A FOURIER SERIES WHICH BELONGS TO THE CLASS H DIVERGES ALMOST EVERYWHERE

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Let

$$F(z) = c_0 + c_1 z + c_2 z^2 + \cdots$$

be a function analytic for $|z| < 1$. If the integral

$$\int_{-\pi}^{\pi} |F(re^{i\theta})| \, d\theta$$

remains bounded when $r \to 1$, the function $F(z)$ and the series $c_0 + c_1 z + \ldots$ are said to belong to the class H and we then write $F(z) \in H$. A necessary and sufficient condition for the function $F(z) \in H$ is that the series

$$c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \cdots\cdots$$

is the Fourier series of a function of the class L. In the famous examples, given by Kolmogoroff [2][3], of function $f(\theta) \in L$ whose Fourier series diverges almost everywhere (or even everywhere), the conjugate series is not a Fourier series, so that the corresponding power series does not belong to H. A.Zygmund [4] has said "the problem whether in the H-class the corresponding power series may converge remains open, but the answer is probably negative". The object of the present note is to solve this problem. In fact we can construct a Fourier series which is real part of the power series belonging to H-class and diverges almost everywhere. This is done by adding some words to Hardy-Rogosinski's example [1]. In fact, in this book the author cannot understand the footnote p.72. For the sake of completeness we shall repeat their arguments.

Theorem 80 of Hardy-Rogosinski's book reads as follows:

There is a sequence of trigonometrical polynomials ϕ_n with the properties

(i) $\phi_n \geq 0$

(ii) $\frac{1}{\pi}\int_0^{2\pi} \phi_n \, d\theta = 1$

(iii) to each ϕ_n correspond

a) an M_n tending to infinity,
b) a set E_n whose measure tends to 2π, and
c) an integer q_n such that

$$|S_{p_n}(\theta, \phi_n)| > M_n$$

for every θ of E_n and a $p_n = p_n(\theta) \leqq q_n$.

Using this theorem, we can easily to prove our assertion. Since $M_n \to \infty$, we can choose a sequence $\{n_s\}$ so that $\sum M_{n_s}^{-\frac{1}{2}} < \infty$. We write ϕ_n as a sum a exponentials, that is

$$\phi_{n_s} = \sum_{\nu=-m_{n_s}}^{m_{n_s}} c_\nu e^{i\nu\theta}$$

and put

$$T_{n_s} = \frac{e^{r_s i\theta}}{M_{n_s}^{1/2}} \phi_{n_s},$$

$$\Phi = T_{n_1} + T_{n_2} + \cdots,$$

$$T = (T_{n_1}) + (T_{n_2}) +$$

(T_{n_s}) being T_{n_s} written out at length as a sum of exponentials. We can plainly choose the r_s so that there is no overlapping between the term of T, when T is a type of power series. Since

$$\frac{1}{\pi}\int_0^{2\pi} |e^{-\phi i\theta}| \sum |T_{n_s}| \, d\theta \leq \frac{1}{\pi}\sum \frac{1}{M_{n_s}^{1/2}} \int_0^{2\pi} \phi_{n_s} \, d\theta = \sum \frac{1}{M_{n_s}^{1/2}} < \infty$$

the series $\sum T_{n_s}$ converges almost everywhere to an integrable Φ, and may be integrated term by term after multiplication by $e^{-\phi i\theta}$. Hence T is the Fourier series of $\Phi \in L$, that is $\Phi \in H$. On the other hand, if

θ is in E_{n_s} , $\{T_{n_s}\}$ contains a block of successive terms whose sum is numerically greater than $M_{n_s}^{1/2}$. Hence T is divergent if θ lies in an infinity of E_{n_s} ; and this is true almost everywhere because $mE_{n_s} \to 2\pi$. If we take the real part of T, it diverges almost everywhere by Kuttner's theorem (see Hardy-Rogosinski [1], p.82). Thus we get the required example.

[1] G.H.Hardy and W.W.Rogosinski; Fourier series, Cambridge Tracts No.38, 2nd ed., 1949.
[2] A.Kolmogoroff; Une série de Fourier-Lebesgue divergente presque partout, Fund. Math., 4(1923); 324-328.
[3] A.Kolmogoroff; Une série de Fourier-Lebesgue divergente partout, Comptes Rendus de Paris, 183(1926), 1327-1328.
[4] A.Zygmund; On the convergence and summability of power series on the circle of convergence (II), Proc. London Math. Soc., 47 (1942), 326-350.

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