NOTE ON LAPLACE-TRANSFORMS, (III)

ON SOME CLASS OF LAPLACE-TRANSFORMS, (II)

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(I) THEOREM. Let $f(x)$ be R-integrable in any finite interval $0 \leqq x \leqq X$, X being an arbitrary positive constant. Let the Laplace-transform of $f(x)$ be

$$(1.1) \quad F(s) = \int_0^\infty \exp(-sx)\, f(x)\, dx \qquad (s = \sigma + it).$$

$F(s)$ has generally four special abscisses, i.e. regularity-abscissa σ_r, simple convergence-abscissa σ_s, uniform convergence-abscissa σ_u, and absolute convergence-abscissa σ_a ($\sigma_r \leqq \sigma_s \leqq \sigma_u \leqq \sigma_a$) In the previous Note ([1] - See references placed at the end -), we have discussed the sufficient conditions for $\sigma_s = \sigma_u = \sigma_a$. In the present Note, we shall study the sufficient conditions for $\sigma_r = \sigma_s = \sigma_u = \sigma_a$. The theorem states as follows.

THEOREM. $\overline{\lim\limits_{t\to\infty}} \frac{1}{t} \log|f(t)| = \sigma_r = \sigma_s = \sigma_u = \sigma_a$, provided that

(a) $f(z)$ ($z = r\exp(i\theta)$) is regular in D : $|\theta| \leqq \vartheta < \frac{\pi}{2}$, except $z = 0$, and $z = \infty$;

(b) for sufficiently large r, $f(z)$ is of exponential type in D;

(c) $\lim\limits_{r\to+0} r\,|f(re^{i\theta})| = 0$ uniformly in D;

(d) $\lim\limits_{\varepsilon_1\to+0} \int_{\varepsilon_1}^{\varepsilon_2} |f(re^{i\theta})|\, dr$ ($\varepsilon_1 < \varepsilon_2$) exists in D.

Furthermore, there exists at least one singular point $s_0 = \sigma_r + it$ ($-\infty < t < +\infty$) on $\sigma = \sigma_r$.

(2) Proof. On account of (a), (b), and (d), $f(t)$ belongs to $C\{I_r\}$ ([1]), so that

$$\sigma_s = \sigma_u = \sigma_a = \overline{\lim_{t\to\infty}} \frac{1}{t} \log|f(t)|$$

By Cauchy's theorem, in D, we have

$$(2.1) \quad \int_{R_1}^{R_2} \exp(-sx) f(x)\, dx = \underset{|x| = R_1}{\int_{R_1}^{R_1 e^{i\theta}}} + \underset{\arg x = \theta}{\int_{R_1 e^{i\theta}}^{R_2 e^{i\theta}}} + \underset{|x| = R_2}{\int_{R_2 e^{i\theta}}^{R_2}} = I_1 + I_2 + I_3, \text{ say.}$$

By (b), there exists a constant c such that

$$(2.2) \quad |f(re^{i\theta})| < \exp(cr)$$

for sufficiently large r.

Suppose that

$$(2.3) \quad 0 < t, \text{ and } \max(c, \sigma_s) < t\cos\theta$$

Then, putting $s = t\exp(-i\theta)$, by (2.2) and (2.3),

$$|I_3| \leqq \int_0^\theta \exp\{-tR_2\cos(\alpha-\theta) + R_2 c\} R_2\, d\alpha$$

$$< \theta R_2 \exp\{R_2(c - t\cos\theta)\}$$

$$\longrightarrow 0 \quad \text{as} \quad R_2 \longrightarrow +\infty$$

By (c),

$$|I_1| \leqq \int_0^\theta \exp\{-tR_1\cos(\alpha-\theta)\} |f(R_1 e^{i\alpha})| R_1\, d\alpha$$

$$< \theta \exp(-tR_1\cos\theta) R_1 \max_{0\leqq\alpha\leqq\theta} |f(R_1 e^{i\alpha})|$$

$$\longrightarrow 0 \quad \text{as} \quad R_1 \longrightarrow +0$$

Hence, by (2.1)

$$(2.4) \quad F(s) = F(te^{-i\theta}) = \int_0^\infty \exp(-sx) f(x)\, dx$$

$$= e^{i\theta} \int_0^\infty \exp(-t|x|)\, f(|x|e^{i\theta})\, d|x|$$

$$(\max(c, \sigma_s) < t\cos\theta).$$

On the other hand,

$$F(s) = \int_0^\infty \exp(-sx) f(x)\, dx$$

is regular for $\Re(s) > \varphi(0) = \overline{\lim\limits_{t\to\infty}} \frac{1}{t} \log|f(t)|$, and

$$G(t) = e^{i\theta} \int_0^\infty \exp(-t|x|) f(|x|e^{i\theta})\, d|x|$$

is regular for $\Re(t) > \varphi(\theta) = \overline{\lim\limits_{r\to\infty}} \frac{1}{r} \log|f(re^{i\theta})|$. For, by (a) and (b), $f(x)$ and $f(|x|e^{i\theta})$ belong to $C\{I_r\}$, so that three convergence-abscisses coincide with $\varphi(\beta)$ ($\beta = 0$ or θ) respectively. By (2.4), for $\max(c, \sigma_s) < t\cos\theta$, $F(te^{-i\theta})$ is equal to $G(t)$. Hence, $F(s)$ is regular in $D_1 \cup D_2$, where D_1: $\Re(s) > \varphi(0)$, D_2: $\Re(se^{i\theta}) > \varphi(\theta)$, and

$$(2\cdot5)\quad F(te^{-i\theta}) = e^{i\theta}\int_0^\infty \exp(-t|x|) f(|x|e^{i\theta})\, d|x|$$

for $\quad \Re(t) > \varphi(\theta)$.

Suppose that

$$(2\cdot6)\quad \sigma_r < \sigma_s = \sigma_u = \sigma_a = \alpha = \varphi(0).$$

For sufficiently small $\varepsilon\ (>0)$, we have

$$\sigma_r < \alpha-\varepsilon < \alpha < \alpha+\varepsilon.$$

Then, $F(s)$ would be regular in $\Re(s) \geqq \alpha-\varepsilon$. $F(s)$ is absolutely convergent for $\Re(s) = \alpha+\varepsilon$, and for $x > c$, $f(x)$ is analytic, so that $f(x)$ is continuous and of bounded variation. Hence, by the inversion-formula of Laplace-transform ([2] p.105),

$$f(x) = \frac{1}{2\pi i}\int_{\alpha+\varepsilon-i\infty}^{\alpha+\varepsilon+i\infty} \exp(sx)\, F(s)\, ds \quad (x>0)$$

By Cauchy's theorem,

$$\int_{\alpha+\varepsilon-iT}^{\alpha+\varepsilon+iT} \exp(sx)\, F(s)\, ds$$

$$= \int_{\alpha+\varepsilon-iT}^{\alpha-\varepsilon-iT} + \int_{\alpha-\varepsilon-iT}^{\alpha-\varepsilon+iT} + \int_{\alpha-\varepsilon+iT}^{\alpha+\varepsilon+iT}$$

$$= I_1 + I_2 + I_3 \quad \text{say}$$

For sufficiently large T, the interval: $\Im(s) = -T$, $\alpha-\varepsilon \leqq \Re(s) \leqq \alpha+\varepsilon$ is contained in the angular domain $|\arg(s-s_0)| \leqq \beta < \pi/2$, where $\Re(s_0e^{i\theta}) > \varphi(\theta)$, $\frac{\pi}{2}-\theta < \beta < \frac{\pi}{2}$. Therefore, by (2.5) and the well-known theorem ([2] p.49), we have

$$(2\cdot8)\quad \lim_{\substack{t\to-\infty \\ \alpha-\varepsilon\leqq\sigma\leqq\alpha+\varepsilon}} |F(\sigma+it)| = 0$$

uniformly with respect to σ.

Accordingly,

$$|I_1| < o(1)\, 2\varepsilon\, \exp((\alpha+\varepsilon)x) \to 0 \quad \text{as } T\to+\infty.$$

Similarly $\quad |I_3| \to 0$

Hence, by (2.7),

$$(2\cdot9)\quad f(x) = \frac{1}{2\pi i}\int_{\alpha-\varepsilon-i\infty}^{\alpha-\varepsilon+i\infty} \exp(sx)\, F(s)\, ds \quad (x>0).$$

By (2.9),

$$(2\cdot10)\quad f(x)$$

$$= \exp((\alpha-\varepsilon)x)\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(itx)\, F(\alpha-\varepsilon+it)\, dt$$

$$(x>0).$$

On account of (2.8)

$$\lim_{t\to-\infty} |F(\alpha-\varepsilon+it)| = 0,$$

so that

$$F(\alpha-\varepsilon+it) = \int_{+\infty}^{t} F'(\alpha-\varepsilon+it)\, i\, dt$$

Hence, $F(\alpha-\varepsilon+it)$ is of bounded variation in $T \leqq t < +\infty$. Therefore, by the well-known theorem ([3] p.7),

$$\int_{-\infty}^{0} \exp(itx)\, F(\alpha-\varepsilon+it)\, dt = O\left(\frac{1}{x}\right)$$

Similarly,

$$\int_0^{+\infty} \exp(itx)\, F(\alpha-\varepsilon+it)\, dt = O\left(\frac{1}{x}\right).$$

Therefore, by (2.10),

$$f(x) = \exp((\alpha-\varepsilon)x)\ O\left(\frac{1}{x}\right),$$

so that

$$\varlimsup_{x\to+\infty} \frac{1}{x}\log|f(x)| \leqq \alpha-\varepsilon < \alpha,$$

which is impossible. Thus, we have

$$\sigma_r = \sigma_s = \sigma_u = \sigma_a = \alpha.$$

By what was proved above, the existence of s_0 immediately follows. This completes our proof.

(*) Received July 28, 1951.

[1] C.Tanaka: Note on Laplace-transforms. (II) On some class of Laplace-transforms. (I) These Rep.

[2] G.Doetsh: Theorie und Anwendung der Laplace-Transformation. 1937.

[3] S.Bochner: Vorlesungen uber Fouriersche Integrale. Leipzig. 1932.

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