)} - N(P)) - B \cap \overline{N(P)}$ corresponds to the peripherie $|z| = 1,\ x > 0$.

ii) If $b(P) \neq 0$, then each connected component of $A_N(P) - P$ corresponds to a radius

$$\theta = -\frac{\pi}{2} + \frac{\pi}{b(P)+1} \cdot i, \quad i = 1, 2, \cdots, b(P).$$

iii) and iv) are the same as in the previous case of inner point, respectively.

We are now in a position to explain our fundamental assumptions.

F.A.I. There is no point on $\overline{G}$ such that $b(P) = \infty$.

F.A.II. There always exists a canonical neighborhood $N(P)$ of each $P \in \overline{G}$. For every $P \in G$, $T^{-1}(|z| < r,\ 0 < r < 1)$ $x > 0)$ is also a canonical neighborhood of P, and for every $P \in B$, $T^{-1}(|z| < r,\ 0 < r < 1,\ x > 0)$ is also a canonical neighborhood of P.

F.A.III. For any point $P \in \overline{G}$, P cannot be a cluster point of the sequence $\{P_n\}$, such that $P_n \in G$ and $b(P_n) \neq 2$.

F.A.IV. For any point $P \in B$, P cannot be a cluster point of the sequence $\{P_n\}$, such that $P_n \in B$ and $b(P_n) \neq 1$.

From F.A.II. and the Heine-Borel's covering theorem we have

Corollary 2. There are a finite number of relative extremum points of u on B, when u is considered as a continuous function defined on B, and

Corollary 3. There are a finite number of subintervals of each proper boundary of each sector in $N(P)$, on which $u(P)$ is monotone and continuous, where the proper boundary of a sector means the set common to the boundary of $N(P)$ and to the sector.

3. Critical points.

We shall now define many sorts of critical points.

Definition 5. Saddle point.

If the number $\sigma_{N_\varepsilon}(P)$ of the sectors below $u(P)$ relating to $N_\varepsilon(P)$ at P is not less than 2, then $\sigma_{N_\varepsilon}(P) - 1$ is called the order of the saddle point P relating to $N_\varepsilon(P)$.

As a remark with respect to the definition of the sector below $u(P)$, we shall only concern that it is always arcwisely connectible to the vertex P along a continuous arc belonging to that sector below $u(P)$.

In general we have the following theorem with regard to the quantity $\sigma_{N_\varepsilon}(P)$

Theorem 2. At each point P of $\bar{G}$, $\lim_{\varepsilon\to 0} \sigma_{N_\varepsilon}(P)$ exists and is either a finite non-negative integer or an infinite number.

Proof can be done in a similar manner as in Theorem 1. But under our F.A. there does not occur that $\lim_{\varepsilon\to 0} \sigma_{N_\varepsilon}(P) = \infty$. (See. Lemma 2, No. 5)

Corollary 4. If $\lim_{\varepsilon\to 0} \sigma_{N_\varepsilon}(P) \neq \infty$, then there is a small positive number ε such that $\sigma_{N_\varepsilon}(P) = \lim_{\varepsilon\to 0} \sigma_{N_\varepsilon}(P)$.

Definition 6. $\lim_{\varepsilon\to 0} \sigma_{N_\varepsilon}(P) - 1 = \sigma(P) - 1$ is called the order of the saddle point P, if $\sigma(P) \geqq 2$.

Definition 7. If $u(P) < u(Q)$ (or $u(P) > u(Q)$) holds for any point Q belonging to $N_\varepsilon(P)$ and being different from P, then P is called a strictly relative minimum point (or maximum point). If $u(P) \leqq u(Q)$ (or $u(P) \geqq u(Q)$) holds for any point Q belonging to $N_\varepsilon(P)$ and being different from P, and there is a point Q for each ε such that $u(P) = u(Q)$, then P is called a non-strictly relative minimum (or maximum) point.

Definition 8. If any point of a one-dimensional connected continuum γ is a non-strictly relative minimum point, then γ is called a minimum locus.

Definition 9. If there is at least one point P_0, named a connecting point relating γ_0, satisfying the following conditions, then the minimum locus γ_0 is called an open minimum locus.

(i) $P_0 \in \gamma_0$,

(ii) P_0 is arcwisely connectible to every point of γ_0 by a subarc of γ_0.

If there is no connecting point on a given minimum locus, then this locus is called a closed minimum locus.

Theorem 3. On a minimum locus γ_0 (and relating connecting point P_0), $u(P)$ remains constant.

Proof. If P is an inner point of γ_0, then $u(Q) \equiv u(P)$, where $Q \in \gamma_0$ and belongs to a canonical neighborhood of P. Thus by Heine-Borel's covering theorem $u(Q) \equiv$ constant at any point of γ_0. Since P_0 is a cluster point of a sequence $\{P_n\}$, $P_n \in \gamma_0$, the relation $u(P_0) = \lim_{n\to\infty} u(P_n) \equiv$ constant remains true.

Definition 10. If there is at least one critical point, that is, either a relative extremum point (strictly or non-strictly) or an interior point having $b(P) \neq 2$ or a boundary point having $b(P) \neq 1$, on the level lying on c, then c is called a critical value. If it is not the case, that is, there are only the ordinary points on the level lying on c, then c is called an ordinary value.

4. Considerations in the small.

Next we shall determine the local aspects of the level line around a point P.

A) Case of an inner point P.

Since the sectors at P are finite in number by F.A.I, it is sufficient to examine the local aspects of the level line in each sector. Moreover we shall be able to choose a canonical neighborhood $N(P)$ satisfying the following condition by F.A.II and III:

In $N(P)$ there is no point Q with $b(Q) \neq 2$ except at most at P.

Let us denote the sector in question by $S(P)$ and its proper boundary by $PB(P)$ and other boundary by $B(P)$.

i) If $b(P) = 0$, then P is a strictly relative minimum point in view of the subharmonicity of u, thus $\sigma(P) = 0$, and $B(P)$ is an empty set. $S(P)$ is a sector above $u(P)$.

Let $R_\varepsilon(P)$ be a domain such that $u(Q) - u(P) < \varepsilon_0$ and $\varepsilon_0 \lneqq u(T) - u(P)$, $T \in PB(P)$, and P and Q are mutually arcwisely connectible on $R_{\varepsilon_0}(P)$. Let the boundary of $R_{\varepsilon_0}(P)$ be denoted by Γ_{ε_0}. Then Γ_{ε_0} is a simple closed curve and $\Gamma_{\varepsilon_0} \cap PB(P)$ is empty. And, on Γ_{ε_0}, $u(Q) = u(P) + \varepsilon_0$. Let ε vary from 0 to ε_0, then $R_\varepsilon(P)$ monotonically expands from a point P to a domain $R_{\varepsilon_0}(P)$, and Γ_ε consists always of a simple closed curve. Strictly speaking, we must prove the following fact:

By Lemma 10 which will be explained in the sequel, there is only a finite number of critical values. Hence we may choose such a canonical neighborhood $N(P)$ that there is no critical value except at P. If there are two disjoint connected components on which $u(Q) < u(P) + \varepsilon_0$, then there is at least one critical point other than P, which leads to a contradiction. Thus there is only one connected component, satisfying $u(Q) < u(P) + \varepsilon_0$, in $N(P)$.

ii) If $b(P) = 1$, then P is a non-strictly relative minimum point in view of the subharmonicity of u; thus $\sigma(P) = 0$, and $B(P)$ is only one simple arc starting from P to a boundary point R_1 of $N(P)$. $S(P)$ is a sector above $u(P)$.

Let $R_{\varepsilon_0}(P)$ be a domain such that $u(Q) - u(P) < \varepsilon_0$ and $\varepsilon_0 < \min_n (u(Q_n) - u(P))$, where $Q_n (\neq R_1)$ is the relative minimum point on $PB(P)$, and P and Q are mutually arcwisely connectible on $R_{\varepsilon_0}(P)$. Let the boundary of $R_{\varepsilon_0}(P)$ belonging to $S(P)$ be denoted by Γ_{ε_0}, being also a simple arc and ending at two boundary points T_1, T_2 on which $u(T_i) = u(P) + \varepsilon_0$ and u is monotone and continuous on $\widehat{R_1 T_i}$.

iii) If $b(P) \geqq 2$, then $S(P)$ is either a sector below $u(P)$ or above $u(P)$, and $B(P)$ consists of two simple arcs $\widehat{PR_1}$, $\widehat{PR_2}$; $R_1, R_2 \in \overline{PB(P)}$.

If $S_j(P)$ is a sector above $u(P)$, then $R_\varepsilon^j(P)$ and Γ_ε^j are similarly defined.

If $S_j(P)$ is a sector below $u(P)$, then $R_\varepsilon^j(P)$ and Γ_ε^j are defined in the following manner:

$R_\varepsilon^j(P)$ is a domain on which $u(Q) < u(P) - \varepsilon$ and any point $Q \in R_\varepsilon^j(P)$ is arcwisely connectible to P along a curve belonging to $R_\varepsilon^j(P)$, where $-\varepsilon > u(Q_n) - u(P)$, Q_n being any relative maximum point on $PB(P)$. Γ_ε^j is a boundary arc belonging to $S_j(P)$.

In each case $R_\varepsilon^j(P)$ is a simply connected domain bounded by Γ_ε^j, by two radii $\widehat{PR_1}$, $\widehat{PR_2}$ and by two subarcs $\widehat{R_1T_1}$, $\widehat{R_2T_2}$ on which u is monotone. $R_\varepsilon^j(P)$ increases monotonically and continuously from $\underline{A_N(P) \cap \overline{S_j(P)}}$, that is, two radii PR_1, PR_2, when ε increases monotonically and continuously from 0. Γ_ε^j is a simple arc on which $u(Q) = u(P) + \varepsilon$ or $u(Q) = u(P) - \varepsilon$ and $b(Q) = 2$ for each ε.

B) Case of a boundary point P.

i) If $b(P) = 0$, then P is a strictly relative maximum or minimum point.

ii) If $b(P) \geq 1$, it is sufficient to consider the circumstances at a boundary sector.

We shall be able to choose a canonical neighborhood $N(P)$ satisfying the following condition by F.A.II, III and IV:

In $N(P) - B \frown \overline{N(P)}$ there is no point Q with $b(Q) \neq 2$, and on $B \frown \overline{N(P)}$ there is no point Q with $b(Q) \neq 1$. Since the definitions of $R_\varepsilon^j(P)$ and Γ_ε^j are similar as in A), so we may omit off the discussions.

Definition 11. $CN(P)_\varepsilon = \sum_{j=1}^{b(P)} R_\varepsilon^j(P) + A_N(P)$ is called a cylindrical ε-neighborhood of P, if $b(P) \geq 1$. If $b(P) = 0$, then $R_\varepsilon(P)$ is called so.

Lemma 1. For any point of $\overline{G}$, $CN(P)_\varepsilon$ is a simply connected domain increasing from $A_N(P)$ with ε, if ε is a sufficiently small positive number. The same conclusion holds for $R_\varepsilon^j(P)$ of j-th sector S_j, and Γ_ε^j is a simple arc and $b(Q) = 2$ for every points $Q \in \Gamma_\varepsilon^j$ for each sufficiently small positive number ε.

We shall often make use of these $R_\varepsilon^j(P)$ and Γ_ε^j each denoting j-th sectorial ε-neighborhood of P and its proper boundary, respectively. Moreover we denote PR_1 and PR_2 on which $u(Q) = u(P)$ by the equally level boundaries of $R_\varepsilon^j(P)$, and R_1T_1 and R_2T_2 by the monotonic boundaries of $R_\varepsilon^j(P)$, where S_j is not a boundary sector. If S_j is a boundary sector, then there is one equally level boundary PR_1 and two monotonic boundaries R_1T_1 and PT_2. Let $V_\varepsilon^j(P) = R_\varepsilon^j(P) + \Gamma_\varepsilon^j$ two monotonic boundaries, and we call it j-th adding sectorial ε-closure of P.

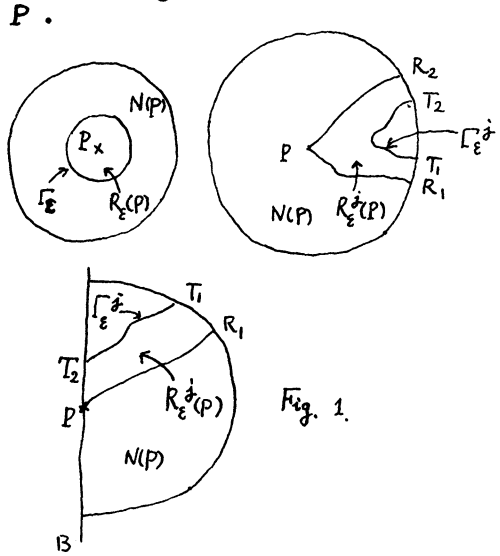

Fig. 1

5. Relation between $b(P)$ and $\sigma(P)$.

Lemma 2. Let P be a non extremum point.

If $P \in G$, then
$$2\sigma(P) \leqq b(P).$$

If $P \in B$, then
$$2(\sigma(P) - 1) \leqq b(P).$$

Proof. We may prove this by choosing a cylindrical ε-neighborhood $CN(P)_\varepsilon$ of P. By the subharmonicity of u, two sectors below $u(P)$ have no common point except only a common vertex P. If $P \in G$, then each sector below $u(P)$ has two distinct boundaries in $CN(P)_\varepsilon$. Therefore the number of the connected components of $A_{CN}(P) - P$ is at least $2\sigma(P)$. If $P \in B$, then each boundary sector below $u(P)$ has a boundary in $CN(P)_\varepsilon$ and each interior sector below $u(P)$ has two distinct boundaries in $CN(P)_\varepsilon$. Therefore we have $2(\sigma(P) - 2) + 2 \leqq b(P)$.

Since u is subharmonic, there is no strictly relative maximum point in G and no non-strictly relative maximum point on $\overline{G}$.

Lemma 3. If $b(P) = 1$, then P is either an ordinary boundary point or a boundary non-strictly relative minimum point or an inner non-strictly relative minimum point.

Proof. If P is not a relative minimum point, then there is a point Q such that $u(P) \gneqq u(Q)$ in $CN(P)_\varepsilon$. Thus there is at least one sector below $u(P)$, that is, $\sigma(P) \geq 1$.

If $P \in G$, then $2 \leqq 2\sigma(P) \leqq b(P) = 1$, which is absurd.

If $P \in B$, then $2\sigma(P) \leqq b(P) + 2 = 3$, that is, $\sigma(P) = 1$. Then there is at least one sector above $u(P)$, for, if it is not the case, we have two possibilities; that is, P is a strictly maximum point, or a subarc of B ending P lies on the level $u(P)$ in $CN(P)_\varepsilon$. These are both contradictory.

Lemma 4. If P is an ordinary inner point, then $b(P) = 2$ and $\sigma(P) = 1$, and, if P is an ordinary boundary point, then $b(P) = 1$ and $\sigma(P) = 1$, and vice versa.

Proof of the above Lemma is easy.

6. Considerations in the large.

Lemma 5. Each closed minimum locus can be divided into a finite number of simple arcs.

Proof. If a closed minimum locus cannot be divided into a finite number of simple arcs, then there is an infinite number of points with $b(P_n) \geqq 3$. These points cluster at a point on $\bar{G}$, which is impossible by F.A.III and IV.

Definition 12. Euler number of a closed minimum locus.

A closed minimum locus can be regarded as a one-dimensional closed complex in the sense of the combinatorial topology. We shall now define the Euler number of this locus by a number $a_0 - a_1$, where a_i $(i = 0, 1)$ are the numbers of the i-dimensional simplices.

Lemma 6. The closed minimum loci are finite in number and each of them is of finite Euler number.

Proof. Each closed minimum locus has either at least a point $b(P) = 1$ or at least a point $b(P) \geqq 3$ or $b(P) = 2$ for every point of that locus. If there is an infinite number of closed minimum loci, then there are an infinite number of either points P_n with $b(P_n) = 1$ or $\geqq 3$, or looping arcs, i.e., simple closed curves in the ordinary sense. If $\{P_n\}$ contains an infinite subsequence $\{P_{n_\nu}\}$ such that $b(P_{n_\nu}) = 1$ or $\geqq 3$ and $P_{n_\nu} \in G$, then this is absurd by F.A.III. Moreover if $\{P_n\}$ contains an infinite subsequence $\{P_{n_\nu}\}$ such that $b(P_{n_\nu}) \geqq 3$ and $P_{n_\nu} \in B$, this is absurd by F.A.IV. If $\{P_n\}$ contains an infinite subsequence $\{P_{n_\nu}\}$ such that $b(P_{n_\nu}) = 1$ and $P_{n_\nu} \in B$, then this is absurd by Corollary 3, since each P_{n_ν} is a relative minimum point on B.

Each looping arc γ_n can correspond to a boundary component of B, and the different looping arcs can correspond to the different boundary components of B by a suitable choice, since u is a subharmonic continuous function. The finiteness of the looping arcs is concluded by the finiteness of the connectivity of G.

The second half of this Lemma follows by the above Lemma 5.

Lemma 7. Strictly relative minimum or maximum points of u are finite in number.

Lemma 8. At each connecting point P_0 relating to an open minimum locus γ_0, $u(P_0) =$ the level of γ_0 and $\sigma(P_0) \geqq 1$. Moreover $b(P_0) \geqq 3$ for $P_0 \in G$ and $b(P_0) \geqq 2$ for $P_0 \in B$.

Proof. If $\sigma(P_0) = 0$, then P_0 is a relative minimum point and arcwisely connectible to γ_0. Thus $P_0 \in \gamma_0$, which is absurd. Thus $\sigma(P_0) \geqq 1$. And moreover there are two sectors above $u(P_0)$ in $CN(P_0)_\varepsilon$ by the existence of γ_0. Hence we have $b(P_0) \geqq 3$ for $P_0 \in G$, and $b(P_0) \geqq 2$ for $P_0 \in B$.

Lemma 9. Each open minimum locus can be divided into a finite number of simple arcs and the open minimum loci are finite in number.

Proof. The first half of the Lemma is easy.

Each open minimum locus has at least one relating connecting point P_n. Thus there are an infinite number of connecting points $\{P_n\}$, if the open minimum loci are infinite in number. If $\{P_n\}$ has an infinite subsequence $\{P_{n_\nu}\}$ all points of which coincide with a point P_0, then $b(P_0) = \infty$, which contradicts F.A.I. Therefore there is an infinite subsequence $\{P_{n_\nu}\}$ all points of which are different. On the other hand, P_{n_ν} satisfies either $b(P_{n_\nu}) \geqq 3$ or $\geqq 2$, according to $P_{n_\nu} \in G$ or B. Each case contains a contradiction by F.A.III or IV, respectively.

Definition 13. Let Γ_0, called a relating curve to a given open minimum locus γ_0, be a curve obtained by adding all the relating connecting points to a given open minimum locus γ_0 .

Obviously Γ_0 is a one-dimensional closed complex.

Definition 14. Euler number $E(\gamma_0)$ of an open minimum locus γ_0 .

$E(\gamma_0) =$ (Euler number of the relating curve Γ_0)
- (the number of the relating connecting points to γ_0).

Lemma 10. There are a finite number of connected components of the critical points on $\overline{G}$.

Lemma 11. There are a finite number of critical values of u .

These Lemmas are evident by F.A.II, III and IV and Lemmas 6, 7 and 9 and the theorem 3.

7. Maximal continuations of a level curve.

By the finiteness of the critical values of u we may put that there is no critical value on the closed interval $[c-\varepsilon, c+\varepsilon]$ except only at a value c which is either a critical value or an ordinary value.

A connected component $L(c)$ can be covered by a finite number of canonical neighborhoods $N(P_i)$, $i = 1$, ..., n , where the sequence $\{P_i\}$ contains all the critical points such that $b(P) \geq 3$ if $P \in G$, $b(P) \geq 2$ if $P \in B$ and the points being $b(P) = 1$, and $P_i \in L(c)$. P_i may be not all different.

If $M(c)$ satisfies the following conditions, then we call it a maximal arrivable level component of $L(c)$ or a maximal continuation.

i) $M(c) \subset L(c)$ and $M(c)$ is a closed set on $L(c)$.

ii) Let $\{P_{i_\nu}\}$ be a subset of $\{P_i\}$ belonging to $M(c)$. And $\{P_{i_\nu}\}$ can be ordered such that $\{P_{i_\nu}\}$ and $\{P_{i_{\nu+1}}\}$ are mutually connectible along a simple continuous arc Z , belonging to a set common to a connected component D of $U(c)$ and to the sum set $N(P_{i_\nu}) \cup N(P_{i_{\nu+1}})$, and on that arc Z $u(R) < c+\varepsilon^\circ$, where D is independent of the choice of the index and ε° is an arbitrary small positive number and $\varepsilon^\circ < \varepsilon$.

iii) The subarc $\widehat{P_{i_\nu} P_{i_{\nu+1}}}$ of $M(c)$ is simple and connected, and it belongs to $N(P_{i_\nu}) \cup N(P_{i_{\nu+1}})$.

iv) Any subarc of $L(c)$ containing $M(c)$ does not satisfy these three conditions.

We may conventionally suppose $P_{i_\nu} \notin \overline{N(P_{i_{\nu+1}})}$ and $P_{i_{\nu+1}} \notin \overline{N(P_{i_\nu})}$

Let $L(c)$ be a maximal continuation, and $\{N(P_i)\}$ be a finite number of canonical neighborhoods covering $L(c)$ and satisfying the above mentioned conditions ii) and iii). We denote further by $\overline{N(P_i)}$ the part of $\overline{N(P_i)}$ belonging to D , that is, $D \cap \overline{N(P_i)}$. Corresponding to each $\overline{N(P_i)}$ we construct the sectorial ε_0 -neighborhood $R_{\varepsilon_0}(P_i)$ satisfying the condition that the two monotonic boundaries of $R_{\varepsilon_0}(P_i)$ are contained in the former and latter sectorial ε_0 -closures, respectively. We put $\Gamma_{\varepsilon_0}(P_i) = \widehat{T_1(i)T_2(i)}$ and $T_1(i) \in \Gamma_{\varepsilon_0}(P_{i-1})$, $T_2(i) \in \Gamma_{\varepsilon_0}(P_{i+1})$ and two monotonic boundaries $\widehat{T_1(i)R_1(i)}$ and $\widehat{T_2(i)R_2(i)}$.

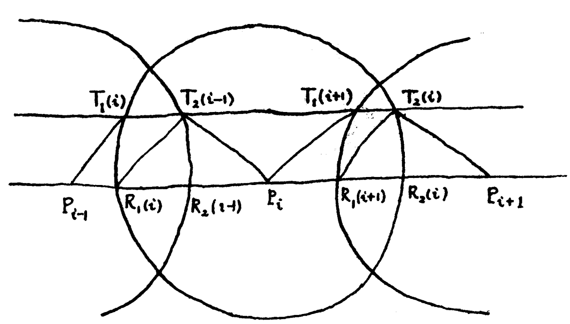

Fig. 2.

Let $RD(c)_{\varepsilon_0} = \sum_{i=1}^{n} R_{\varepsilon_0}(P_i)$

and we call it a rectangular-like or ring-like domain. On $\overline{RD(c)_{\varepsilon_0}}$ we can introduce a triangulation: The arcs $\widehat{T_2(i-1)P_i}$, $\widehat{R_2(i-1)P_i}$, $\widehat{T_2(i-1)T_1(i+1)}$, $\widehat{P_i T_1(i+1)}$, $\widehat{R_1(i+1)}$

$\overline{T_i(i+1)}$, $\overline{P_i R_1(i+1)}$, $\overline{T_1(i+1) T_2(i)}$, $\overline{R_2(i+1) R_2(i)}$, $\overline{R_1(i+1) T_2(i)}$ and $\overline{T_2(i) R_2(i)}$ ($i = 1, \ldots, n$) constitute all the edges of $\overline{RD(c)_{\varepsilon_0}}$. Here we have to remark that the notations of the above listed ones are suitably changed for $i = 1$ and n. The set $\overline{RD(c)_{\varepsilon_0}} - L(c)$ is denoted by ε_0-field of $L(c)$: $\mathfrak{F}_{\varepsilon_0}(L(c))$.

When $\overline{RD(c)_{\varepsilon_0}}$ is triangulated, we must consider all the vertices and edges belonging to $L(c)$ with their orders, or, more precisely, with their incidence relations (in the sense of the combinatorial topology) to the other edges and triangles, and, if an edge occurs two times in the different orders, then we shall consider that these ordered edges are different and are separated mutually into two different edges having the same incidence relations for the edges and triangles belonging to $\overline{RD(c)_{\varepsilon_0}} - L(c)$ as that of the original edge, respectively. And we call this triangulated $L(c)$ and $\overline{RD(c)_{\varepsilon_0}}$ the ordered $L(c)$ and the ordered $\overline{RD(c)_{\varepsilon_0}}$, respectively. Then ε_0-field $\mathfrak{F}_{\varepsilon_0}(L(c))$ itself is considered as an ordered triangulated ε_0-field.

An example of the ordered $L(c)$ and the ordered $\overline{RD(c)_{\varepsilon_0}}$.

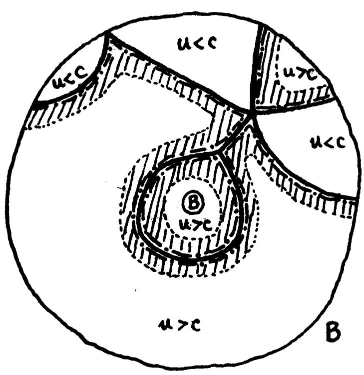

Fig. 3.

▬ shows a level curve lying on c.
—·— shows an ordered maximal continuation.
▨ shows an ε_0-field $\mathfrak{F}_{\varepsilon_0}(L(c))$.
.... shows a level curve lying on $c + \varepsilon_0$.

The Euler number of $\mathfrak{F}_{\varepsilon_0}(L(c))$ is equal to zero. In fact, the Euler number of $\mathfrak{F}_{\varepsilon_0}(L(c))$ is equal to (the Euler number of ordered $\overline{RD(c)_{\varepsilon_0}}$) – (the Euler number of ordered $L(c)$), and we can easily calculate this by the above triangulation. Thus we have

Lemma A1. Any ε_0-field $\mathfrak{F}_{\varepsilon_0}(L(c))$ of a maximal continuation $L(c)$ is of Euler number zero.

We shall next prove that the maximal continuations lying on the level c are finite in number for any value c.

Let c be an ordinary value, then each connected component is a simple arc. And there are only two possibilities, that is, one is the case of a looping simple arc (simple closed curve) and the other is that of a simple curve ending at two different boundary points. Moreover $\sigma(P) = 1$ and $b(P) = 2$ hold for any point $P \in G$, and $\sigma(P) = 1$ and $b(P) = 1$ hold for boundary points P. Thus each connected component lying on c is a maximal continuation.

a) If γ is a looping simple arc, then γ bounds a domain D_γ belonging to G.

If $u(Q) > c$, where $Q \in D_\gamma$ and Q is a neighboring point of γ, then there is a boundary component of B in D_γ. If $u(Q) < c$, where $Q \in D_\gamma$ and Q is a neighboring point of γ, then there is either a minimum point of u or a boundary component of B in $\overline{D}_\gamma - \gamma$. These facts due to the subharmonicity of u.

Thus a looping simple arc γ, lying on an ordinary value, corresponds to either a minimum point or a boundary component of B. Therefore the looping simple arcs lying on the same ordinary value are finite in number.

b) If there are an infinite number of non-looping simple arcs lying on the level c, then there is a boundary component of B on which there exist an infinite number of points lying on c. Then $\{P_n\}$ has an infinite subsequence $\{P_{n_\nu}\}$ lying on B_j such that the subarc $\widehat{P_{n_\nu} P_{n_{\nu+1}}}$ of B_j has no point lying on c. On $\widehat{P_{n_\nu} P_{n_{\nu+1}}}$ there is at least either a maximum point or a minimum point, which leads to a contradiction. For there are only a finite number of extremum points on B .

Next let c be a critical value of u .

Let $L(c)$ be an ordered maximal continuation lying on c . If we consider the level lines lying on $c+\varepsilon_0$, that is, the proper boundaries of ε_0 -field of all $L(c)$, then there are a finite number of the level lines lying on $c+\varepsilon_0$. The number of the maximal continuations lying on c is less than or equal to the number of the level lines lying on $c+\varepsilon_0$. And each of the isolated points lying on c is also considered as a maximal continuation, but this case is evident. Thus we have

Lemma A2. There are a finite number of maximal continuations lying on any value of u on $\overline{G}$.

8. Lemmas concerning the variations of the Euler numbers.

Let $E(S)$ be the Euler number of a closed set S . We put $E_1(c)=E(U(c))$, $E_2(c)=E(\overline{(U(c))})$ and $E_3(c) = \lim_{\varepsilon\to 0} E_1(c-\varepsilon)$.

If c is an ordinary value of u , then $U(c)=\overline{(U(c))}$.

Proof. Each connected component of the levels lying on c is a simple curve and two possibilities can occur, one is the case of a simple closed curve and the other is that of a curve ending to two different boundary points.

Moreover $\sigma(P)=1$ for any point P with $u(P)=c$. Thus for any point P with $u(P)=c$ there is only one sector below c . Therefore P is a cluster point of P_n , $u(P_n)<c$. Thus $U(c)\subseteqq\overline{(U(c))}$. On the other hand $U(c)\supseteqq\overline{(U(c))}$ is evident for every c , either ordinary or critical. Thus we have

Lemma B. If c is an ordinary value of u , then $E_1(c)=E_2(c)$.

Lemma C. If c is an ordinary value of u and ε is a sufficiently small positive number, then $E_1(c) = E_3(c) = E_1(c-\varepsilon) = E_1(c+\varepsilon)$.

Corollary 5. If there is no critical value on a closed interval $[a,b]$, then $E_i(c) =$ a constant for $a \leqq c \leqq b$.

By the definition of $U(c)$ and $\overline{(U(c))}$, $U(c)-\overline{(U(c))}$ consists of some number of open minimum loci, of closed minimum loci and of the isolated minimum points lying on the level c . These are finite in number.

Let $C_{-i}(c)$ and $O_{-i}(c)$ be the numbers of the closed minimum loci and the open minimum loci whose Euler number equal to $-i$, respectively, and both lie on the level c . Let $C(c)$ be the sum of the number of the closed minimum loci whose Euler number are equal to 1 and the number of the isolated minimum point lying on the level c .

Then we have

Lemma D.

$$E_1(c) - E_2(c) = \sum_{i=0}^{\infty}(-i)\,C_{-i}(c) + \sum_{i=0}^{\infty}(-i)\,O_{-i}(c) + C(c).$$

If we construct a closed covering domain $C(\overline{(U(c))})$ of $\overline{(U(c))}$ by separating the $\sigma(P)$ sectors below $u(P)$ at each saddle point P , then the every level lying on c is simple in $C(\overline{(U(c))})$. Here we remark that the method of the separation is the following manner: Two different sectors below $u(P)$, P being a saddle point of u , are not mutually connectible in a canonical neighborhood $N(P)$ of P .

Thus we have the following lemma in the similar manner as in Lemma A_1.

Lemma E. $E(C(\overline{(U(c))})) = E_1(c-\varepsilon)$ for a sufficiently small positive ε .

Let $S(c)$ be the sum of the orders of all saddle points lying on c .

Lemma F.

$$E(C(\overline{(U(c))})) - E_2(c) = S(c).$$

Proof. For each saddle point the Euler number decreases with the order of that saddle point. For a point P with either $b(P)=0$, $\sigma(P)=1$ (boundary maximum point) or $b(P)=2$, $\sigma(P)=1$, no change occurs. For minimum loci and minimum points and such a connecting point that $\sigma(P)=1$, we need not pay our attentions.

Lemma G. For a sufficiently small positive number ε and for any c,

$$E_1(c+\varepsilon) = E_1(c).$$

Proof. By the finiteness of the maximal continuations lying on the level c and their relating ε-fields and the Lemma A_1, this is evident.

9. The first main theorem.

Theorem A. Let u satisfy F.A.I, II, III and IV and be subharmonic, then we have a relation

$$2-\nu = C_1 + C_2 + \theta - S,$$

where $C_1 = \sum_c C(c)$, $C_2 = \sum_c \sum_{i \geq 0} (-i)\, C_{-i}(c)$, $\theta = \sum_c \sum_{i \geq 0} (-i)\, O_{-i}(c)$ and $S = \sum_c S(c)$.

Proof. There are a finite number of the critical values of u, denoted by $c_1, \ldots\ldots, c_n$ according to their orders, and we suppose that there is no other critical value. Let $m < u \leqq M$ hold on $\overline{G}$, then $E_1(m) = 0$ and $E_1(M) = 2-\nu$. By the Lemmas mentioned above we can calculate all the jumps of the Euler numbers $E_1(c)$ at all the critical values c_i. Therefore we have the Theorem A.

Theorem B. Under the same assumptions as in Theorem A, then we have an inequality

$$-C_2 + C_3 \leqq \nu - 1,$$

where $C_3 = \sum_c \sum_{i \geq 0} C_{-i}(c)$.

Proof. The theorem is deduced by the subharmonicity of u. One may refer to the proofs of Lemma 6 and Lemma A_2. More precisely, if γ has the Euler number $-n\ (n \geqq 0)$ and is a closed minimum locus, then there correspond to at least $n+1$ different boundary components. And the different closed minimum loci correspond to the different collections of the corresponding boundary components.

10. Supplementary note.

1. In the case where u can reduce to a constant on any subinterval of the boundary curve , we modify our assumptions as follows:

F.A.I, II, III are unaltered.

F.A.IV'. On B, there are a finite number of maximal intervals on which u is constant, where the word "maximal" means that there is no interval containing the interval and satisfying the constancy of u. These intervals are denoted by $I_1, \ldots\ldots, I_n$. For any point $P \in B - \sum_{j=1}^{n} I_j$, P cannot be a cluster point of a sequence $\{P_n\}$, such that $P_n \in B$ and $b(P_n) \neq 1$. For any end point P of I_j, P cannot be a cluster point of a sequence $\{P_n\}$, such that $P_n \in B,\ \notin I_j$ and $b(P_n) \neq 1$. For any point $P \in I_j$, P cannot be a cluster point of a sequence $\{P_n\}$, such that $P_n \in I_j$ and $b(P_n) \neq 2$.

Under these fundamental assumptions, we have the same results, formally, and we can proceed to our discussions with some unessential modifications. The concept of the "level index" due to Morse-Heins [1] or Morse [2] is contained in our definition of the Euler number of the open minimum locus.

11. Another application and examples.

Corollary 6. $C_1 - S \geqq 2-\nu$.

Proof. Since $C_2 \leqq 0$ and $\theta \leqq 0$, we have the desired result.

Corollary 7. If $\nu = 1$ and $C_1 = 1$, then $S = 0$.

i) Let $u = |f(z)|^2$, where $f(z)$ is a regular analytic function of z on $\overline{G}$. In this case there is no non-isolated differential critical point on $\overline{G}$, and moreover we can transfer the Morse-Heins' theory by introducing the poles of u suitably and taking the logarithm of u. The introducing method of the poles of u will be explained in No. 14.

ii) The case where $\Delta u = Pu$ and either $P \geqq 0$ or $\leqq 0$, definitely, on $\overline{G}$.

iii) The case where $\Delta u =$ const. on $\overline{G}$.

There are many examples other than those listed above.

12. Topology of superharmonic functions.

Let $-v$ be subharmonic and continuous and satisfy the F.A.I, II, III and IV. We shall now directly examine the topology of superharmonic function v without passing through that of subharmonic function $-v$. Here we should explain some needed modifications of several definitions.

i) Saddle points: Let $N(P)$ be a canonical neighborhood of P. Let $v(Q) < v(P)$, $Q \in N(P)$, $\neq P$, then we construct a component, called an extended sector below $v(P)$, each point of which is arcwisely connectible with Q by a continuous arc belonging to $N(P) - P$, and on which $v(R) \leqq v(P)$. If the number $\tau_N(P)$ of the extended sectors below $v(P)$ is not infinite, then there exists a limit of $\tau_N(P)$ when N decreases to a point P. If $\tau(P) = \lim_{N \to P} \tau_N(P) \geqq 2$, then we say that $\tau(P) - 1$ is the order of saddle point of v or P is a saddle point of order $\tau(P) - 1$.

ii) Maximum locus of v: The maximum locus of v is the minimum locus of $-v$.

iii) Open or closed maximum locus. If a maximum locus γ_0 of v has either a connecting point P_0 when γ_0 is considered as a minimum locus of $-v$ or a boundary point, then γ_0 is called an open maximum locus of v. If it is not the case, then γ_0 is called a closed maximum locus of v.

The definitions of the critical points, critical values, ordinary points and ordinary values are the same as in the subharmonic case. We have already proved the finiteness of the critical values of $-v$, and hence we have simultaneously the finiteness of the critical values of v.

iv) The Euler number of a closed maximum locus γ.

We shall now define that the Euler number $E(\gamma)$ of a closed maximum locus γ is equal to the number of a closed minimum locus γ of $-v$.

v) The Euler number of an open maximum locus γ_0.

Let Γ_0 be a relating curve of γ_0 obtained by adding all the relating connecting points and the relating boundary points of γ_0. We then define the Euler number $E(\gamma_0)$ of γ_0 in the following manner:

$$E(\gamma_0) = E(\Gamma_0) - (\text{the number of all the relating connecting points and the relating boundary points of } \gamma_0).$$

Then we have the second main theorem:

<u>Theorem F.</u> Let v be superharmonic and continuous on G, continuous on B, and satisfy the F.A.I, II, III and IV. Then we have a relation

$$2 - \nu = C_{0m} + C_{1M} + C_{2M} + O_M - S_M,$$

where C_{0m} is the number of the isolated boundary minimum points, C_{1M} is the number of the closed maximum loci having the Euler number 1, C_{2M} is the sum of the Euler numbers of the closed maximum loci having the Euler number $\leqq 0$, O_M is the sum of the Euler numbers of the open maximum loci and S_M is the sum of the orders of the extended saddle points.

Let $E(\gamma(f))$ be the number of γ of the function f. Let M and m be the maximum locus and minimum locus, respectively, where the word "locus" contain the isolated point. Let C and O denote the closed and open locus, respectively. Let indices 1 and 2 indicate the first and the second kinds, respectively, there the first kind or the second kind means the locus whose Euler number is 1 or is less than or equal to 0, respectively.

Let $E_m(v) = \sum E(m_1(v))$,

$E_1(v) = \sum E(CM_1(v))$, $E_2(v) = \sum E(CM_2(v))$,

$O_1(v) = \sum E(O_1 M(v) \cap Cm(-v))$,

$O_2(v) = \sum E(O_2 M(v) \cap Cm(-v))$,

$O_3(v) = \sum E(O_2 M(v) \cap O_2 m(-v))$,

$E_1(-v) = \sum E(C_1 m(-v))$, $E_2(-v) = \sum E(C_2 m(-v))$.

Let n_{cb} be the total number of the relating boundary points of such a locus that belongs to the set of $O_2 M(v) \cup O_1 M(v) \cap Cm(-v)$, and n_{ob} be the total number of the relating boundary points of such a locus that belongs to the set of $O_2 M(v) \cap O_2 m(-v)$ and does not belong to a set of relating connecting points of the given maximum locus of v.

Then we have the following four relations:

1) $$2 - \nu = E_m(v) + E_1(v) + O_1(v) + O_2(v) + O_3(v) + E_2(v) - S_i(v) - S_b(v),$$

2) $$2 - \nu = E_1(-v) + E_2(-v) + O_2(-v) - S_i(-v) - S_b(-v),$$

3) $$E_1(v) + E_2(v) + O_1(v) + O_2(v) + n_{cb} = E_1(-v) + E_2(-v),$$

4) $$O_3(v) + n_{ob} = O_2(-v) = \sum E(O_2 m(-v)),$$

where $S_i(f)$ and $S_b(f)$ are the sum of orders of the inner saddle points and that of the boundary saddle points of f, respectively. Then it follows that

5) $$S_i(-v) = S_i(v).$$

Therefore we have a curious boundary relation, that is,

$$E_m(v) - (n_{cb} + n_{ob}) = S_b(v) - S_b(-v)$$

Theorem G. Under the same hypotheses as in Theorem A, we have a relation

$$M_b(u) - m_b(u) = S_b(-u) - S_b(u),$$

where $M_b(u)$ and $m_b(u)$ denote the numbers of the maximum and minimum boundary points of u, respectively.

13. Proof of the Theorem F.

Let $V(c)$, $\underset{\sim}{V}(c)$, $\hat{V}(c)$ and $\bar{V}(c)$ be the sets of points satisfying the conditions $v < c$, $v \leqq c$, $v > c$ and $v \geqq c$, respectively.

The set $\underset{\sim}{V}(c) - \overline{(V(c))}$ consists of a finite number of the isolated minimum boundary points. Thus we have

Lemma A'. $E(\underset{\sim}{V}(c)) - E(\overline{(V(c))}) = C_{om}(c)$, where $C_{om}(c)$ is the number of the isolated minimum boundary points lying on the level c.

Let $CD(c)$ be the compact covering domain of $\overline{(V(c))}$ which is constructed by separating the respective extended saddle points with their orders. Here we remark that the method of the separation is the following manner:

Two different extended sectors below $v(P)$, P being a saddle point of v, are not mutually connectible in a canonical neighborhood $N(P)$ of P.

Then we have

Lemma B'.

$$E(CD(c)) - E(\overline{(V(c))}) = S(c)$$

where $S(c)$ is the sum of orders of the extended saddle points lying on c.

Covering domain of the second kind $KD(c)$ is defined in the following manner:

In the first place we construct all the relating maximal continuations of the proper boundaries of $CD(c)$, by considering the function $-v$. Moreover we separate all the components of $\underset{\sim}{V}(c) \cap CD(c)$ along the relating maximal continuations. The union of all the components mentioned above is called the covering domain of the second kind and is denoted by $KD(c)$.

Evidently we have

Lemma C'. $E(KD(c)) = E(\underset{\sim}{V}(c-\varepsilon))$.

Next we shall calculate the Euler numbers of the closed or open maximum loci of v.

i) In case of the closed maximum locus γ_c, all the relating maximal continuations of γ_c are the looping simple closed curves. The Euler number of the sum sets of all the relating maximal continuations is then equal to zero. Let p_0^i ($i = 1, \dots, n$) and p_1^j ($j = 1, \dots, m$) be all the vertices and the edges belonging to γ_c according to the triangulation already mentioned, respectively. Then we have

$$E(\gamma_c) = n - m = -\left(\sum_{i=1}^{n}(b(p_0^i) - 1) - m\right),$$

since

$$\sum_{i=1}^{n} b(p_0^i) - 2m = 0.$$

This shows that $E(\gamma_c)$ is equal to the following number:

$-\Big[\Big\{\sum_{i=1}^{n}$ (the multiplicities of the vertices p_0^i $- 1$)$\Big\}$
$- \Big\{\sum_{j=1}^{m}$ (the multiplicities of the edges $p_1^j (=2)$ $- 1$)$\Big\}\Big]$

in all the relating maximal continuations of γ_c.

ii) In case of the open maximum locus γ_o, each relating maximal continuation of γ_o is either a looping simple closed curve or a non-looping simple curve ending at two boundary points.

All the open subarcs of the relating maximal continuations, at any point P of which there is no other maximal continuation in the sufficiently small canonical neighborhood $N(P)$ of P, are not our present problems. Thus our attentions may be paid to the maximal connected simple subarcs of the maximal continuations, at any point P of which there is another maximal continuation in $N(P)$.

Then we have

$$E(\gamma_o) = E(\Gamma_o) - n_c - n_b = n - m - (n_c + n_b),$$

where Γ_o is a relating curve of γ_o and n, m are the numbers of the vertices and edges of Γ_o, respectively, and n_c, n_b are the numbers of the relating connecting points and the relating boundary points of γ_o, respectively. On the other hand we have

$$\sum_{i=1}^{n-n_c-n_b} b(p_0^i) - 2m + N_c + N_b = 0,$$

where p_0^i is any vertex belonging to the inner part of γ_o, N_c and N_b the sums of the branching orders of the relating connecting points and the relating boundary points of γ_o with respect to γ_o, respectively. The saltus of the Euler numbers caused by a γ_o is equal to

$$-\left\{\sum_{i=1}^{n-n_c-n_b}(b(p_0^i) - 1) - m + N_c + N_b\right\}.$$

However, by the above arguments, this saltus is equal to the following:

$$n - m - n_c - n_b = E(\gamma_o).$$

Therefore we have the following Lemma:

Lemma D'.

$$E(CD(c)) = E(KD(c)) + C_M(c) + O_M(c),$$

where $C_M(c)$ and $O_M(c)$ are the sums of the Euler numbers of all the closed and all the open maximum loci of v lying on the level c.

For any ordinary value c of v, we have

Lemma E'.

$$E(\underline{V}(c-\varepsilon)) = E(\underline{V}(c)) = E(\underline{V}(c+\varepsilon)).$$

We are now able to obtain the Theorem F by the above Lemmas A' - E', immediately.

Each closed minimum locus of $-v$ is either a closed maximum locus of v, or an open maximum locus of v containing no connecting point but having at least one boundary point. Thus we have the relation (3):

$$E_1(v) + E_2(v) + O_1(v) + O_2(v) + n_{cb} = E_1(-v) + E_2(-v).$$

A relation (4) can be similarly obtained.

14. Polar linear submanifolds.

We shall now introduce the poles of u.

Definition 15. If either $\lim_{z\to z_0} u(z) = +\infty$ or $\lim_{z\to z_0} u(z) = -\infty$ holds definitely for all the approaching paths to z_0, then we call z_0 a pole of u either of the first kind or of the second kind, respectively.

Definition 16. Polar linear submanifolds of the first kind $\mathcal{L}(u=+\infty)$ and of the second kind $\mathcal{L}(u=-\infty)$.

Let $\mathcal{L}(u=+\infty)$ be a one-dimensional connected continuum on which $u \equiv +\infty$, and the F.A.I, II, III and IV hold on $\mathcal{L}(u=+\infty)$, then we call this $\mathcal{L}(u=+\infty)$ a polar linear submanifold of the first kind.

We also define a polar linear submanifold of the second kind $\mathcal{L}(u=-\infty)$ in a similar manner.

Now we should postulate a new fundamental assumption:

F.A.V There are only a finite number of polar linear submanifolds on $\bar{G}$.

Definition 17. Euler number of a polar linear submanifold of the second kind: $E(\mathcal{L}(u=-\infty))$.

$\mathcal{L}(u=-\infty)$ plays the same role as that of the closed minimum locus of u. So we shall define that $E(\mathcal{L}(u=-\infty))$ is equal to $a_0 - a_1$, where a_0 and a_1 denote the numbers of the vertices and edges on $\mathcal{L}(u=-\infty)$.

Definition 18. Euler number of a polar linear submanifold of the first kind: $E(\mathcal{L}(u=+\infty))$.

$\mathcal{L}(u=+\infty)$ plays the same role as that of the maximum locus of $-u$ in the Theorem F. If $\mathcal{L}(u=+\infty)$ has $n_{+\infty}(\mathcal{L})$ relating boundary points, then we suppose that $E(\mathcal{L}(u=+\infty))$ is equal to $a_0 - a_1 - n_{+\infty}(\mathcal{L})$, where a_0 and a_1 denote the numbers of the vertices and the edges on $\mathcal{L}(u=+\infty)$.

Then we have the following theorem.

Theorem A'.

$$2-\nu = L_1(u) + L_2(u) + C_1 + C_2 + \theta - S$$

where $L_1(u) = \sum_{\mathcal{L}} E(\mathcal{L}(u=+\infty))$ and $L_2(u) = \sum_{\mathcal{L}} E(\mathcal{L}(u=-\infty))$.

Similarly we can easily calculate the variations of the Euler numbers by considering the superharmonic function $v=-u$. Then we have

Theorem F'.

$$2-\nu = L_1(-u) + L_2(-u) + C_{0M} + C_{1M} + C_{2M} + O_M - S_M,$$

where $E(\mathcal{L}(-u=+\infty))$ is the Euler number of the polar linear submanifold $\mathcal{L}(-u=+\infty)$ of v of the first kind, which is similarly defined as in the Definition 18, and $E(\mathcal{L}(-u=-\infty))$ is that of the polar linear submanifold $\mathcal{L}(-u=-\infty)$ of v of the second kind, which is similarly defined as in the Definition 17, and $L_1(-u) = \sum_{\mathcal{L}} E(\mathcal{L}(-u=+\infty))$ and $L_2(-u) = \sum_{\mathcal{L}} E(\mathcal{L}(-u=-\infty))$.

By the definitions, we have

$$L_2(u) = L_1(-u) + n_{-\infty}$$

and

$$L_1(u) + n_{+\infty} = L_2(-u),$$

where $n_{-\infty}$ and $n_{+\infty}$ denote the numbers of the boundary poles of u of the second kind and of the first kind, respectively.

Thus we have

Theorem G'.

$$m_\ell(u) + n_{-\infty} - M_\ell(u) - n_{+\infty}$$
$$= S_\ell(u) - S_\ell(-u).$$

If we suppose that the number of the connected components of the closed polar linear submanifolds of the first kind is L_c, where the word "closed" means that there is no relating boundary point on that submanifold, then we have the following inequality:

Theorem B'.

$$C_3 - C_2 \leq \nu - 1 + L_c .$$

15. A family of open minimum loci.

In Theorem B, we have an estimation of the numbers of closed minimum loci, but this is not impartiality for the treatment of open minimum loci. Three numbers in the sequel are devoted to maintain impartiality for both types of minimum loci.

Definition 19. Arrivability of two open minimum loci.

Let γ_i $(i=1,2)$ be two distinct open minimum loci. If we can select at least a relating connecting point P_i of each γ_i satisfying the following conditions, then we call that γ_1 and γ_2 are mutually arrivable and denote this relation by $\gamma_1 \longleftrightarrow \gamma_2$.

1) Both P_1 and P_2 lie on the boundary $R_c^{(i)}$ of the same connected component $\overline{U}^{(i)}(c)$ of $\overline{U}(c)$, where c is the common level of γ_1 and γ_2.

2) P_1 and P_2 are arcwisely connectible along a subarc $\overline{Y}$ (it may reduce to a point or need not be a simple curve) of $R_c^{(i)}$, and $\overline{Y}$ contains P_1 and P_2 as a starting and final points, respectively.

3) If $\overline{Y}$ is simple, then we put $Y = \overline{Y}$. Otherwise, we construct a simple curve Y from $\overline{Y}$ by the separation of the double points of $\overline{Y}$, that is, by considering that some number of different points eventually coincide and constitute a multiple point with some orders of $\overline{Y}$.

4) Y can be homotopically deformable to a continuous simple curve Z defined by the following conditions:

a) Let $N(P_1)$ and $N(P_2)$ be two distinct canonical neighborhoods of P_1 and P_2, respectively. For each γ_i there is a subarc Γ_i, with the ends P_i and Q_i, of γ_i belonging to $N(P_i)$, and moreover satisfying that Q_1 and Q_2 are arcwisely connectible along a simple Jordan curve X belonging to a component $\overline{U}^{(j)}(c)$ of $\overline{U}(c)$ except only at two points Q_1 and Q_2. Let Γ_1 and Γ_2 be sensed by the order P_1, Q_1 and P_2, Q_2, respectively.

b) Let Z be $\Gamma_1 X \Gamma_2^{-1}$, where Γ_2^{-1} is the same arc, inversely sensed, as Γ_2.

For an open minimum locus there holds $\gamma_1 \longleftrightarrow \gamma_1$. In fact, we may only choose $P_1 \equiv P_2$, $Q_1 \equiv Q_2$ and $X \equiv Q_1$, but this is trivial.

Definition 20. Self-arrivability, that is, $\gamma_1 \longleftrightarrow \gamma_1$, (but non-trivial).

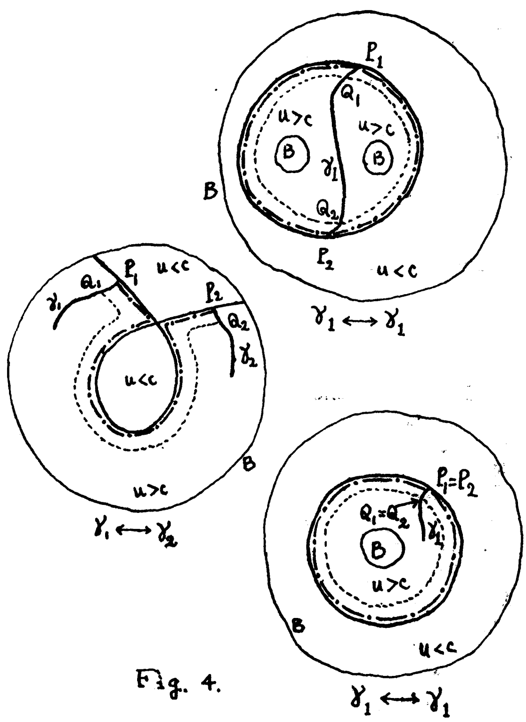

Fig. 4.

—·— shows a connecting path.
······ shows an X.

If we can select the relating connecting points P_1 and P_2 of γ_1 satisfying the conditions of Definition 19, then we call that γ_1 satisfies the self-arrivability condition.

A curve Υ defined in Definitions 19 and 20 is called a connecting path between minimum loci. Some examples will clarify these Definitions.

Definition 21. Familiarity relation of two open minimum loci γ_1 and γ_n. Notation $\gamma_1 \approx \gamma_n$.

If there is a finite chain γ_j ($j = 1, 2, \ldots, n-1, n$), where γ_j is an open minimum locus for each j and that there holds successively the relations $\gamma_1 \leftrightarrow \gamma_2$, $\gamma_2 \leftrightarrow \gamma_3$, $\ldots$, $\gamma_{n-1} \leftrightarrow \gamma_n$, then we say that γ_1 and γ_n are familiar.

Lemma 12. At each point P on $\bar{\Upsilon}$ of two open minimum loci except at two end points, there hold $\sigma(P) \geqq 1$ and $b(P) \geqq 2$, in which the inequalities appear only at a finite number of points on $\bar{\Upsilon}$.

Proof. If it is not the case, then $\sigma(P) = 0$. Thus P must be a relative maximum boundary point, and hence P is a strictly relative maximum boundary point, that is, $b(P) = 0$. On the other hand, $P \in \bar{\Upsilon}$ implies that $b(P) \geqq 1$, which is absurd. Thus $\sigma(P) \geqq 1$. If $\sigma(P) \geqq 2$, then $b(P) \geqq 4$ for $P \in G$ and $b(P) \geqq 2$ for $P \in B$. By F.A.III and IV, these points are finite in number.

If $P \in G$ and $\sigma(P) \geqq 1$, then $b(P) \geqq 2\sigma(P) \geqq 2$.

If $P \in B$ and $\sigma(P) \geqq 1$, then P is an end point of $\bar{\Upsilon}$. For, if P is not an end of $\bar{\Upsilon}$, then $b(P) \geqq 2$. On $\bar{\Upsilon}$, u is constant, thus $\bar{\Upsilon}$ does not contain a subinterval of B. By $\sigma(P) = 1$, there is only one sector $S_-(P)$ below $u(P)$ in $CN(P)_\varepsilon$. One component Γ_1 of $\bar{\Upsilon}$ is a boundary of $S_-(P)$, and the other component Γ_2 of $\bar{\Upsilon}$ exists in $G \frown CN(P)_\varepsilon$. Any point Q of $\Gamma_2 \frown G \frown CN(P)_\varepsilon$ except at P has $b(Q) = 2$ and $\sigma(Q) = 1$. Thus $S_-(P)$ has Γ_2 as the boundary. In local at P, Υ consists of Γ_1, P, Γ_2 with this order. But in this case Υ is not able to deform into a continuous curve Z homotopically, which is a contradiction. Thus, if $P \in B$ and $\in \bar{\Upsilon}$, then $\sigma(P) \geqq 2$ except at two end points of Υ. Thus at this point P, $b(P) \geqq 2$ and P is a saddle point.

The familiarity relation $\approx$ satisfies the equivalence relations, that is, $\gamma \approx \gamma$, $\gamma_1 \approx \gamma_2 \Rightarrow \gamma_2 \approx \gamma_1$ and $\gamma_1 \approx \gamma_2$, $\gamma_2 \approx \gamma_3 \Rightarrow \gamma_1 \approx \gamma_3$.

Definition 22. A family of the open minimum loci.

A class classified by the above familiarity relation is called a family. We suppose that a family, as a point set, consists of all the open minimum loci, any two of which are familiar.

Definition 23. Range set of a family.

Union of a family (considered as a point set) and all the possible relating connecting paths (including all the relating connecting points) are called the range set of a given family (considered as a class).

Here we must remark that the range set of a family has no multiple point except at most at the relating connecting points and at the finite number of multiple points on the open minimum loci.

Lemma 13. The range set of a family can be divided into a finite number of simple arcs and is a closed complex of one-dimension.

Definition 24. Euler number of a family.

Euler number of a family (considered as a point set) is defined by the following number: (Euler number of a range set of the given family)
- (sum of Euler numbers of the connected unions of all the possible connecting paths).

The connected unions of all the possible connecting paths are defined as the connected collections of all the connecting paths, of which any two successive paths have a common connecting point. We call this connected union a hedge.

Lemma 14. The different families are finite in number and each family is of finite Euler number.

16. Restatement of the first main Theorem A.

Let $R(c)$ be the number of the range sets of a family lying on c and having the Euler number 1. Let $R_{-i}(c)$ be the number of the range sets of a family lying on c and having the Euler number $-i$. Let $T_j(c)$ be the number of the connected unions (hedges) lying on c and having the Euler number j. Evidently j is either 1 or 0, since any connected union (hedge) is simple. And we put

$$\sum_c \sum_{i \geq 0} (i+1) R_{-i}(c) = R_\ell ,$$

$$\sum_c \sum_{i \geq 0} (-i) R_{-i}(c) = R_2 ,$$

$$\sum_c R(c) = R_1 ,$$

$$\sum_c \sum_{i \geq 0} R_{-i}(c) = R_3$$

and

$$\sum_c T_j(c) = T_j , \ (j = 0, 1).$$

Lemma 15. Let $\mathfrak{F}$ be a family consisting of the open minimum loci γ_1, ..., γ_n. Then

$\sum_{i=1}^{n} E(\gamma_i) =$ The Euler number of a family $\mathfrak{F}$ by the second definition: $\mathcal{E}(\mathfrak{F})$.

Proof. Let γ_i have a_0^i inner vertices, a_1^i edges and m_0^i connecting points, then

$$E(\gamma_i) = (a_0^i + m_0^i - a_1^i) - m_0^i .$$

Thus

$$\sum_{i=1}^{n} E(\gamma_i) = \sum_{i=1}^{n} (a_0^i - a_1^i) .$$

By the definition of the connecting path each connecting path is a simple curve. Hence we have

$$\mathcal{E}(\mathfrak{F}) = \left(\left(\sum_{i=1}^{n} a_0^i + q_0\right) - \left(\sum_{i=1}^{n} a_1^i + s_1\right)\right)$$

$$-(q_0 - s_1)$$

$$= \sum_{i=1}^{n} (a_0^i - a_1^i)$$

where q_0 is the number of the vertices on the connecting paths and s_1 is the number of the edges on the connecting paths.

By the Lemma 15, we have $R_1 + R_2 - T_1 = \theta$. Thus we have

Theorem A".

$$C_1 + C_2 + R_1 + R_2 - T_1 - S = 2 - \nu .$$

17. Causality for occurrence of a boundary component.

Each ordered maximal continuation lying on a level c is a simple arc, being either of looping one or of non-looping one. Every looping simple ordered maximal continuation is classified into two types. Let be a looping simple ordered maximal continuation lying on c. $\mathfrak{F}_\varepsilon(L(c))$ and $\Gamma_\varepsilon(L(c))$ are relating ordered ε-field and its proper boundary, respectively, where ε is a sufficiently small positive number.

$\Gamma_\varepsilon(L(c))$ is also a looping simple curve and bounds a domain $D_\varepsilon(L(c))$. If $D_\varepsilon(L(c))$ contains $L(c)$ in its interior, then we say that $L(c)$ belongs to the divergent type. If it is not the case, then we say that $L(c)$ belongs to the convergent type.

Next we shall define a figure of boundary type.

If $\mathfrak{L}$ satisfies the following conditions, then we call $\mathfrak{L}$ the boundary type.

i) On $\mathfrak{L}$, $u = c$ holds and $\mathfrak{L}$ is connected.

ii) $\mathfrak{L}$ and a finite number of boundary components bound a number of connected domains $D_i(\mathfrak{L})$.

iii) $\overline{\sum_i D_i(\mathfrak{L})} = \overline{D(\mathfrak{L})}$ contains a finite number of boundary components B_1, ..., B_n, any two of which are arcwisely connectible along a part of $\mathfrak{L}$.

iv) For every i, all the relating maximal continuations, if exist, end always at the boundary points, and the relating Γ_ε (proper boundary of ε-field) end always at the different boundary points, and every Γ_ε is contained in $\overline{D(\mathcal{G})}$.

v) $\mathcal{G}$ is a minimal one among all the linear connected graphs satisfying i) - iv), that is,

$$\overline{D(\mathcal{G})} \subsetneqq \overline{D(\mathcal{G}_1)}$$

holds if $\mathcal{G}_1$ satisfies i) - iv).

Now we should classify all the relating maximal continuations ending to the boundary points but constituting a simple closed curve in total. We now construct a simple closed curve from all the relating Γ_ε by connecting the pairs of end points along the parts of boundary curve contained in each $N(P)$, where P is a boundary point on $L(c)$. If a simple closed curve thus constructed bounds a domain $D_\varepsilon(L(c))$ containing $L(c)$ in its interior, then we say that $L(c)$ belongs to the divergent type. If it is not the case, then $L(c)$ belongs to the boundary type.

A figure of any type is a causality of the occurrence of a sort of critical point or of a boundary component, in view of the subharmonicity of u. Exact causality for the occurrence of a boundary component arises from the occurrence of the figure of boundary type or of convergent type.

If c is a critical value of u, and $L(c)$ is a looping simple maximal continuation relating to a range set $R(c)$ of Euler number 0, and the number of hedges is 1, and further this hedge is of Euler number 0, then there is a slight disturbance in the flows of the level lines but no efficiency for the causality in $[c-\varepsilon, c+\varepsilon]$. If c is an ordinary value of u, then each level curve lying on c is a simple curve, either looping or non-looping. If $L(c)$ is a looping level curve, then the relating $\Gamma_\varepsilon(L(c))$ and $\Gamma_{-\varepsilon}(L(c))$ are also looping level curves and $\mathfrak{F}_\varepsilon(L(c))$ and $\mathfrak{F}_{\varepsilon}(L(c))$ have no critical point or a boundary component. Hence $\Gamma_{\varepsilon_1}(L(c))$, $-\varepsilon \leqq \varepsilon_1 \leqq \varepsilon$, has the same causality as $L(c)$.

Next we shall calculate the number of different causalities for the occurrence of a boundary component which relates to the closed minimum loci and the range sets of the families.

Let b, ℓ, and t_0 be three numbers defined as follows:

b is equal to the number of the figures of boundary type in a given connected level, ℓ the number of the looping simple maximal continuation of convergent type, and t_0 the number of the hedges of Euler number 0 in the same figure.

Let γ_c be a closed minimum locus of u with the Euler number $-n$, then the relations $b+\ell = n+1$ and $t_0 = 0$ remain true. Therefore there must exist at least $n+1$ boundary components.

Lemma 16. In any connected collection of level curves, there is at most one maximal continuation of divergent type.

Proof. If there are two such maximal continuations $L_1(c)$, $L_2(c)$ of divergent type lying on c, then there are two relating bounded domains $D_\varepsilon(L_1(c))$, $D_\varepsilon(L_2(c))$ containing $L_1(c)$, $L_2(c)$, respectively. Then either $D_\varepsilon(L_1(c)) \supset D_\varepsilon(L_2(c))$ or $D_\varepsilon(L_1(c)) \cap D_\varepsilon(L_2(c)) = \phi$ (empty set) remains true. But $L_1(c)$ cannot be connected with $L_2(c)$ along a curve, on which $u \equiv c$. Both cases contradict the connectibility of the original figure.

The outest hedge of Euler number 0 is the hedge bounding a domain which contains all the points of the given range set.

Let R_d be a range set of a family having one looping maximal continuation of divergent type. On R_d, there is no outest hedge.

Let R_h be a range set of a family having the outest hedge and let R_c be a range set which does not belong to $R_d \cap R_h$.

Lemma 17. In an R_h, there holds a relation

$$b + \ell + t_0 = i + 2,$$

where $-i$ is the Euler number of the given range set.

Proof. In the first place we shall consider that R_h has no figure of boundary type. Conventionally, we consider that the given range set is a closed minimum locus, having no figure of boundary type, then there are $i+1$ looping simple maximal continuations of convergent type and 1 looping simple maximal continuation of divergent type. Since there are $t_0 - 1$ inner hedges of Euler number 0 and 1 outest hedge of Euler number 0, we have $i+1-(t_0-1)$ convergent maximal continuations.

In the second place if R_h has b figures of boundary type, then any figure of boundary type can be took place by a looping maximal continuation of convergent type. Therefore we have the desired result.

Lemma 18. In an R_c or an R_d, there holds a relation

$$b + \ell + t_0 = i + 1.$$

Proof. Proof is similar as in the above Lemma.

For an R_h of Euler number 0, we shall exclude this case for our calculation of the causalities.

Lemma 19. The $\sum(b+\ell)$ causalities thus calculated for the occurrence of the boundary component are all different.

Proof. Evident.

Let $N_{u,-i,t_0}$ be the number of the range sets of R_u ($u = d, h, c$) type, of Euler number $-i$ and having t_0 hedges of Euler number 0. Thus we have an inequality:

$$\sum_{i\geq 0}\sum_{t_0=0}^{i+1}(i+1-t_0)N_{d,-i,t_0} + \sum_{i\geq 1}\sum_{t_0=1}^{i+1}(i+2-t_0)N_{h,-i,t_0} + \sum_{i\geq 0}\sum_{t_0=0}^{i+1}(i+1-t_0)N_{c,-i,t_0} + C_3 - C_2 \leqq \nu - 1.$$

From this we have

$$\sum_{i\geq 0}(i+1)R_{-i} + \sum_{i\geq 1}\sum_{t_0=1}^{i+1}N_{h,-i,t_0} - T_0 + C_3 - C_2 \leqq \nu - 1,$$

and hence

$$R_3 - R_2 + N_h + C_3 - C_2 - T_0 \leqq \nu - 1, \qquad N_h = \sum_{i\geq 1}\sum_{t_0=1}^{i+1}N_{h,-i,t_0}.$$

Thus we have the third main theorem.

Theorem C.

$$R_3 + C_3 - R_2 - C_2 + N_h \leqq T_0 + \nu - 1.$$

By this theorem C we have

Theorem D.

$$C_1 + C_3 + R_1 + R_3 + N_h \leqq S + T_0 + T_1 + 1.$$

18. Applications and a concluded remark.

Analogues of minimax principle due to Whyburn [1].

Theorem E. Let u be a subharmonic function which attains a constant relative maximum on each component of B; the constants may differ for different components. If u has at least two different minimum loci in G, then there is at least a saddle point or an open minimum locus in G.

Proof. By Theorem D, we have

$$C_1 + C_3 + R_1 + R_3 \leqq C_1 + C_3 + R_1 + R_3 + N_h \leqq S + T_0 + T_1 + 1.$$

By the assumptions of this theorem we have

$$S = S_i\ (S_b = 0) \quad \text{and}$$

$$C_1 + C_3 + R_1 + R_3 \geqq 2,$$

which lead to the fact $S_i \geqq 1$ or $T_0 \geqq 1$ or $T_1 \geqq 1$. Moreover $T_j \geqq 1$ implies that there is at least an open minimum locus in G.

Corollary 8. If G is a simply connected domain and there is no hedge of Euler number 0, then $R_3 = C_3 = 0$.

Proof. By Theorem C, we have

$$0 \leqq R_3 - R_2 + C_3 - C_2 + N_h$$

$$\leqq \nu - 1 + T_0,$$

and by the assumptions, we have $\nu = 1$ and $T_0 = 0$, thus $R_3 = C_3 = 0$.

Let the flows of the level lines be defined by Fig. 5, the arrows showing the directions of the increasing levels. What facts can we conclude by means of Fig. 5?

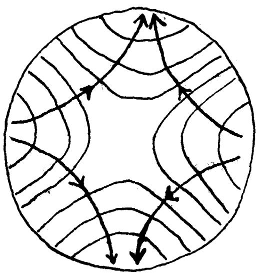

Fig. 5.

In the harmonic case, we have $S = 1$, but in our case we have either $S = 1$ or $\theta = -1$ and $R_3 = C_3 = 0$. Because we have by the first main theorem A

$$1 = C_1 + C_2 + \theta - S.$$

By the assumption $C_1 = 2$, we have

$$0 \geqq C_2 + \theta - S = -1$$

but by the last corollary, $C_3 = R_3 = C_2 = R_2 = 0$, thus $\theta = -1$ or $S = 1$ remain true.

This fact shows that the similar causality does not imply the same conclusion, and moreover we cannot distinguish these by our given datas.

(*) Received June 15, 1953.

Morse, M.: [1] The topology of pseudo-harmonic functions. Duke Math. Journ. 13 (1946), pp. 21-42.
[2] Topological methods in the theory of functions of a complex variable. Princeton University Press. 1947.

Morse, M. and M, Heins : [1] Topological methods in the theory of functions of a complex variable:
I. Deformation types of locally simple curves. Ann. of Math. 46 (1945) pp. 600-624.
II. Boundary values and integral characteristic of interior transformations and pseudo-harmonic functions. Idem. pp. 625-666.
III. Causal isomorphisms in the theory of pseudo-harmonic functions. Idem. 47 (1946) pp. 233-274.

Morse, M. and van Schaack : [1] The critical point theory under general boundary conditions. Ann. of Math. 35 (1934) pp. 545-571.

Whyburn, W. M.: [1] Non-isolated critical points of functions. Bull. Amer. Math. Soc. 35 (1929) pp. 701-708.

Department of Mathematics,
Tokyo Institute of Technology.