# ON SOME FAMILY OF MULTIVALENT FUNCTIONS

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## 1. Quasi - convex Functions.

Let

$$(1) \quad F(z) = z^p + \sum_{K=1}^{\infty} a_{p+K} z^{p+K}$$

be any function regular in $|z| < 1$, where $p$ is a positive integer.[1] If we denote a family of functions of the form (1), by which $|z| < 1$ is transformed into a starshaped (with the center at the origin) or a convex region of p-valence, respectively denoted by $\mathfrak{S}_p$ or $\mathfrak{K}_p$, then the following theorem is well known.[2]

Theorem 1.

The necessary and sufficient condition that $F(z)$ should belong to $\mathfrak{S}_p$ or $\mathfrak{K}_p$ is that

$$R\left[z\frac{F'(z)}{F(z)}\right] > 0$$

or

$$1 + R\left[z\frac{F''(z)}{F'(z)}\right] - \frac{p-1}{p} R\left[z\frac{F'(z)}{F(z)}\right] > 0$$

holds respectively in $|z| < 1$.

Now we denote by $\mathfrak{Q}_p$ a family of functions of the form (1) which is characterized by the following properties:

1° The mapped region of $|z| < 1$ by $w = F(z)$ is p-valent,

2° The curvature at any point on the mapped curve of $|z| = r$ by $w = F(z)$ is positive and finite, where $r$ is an arbitrary positive number less than unity.

And we say that $F(z)$ in $\mathfrak{Q}_p$ is a quasi-convex function, then we have the theorem as follows:

Theorem 2.

The necessary and sufficient condition that $F(z)$ should belong to $\mathfrak{Q}_p$ is

$$1 + R\left[z\frac{F''(z)}{F'(z)}\right] > 0 \quad (|z| < 1).$$

Proof. We have, by (1),

$$\left[\frac{F'(z)}{z^{p-1}}\right]_{z=0} = p \neq 0.$$

Therefore, if

$$(2) \quad R\left[z\frac{(zF'(z))'}{zF'(z)}\right] = 1 + R\left[z\frac{F''(z)}{F'(z)}\right] > 0,$$

then $\frac{F'(z)}{z^{p-1}} \neq 0$ in $|z| < 1$ and $F'(z) \neq 0$ in $0 < |z| < 1$.[3] Denoting by $\rho$ the curvature at any point given in 2°, we have

$$\rho = \frac{1}{|zF'(z)|} \cdot R\left[1 + z\frac{F''(z)}{F'(z)}\right] > 0.$$

The mapped curve C of $|z| = r$ by $w = F(z)$ is regular and the angle $\varphi$ between the real axis and the tangent to the curve C at any point on C is given by $\arg izF'(z)$. Hence we have, as $z$ describe $|z| = r$ in the positive direction,

$$\int d\arg izF'(z) = \int d\arg z^p + \int d\arg \frac{F'(z)}{z^{p-1}}$$
$$= \int d\arg z^p = 2p\pi,$$

and consequently the curve C is closed and p-valent. Here $r$ being arbitrary, the mapped region of $|z| < 1$ is p-valent.

Conversely, if $\rho > 0$, then $1 + R\left[z\frac{F''(z)}{F'(z)}\right] > 0$ follows directly from the equality for $\rho$ cited above. Our theorem is thus proved.

## 2. Relations among $\mathfrak{S}_p$, $\mathfrak{K}_p$ and $\mathfrak{Q}_p$.

Let $F(z)$ be any function regular and p-valent in $|z| < 1$, and then $F(z)$ does not vanish in $0<|z|<1$. Therefore there exists a function $h(z)$ which is regular in $|z|<1$ and satisfies

$$h(z)=z\sqrt[p]{\frac{F(z)}{z^p}},\quad F(z)=[h(z)]^p,$$
$$h(0)=0,\quad h'(0)=1.$$

Consequently, we have

$$(3)\begin{cases} R\left[z\dfrac{F'(z)}{F(z)}\right]=p\,R\left[z\dfrac{h'(z)}{h(z)}\right], \\ 1+R\left[z\dfrac{F''(z)}{F'(z)}\right]=1+R\left[z\dfrac{h''(z)}{h'(z)}\right]+(p-1)R\left[z\dfrac{h'(z)}{h(z)}\right], \\ 1+R\left[z\dfrac{F''(z)}{F'(z)}\right]-\dfrac{p-1}{p}R\left[z\dfrac{F'(z)}{F(z)}\right]=1+R\left[z\dfrac{h''(z)}{h'(z)}\right]. \end{cases}$$

From these equalities we obtain immediately that if $F(z)\in\gamma_p$, then $h(z)\in\gamma_1$ and if $F(z)\in\mathfrak{K}_p$, then $h(z)\in\mathfrak{K}_1$. Hence if $F(z)\in\mathfrak{K}_p$, then $h(z)\in\mathfrak{K}_1$ and, by the theorem due to E. Strohhäcker[4] we have

$$R\left[z\frac{h'(z)}{h(z)}\right]>\frac{1}{2}$$

and, by (3),

$$1+R\left[z\frac{F''(z)}{F'(z)}\right]>\frac{1}{2}(p-1)\geqq 0,$$
$$R\left[z\frac{F'(z)}{F(z)}\right]>\frac{p}{2}.$$

These inequalities conclude that if $F(z)\in\mathfrak{K}_p$, then $F(z)$ belongs both to $\mathfrak{O}_p$ and $\gamma_p$.

Next we investigate the relation between $\mathfrak{O}_p$ and $\gamma_p$. If $F(z)\in\mathfrak{O}_p$ and $\varphi$ is the angle defined in § 1, then

$$\frac{d\varphi}{d\theta}=\frac{d}{d\theta}\arg iz F'(z)=1+R\left[z\frac{F''(z)}{F'(z)}\right]>0.$$

Therefore the tangent to the mapped curve of $|z|=r$ by $w=F(z)$ rotates so as to increases the angle $\varphi$, as $z$ moves in the positive sense on $|z|=r$. And, as the curvature $\rho$ at the point $F(z)$ is positive, the radius vector $F(z)$ from the origin rotates in the positive sense. Consequently we have

$$R\left[z\frac{F'(z)}{F(z)}\right]=\frac{d}{d\theta}\arg F(z)>0.$$

Thus, we get the result that if $F(z)\in\mathfrak{O}_p$, then $F(z)\in\gamma_p$

From the above arguments, we have the following theorem:

Theorem 3.

$$\mathfrak{K}_p\subset\mathfrak{O}_p\subset\gamma_p.$$

N. B. In the case of $p=1$, this theorem is reduced to the fact,

$$\mathfrak{K}_1\equiv\mathfrak{O}_1\subset\gamma_1$$

by (3).

Finally, from the equality (2) in § 1, we have

Theorem 4.

If $F(z)\in\mathfrak{O}_p$, then $\frac{1}{p}zF'(z)\in\gamma_p$, and if $F(z)\in\gamma_p$, then

$$p\int_0^z\frac{F(z)}{z}dz\in\mathfrak{O}_p.$$

### 3. Distortion theorem and coefficient problem for $\mathfrak{O}_p$.

Let $F(z)$ belong to $\mathfrak{O}_p$, then, by the Theorem 4, $\frac{1}{p}zF'(z)\in\gamma_p$ and therefore

$$\frac{|z|^p}{(1+|z|)^{2p}}\leqq\frac{1}{p}|zF'(z)|\leqq\frac{|z|^p}{(1-|z|)^{2p}}.$$

From these inequalities, we have, by the similar method as in the case $p=1$, the following Distortion Theorem;

Theorem 5.

Let $F(z)$ belong to $\mathfrak{O}_p$, then

$$\frac{p|z|^{p-1}}{(1+|z|)^{2p}}\leqq|F'(z)|\leqq\frac{p|z|^{p-1}}{(1-|z|)^{2p}},$$
$$p\int_0^{|z|}\frac{z^{p-1}}{(1+z)^{2p}}dz\leqq|F(z)|\leqq p\int_0^{|z|}\frac{z^{p-1}}{(1-z)^{2p}}dz$$

And the equality sign holds for the function of $\mathfrak{O}_p$ defined by

$$F(z)=p\int_0^z\frac{z^{p-1}}{(1-z)^{2p}}dz.$$

Next, if $F(z)\in\gamma_p$, then

$$|a_{p+k}|\leqq\frac{2p(2p+1)\cdots(2p+k-1)}{k!},\quad k=1,2,3,\cdots$$

and the equality holds for the function $F(z)=\dfrac{z^p}{(1-z)^{2p}}$ [5]. Hence we have, for any function $F(z)\in \mathfrak{O}_p$

$$\frac{1}{p}\,zF'(z) \ll \frac{z^p}{(1-z)^{2p}}\ .$$

From this relation, we have immediately the theorem as follows:

Theorem 6.

If $F(z)=z^p+\sum\limits_{k=1}^{\infty} a_{p+k}z^{p+k}\in \mathfrak{O}_p$, then

$$|a_{p+k}|\leqq \frac{2p(2p+1)\cdots(2p+k-1)}{k!\,(p+k)}\ ,\quad k=1,2,3,\cdots$$

The equality sign holds only for the function

$$F(z)=p\int_0^z \frac{z^{p-1}}{(1-z)^{2p}}\,dz\,.$$

4. The Radius of quasi - convexity for $\gamma_p$.

Let $F(z)$ belong to $\gamma_p$, then we have

$$\Re\left[z\frac{F'(z)}{F(z)}\right]>0 \quad \text{and} \quad \left[z\frac{F'(z)}{F(z)}\right]_{z=0}=p.$$

Hence

$$p\,\frac{1-r}{1+r}\leqq \Re\left[z\frac{F'(z)}{F(z)}\right]\leqq p\,\frac{1+r}{1-r},\quad (|z|\leqq r)$$

and, by the well known theorem of G. Julia, we have the following inequality:

$$\left|z\frac{F''(z)}{F'(z)}-z\frac{F'(z)}{F(z)}+1\right|\leqq 2\,\frac{|z|\,\Re\left[z\dfrac{F'(z)}{F(z)}\right]}{(1-|z|^2)\left|z\dfrac{F'(z)}{F(z)}\right|}\,.$$

Combining these inequalities with each other, we have, in $|z|\leqq r$.

$$1+\Re\left[z\frac{F''(z)}{F'(z)}\right]\geqq \frac{pr^2-2(p+1)r+p}{1-r^2}>0\,,$$

$$\left(r<\sigma_p=\frac{p+1-\sqrt{2p+1}}{p}\right)$$

and, in the case of the function $F(z)=\dfrac{z^p}{(1-z)^{2p}}\in\gamma_p$, we have, at

$$z=-\sigma_p\ ,$$

$$1+\Re\left[z\frac{F''(z)}{F'(z)}\right]=0.$$

Hence we have the following:

Theorem 7.

If $F(z)\in\gamma_p$, then we have, in $|z|<\sigma_p$, $F(z)\in\mathfrak{O}_p$, where the number $\sigma_p$ cannot be replaced by any greater one.

5. The Radius of quasi - convexity for the bounded functions.

Let $F(z)$ be bounded, i.e. $|F(z)|<M$, then we have

$$(4)\qquad Mr^p\frac{1-Mr}{M-r}\leqq|F(z)|\leqq Mr^p\frac{1+Mr}{M+r},\ (|z|\leqq r).$$

If we put $\varphi(z)=F(z)/z^p$, then we have $|\varphi(z)|\leqq M$, $\varphi(0)=1$.

Hence

$$|\varphi'(z)|\leqq\frac{M^2-|\varphi(z)|^2}{M(1-r^2)}\,,$$

or

$$(5)\qquad\left|\frac{F'(z)}{z^p}-p\frac{F(z)}{z^{p+1}}\right|\leqq\frac{M^2r^{2p}-|F(z)|^2}{Mr^{2p}(1-r^2)},\ (|z|\leqq r).$$

From this fundamental inequality and (4), we have, in $|z|\leqq r$, the following inequalities:

$$(6)\qquad\left|\frac{\lambda(z)}{zF'(z)}\right|\geqq\Re\left[\frac{\lambda(z)}{zF'(z)}\right]\geqq M\,\frac{1-2Mr+r^2}{P(r)},$$

$$(7)\qquad\frac{|\lambda(z)|}{Mr^{p-1}+|F(z)|}\geqq\frac{r(1-2Mr+r^2)}{(1-r^2)(M-r)},$$

$$(8)\qquad\frac{\nu(z)}{M(1-r^2)r^{p-2}|F'(z)|}\leqq\frac{Mr(M-2r+Mr^2)}{(1-r^2)P(r)}$$

where $\lambda(z)=zF'(z)-(p-1)F(z)$, $\nu(z)=M^2r^{2(p-1)}-|F(z)|^2$ and $P(r)=pM-\{(p+1)M^2+(p-1)\}r+pMr^2$.

The equality sign in (4) - (8) holds at $z=r$ for the function

$$(9)\qquad F(z)=Mz^p\,\frac{1-Mz}{M-z}\,.$$

H. Loomis generalized the theorem of J. Dieudonné for the starshapedness of the bounded functions for $\gamma_1$ to the case of $\gamma_p$ [6]. Now we generalize the theorem for the convexity of the bounded functions for $\mathfrak{O}_1=\mathfrak{K}_1$ to the case of $\mathfrak{O}_p$ [7].

Putting

$$\phi(\zeta)=M^2\,\frac{z^{p-1}F(w)-w^{p-1}F(z)}{M^2\bar{z}^{p-1}w^{p-1}-\overline{F(z)}F(w)},$$

$$\left(w=\frac{-\zeta+z}{1-\bar{z}\zeta}\ ,\quad |z|<1\right)$$

we have, in $|z|<1$,

$$|\phi(\zeta)| < M, \quad \phi(0) = 0,$$

and, by the theorem of the bounded functions,

$$M - \frac{|\phi'(0)|^2}{M} \geq \frac{1}{2}|\phi''(0)|.$$

This is reduced to

$$(10) \quad \Re\left[z\frac{F''(z)}{F'(z)}\right] \geq (p-1)(p-2)\Re\left[z\frac{F(z)}{F'(z)}\right] + \frac{2|z|^2}{1-|z|^2}\Re\left[\frac{\lambda(z)}{zF'(z)}\right] + 2M\frac{|\lambda(z)|^2|z|^{p-2}}{\nu(z)|F'(z)|}$$
$$+ 2\Re\left[\frac{(p-1)M^2|z|^{2(p-1)} - z\overline{F(z)}F'(z)}{\nu(z)}\cdot\frac{\lambda(z)}{zF'(z)}\right]$$
$$-2\frac{\nu(z)}{M(1-r^2)^2|z|^{p-2}|F'(z)|}.$$

Since the third term of the right hand is not less than $2(p-1)\Re\left[\frac{\lambda(z)}{zF'(z)}\right] - 2\left|\frac{F(z)\lambda(z)}{\nu(z)}\right|\left|\frac{\lambda(z)}{zF'(z)}\right|$ and $(p-1)\frac{F(z)}{zF'(z)} = 1 - \frac{\lambda(z)}{zF'(z)}$, we have, from (10),

$$(11) \quad \Re\left[z\frac{F''(z)}{F'(z)}\right] \geq p-2 + \left[\frac{2|z|^2}{1-|z|^2}+p\right]\Re\left[\frac{\lambda(z)}{zF'(z)}\right]$$
$$+2\left|\frac{\lambda(z)}{zF'(z)}\right|\frac{|\lambda(z)|}{Mr^{p-1}+|F(z)|}$$
$$-2\frac{\nu(z)}{M(1-|z|^2)^2|z|^{p-2}|F'(z)|}.$$

Putting (4) - (8) into (11), we obtain

$$(12) \quad 1+\Re\left[z\frac{F''(z)}{F'(z)}\right] \geq \frac{R(r)}{R(r)(M-r)} \quad (|z| \leqq r),$$

where $R(r) = M^2p^2 - M\{(p+1)^2M^2+2p^2-2p-1\}r + \{(2p^2+2p-1)M^2+(p-1)^2\}r^2 - Mp^2r^3.$

The equality sign in (12) holds at $z=r$ for the function defined by (9). It is easily proved that the equation $R(r)=0$ has only one positive root $\rho_p$ less than unity and, $R(r)$ and $P(r)$ are positive for $0 \leqq r \leqq \rho_p$. Therefore we can establish the following theorem:

Theorem 8.

Let $F(z)$ be any function regular and bounded $(|F(z)| < M)$ in $|z|<1$, then $F(z)$ belongs to $\mathfrak{C}_p$ in $|z| < \rho_p$, where $\rho_p$ is the positive root of the equation $R(r)=0$. And $\rho_p$ cannot be replaced by any greater number.

In conclusion I wish to express my hearty thanks to Professor Akira Kobori of the Kyoto University for his kind advice throughout the completion of this work.

(*) Received July 3, 1952.

(1) In this note the function $F(z)$ is always of the form (1), and so we omit this notice hereafter.

(2) A.Kobori, Sur les Fonctions Multivalentes, Proc. Phys.-Math. Soc. Japan. Vol.23, No.6 (1941).

(3) A.Kobori, Über die notwendige und hinreichende Bedingung dafür, dass eine Potenzreihe den Kreisbereich auf den schlichten sternigen bzw. Konvexen Bereich abbildet. Mem. Coll. Sci. Kyoto Imp. Univ. (A) 15 (1932).

(4) E.Strohhäcker, Beiträge zur Theorie der schlichten Funktionen. Math. Zeitschr. 37 (1933).

(5) A.Kobori, Loc. cit. (2).

(6) H.Loomis, The radius and modulus of n-valence for analytic functions whose first n-1 derivatives vanish at a point. Bull. Amer. Math. Soc. Vol.46, No.6 (1940) or see E.Sakai, On the Multivalency of Analytic Functions. Journ. Math. Soc. Japan Vol.2(1950).

(7) Y.Sasaki, Theorems on the Convexity of Bounded Functions. Proc. Jap. Acad. Vol.27 (1951).

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