ON THE COMPACTNESS OF SPACE L^p $(p>0)$ AND ITS APPLICATION TO INTEGRAL EQUATIONS

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In I, we will prove a theorem on the compactness of space L^p $(p>0)$ and in II, we will apply it to prove simply Carleman's theorem on integral equations.

I. Compactness of space L^p $(p>0)$.

1. Let F be a set of functions $f(x)$ defined for $-\infty < x < \infty$, such that

$$\int_{-\infty}^{\infty} |f(x)|^p dx < \infty \quad (p>0).$$

If from any sequence $f_n(x) \in F$, we can find a partial sequence $f_{n_k}(x)$, which converges to $f(x) \subset L^p$ almost everywhere in $(-\infty, \infty)$, such that

$$\lim_k \int_{-\infty}^{\infty} |f_{n_k}(x) - f(x)|^p dx = 0,$$

$$\int_{-\infty}^{\infty} |f(x)|^p dx = \lim_k \int_{-\infty}^{\infty} |f_{n_k}(x)|^p dx,$$

then we say that F is compact. We will prove

Theorem 1. The necessary and sufficient condition that F is compact is that the the following condition (A) is satisfied:

(i) $\int_{-\infty}^{\infty} |f(x)|^p dx \leq M$, for any $f(x) \in F$;

(ii) For any $\varepsilon > 0$, we can find $N_0 > 0$, such that for any $f(x) \in F$,

$$\int_{|x| \geq N} |f(x)|^p dx < \varepsilon, \text{ if } N \geq N_0;$$

(iii) For any $\varepsilon > 0$, we can find $\delta > 0$, such that for any $f(x) \in F$,

$$\int_{-\infty}^{\infty} |f(x+t) - f(x)|^p dx < \varepsilon, \text{ if } |t| < \delta.$$

The case $p \geq 1$ was proved by M. Riesz,[1] so that we will prove the case $0 < p < 1$.

Proof. Since the necessity can be provec easily, we will prove the sufficiency in case $0 < p < 1$, where we assume that the theorem holds for $p \geq 1$.

Now if $0 < p < 1$, $x > 0$, $\delta > 0$,

$$(x+\delta)^p - x^p = p \int_0^\delta \frac{dt}{(x+t)^{1-p}} \leq p \int_0^\delta \frac{dt}{t^{1-p}} = \delta^p,$$

so that

$$|x^p - y^p| \leq |x-y|^p, \quad (x \geq 0, y \geq 0, 0 < p < 1). \quad (1)$$

$$(x+\delta)^p - x^p \geq p \frac{\delta}{(x+\delta)^{1-p}} = p \left(\frac{\delta}{x+\delta}\right)^{1-p} \delta^p,$$

$$\delta^p \leq \frac{1}{p} \left(\frac{x+\delta+x}{\delta}\right)^{1-p} ((x+\delta)^p - x^p),$$

hence

$$|x-y|^p \leq \frac{1}{p}\left(\frac{x+y}{|x-y|}\right)^{1-p} |x^p - y^p|,$$

$$(x>0, y>0, \quad 0<p<1). \quad (2)$$

First we suppose that $f(x) \geq 0$ for all $f(x) \in F$. Then from (1),

$$|f^p(x+t) - f^p(x)| \leq |f(x+t) - f(x)|^p,$$

hence $f^p(x)$ satisfy the condition (A) with $p=1$, so that by M. Riesz' theorem, from any sequence $f_n(x) \in F$, we can find a partial sequence (which we denote $f_n(x)$), such that

$$\int_{-\infty}^{\infty} |f_m^p(x) - f_n^p(x)| dx < \varepsilon, \text{ if } \quad m \geq n \geq n(\varepsilon) \quad (3)$$

By (2), we have

$$|f_m(x) - f_n(x)|^p \leq \frac{1}{p}\left(\frac{f_m(x) + f_n(x)}{|f_m(x) - f_n(x)|}\right)^{1-p} |f_m^p(x) - f_n^p(x)|$$

Let E be the set of x, such that

$$\frac{f_m(x) + f_n(x)}{|f_m(x) - f_n(x)|} \leq K,$$

then by (3),

$$\int_E |f_m(x) - f_n(x)|^p dx \leq \frac{K^{1-p}}{p} \int_E |f_m^p(x) - f_n^p(x)| dx \leq \frac{K^{1-p}}{p} \varepsilon.$$

Let E' be the complement of E, then in E',

$$|f_m(x) - f_n(x)| \leq \frac{f_m(x) + f_n(x)}{K},$$

so that

$$\int_{E'} |f_m(x) - f_n(x)|^p dx \leq \frac{1}{K^p}\left(\int_{E'} f_m^p(x) dx + \int_{E'} f_n^p(x) dx\right) \leq \frac{2M}{K^p},$$

hence

$$\int_{-\infty}^{\infty} | f_m(x) - f_n(x) |^p dx \leqq \frac{K^{1-p}\varepsilon}{p} + \frac{2M}{K^p}. \quad (5)$$

Hence if we choose $K = \frac{1}{p}$, then

$$\int_{-\infty}^{\infty} | f_m(x) - f_n(x) |^p dx \leqq \left(\frac{1}{p} + 2M\right)\varepsilon^p, \quad (6)$$

so that $f_n(x)$ converges in the mean, hence by Hobson's theorem,[2] we can find a partial sequence, which satisfies the condition of the theorem.

In the general case, we put

$$f^{(1)}(x) = \frac{1}{2}\left(|f(x)| + f(x)\right), \quad f^{(2)}(x) = \frac{1}{2}\left(|f(x)| - f(x)\right)$$

$$f(x) = f^{(1)}(x) - f^{(2)}(x),$$

then

$$0 \leqq f^{(1)}(x) \leqq |f(x)|, \quad 0 \leqq f^{(2)}(x) \leqq |f(x)|.$$

We can easily prove that

$$|f^{(1)}(x+t) - f^{(1)}(x)| \leqq |f(x+t) - f(x)|,$$

$$|f^{(2)}(x+t) - f^{(2)}(x)| \leqq |f(x+t) - f(x)|,$$

so that $f^{(1)}(x)$, $f^{(2)}(x)$ satisfy the condition (A), hence if $f_n(x) = f_n^{(1)}(x) - f_n^{(2)}(x)$ be any sequence from F, then we can find a partial sequence (which we denote $f_n(x)$), such that

$$\int_{-\infty}^{\infty} |f_m^{(1)}(x) - f_n^{(1)}(x)|^p dx < \varepsilon,$$

$$\int_{-\infty}^{\infty} |f_m^{(2)}(x) - f_n^{(2)}(x)|^p dx < \varepsilon, \quad \text{if } m \geqq n \geqq n(\varepsilon).$$

Since

$$|f_m(x) - f_n(x)|^p \leqq |f_m^{(1)}(x) - f_n^{(1)}(x)|^p + |f_m^{(2)}(x) - f_n^{(2)}(x)|^p,$$

$f_n(x)$ converges in the mean, so that we can find a partial sequence, which satisfies the condition of the theorem., q.e.d.

2. As an application of Theorem 1, we will prove the following Carleman's theorem.[3]

Theorem 2. Let $f_n(x, y) \geqq 0$ be integrable in $a \leqq x \leqq b$, $a \leqq y \leqq b$, such that

$$\lim_n \int_a^b \int_a^b f_n(x, y)\, dx\, dy = 0$$

We put

$$\varphi_n(x) = \int_a^b f_n(x, y)\, dy.$$

Then we can find a partial sequence $\varphi_{n_k}(x)$, which converges to zero almost everywhere in $[a, b]$.

Proof. By the hypothesis,

$$\int_a^b \varphi_n(x)\, dx = \int_a^b \int_a^b f_n(x, y)\, dx\, dy \leqq M, \quad (n = 1, 2, \cdots), \quad (1)$$

$$\int_a^b |\varphi_n(x+t) - \varphi_n(x)|\, dx \leqq \int_a^b \int_a^b |f_n(x+t, y) - f_n(x, y)|\, dx\, dy$$

$$\leqq 2 \int_a^b \int_a^b f_n(x, y)\, dx\, dy < \varepsilon$$

$$(n \geqq n_0).$$

If we choose $\delta > 0$, such that if $|t| < \delta$,

$$\int_a^b |\varphi_n(x+t) - \varphi_n(x)|\, dx < \varepsilon \quad (n = 1, 2, \cdots, n_0 - 1),$$

then

$$\int_a^b |\varphi_n(x+t) - \varphi_n(x)|\, dx < \varepsilon, \text{ if } |t| < \delta, (n = 1, 2, \cdots). \quad (2)$$

Hence $\varphi_n(x)$ satisfy the condition (A), so that we can find a partial sequence $\varphi_{n_k}(x)$, which converges to $\varphi(x) \subset L$ almost everywhere in $[a, b]$, such that

$$\int_a^b \varphi(x)\, dx = \lim_k \int_a^b \varphi_{n_k}(x)\, dx$$

$$= \lim_k \int_a^b \int_a^b f_{n_k}(x, y)\, dx\, dy = 0.$$

Since $\varphi(x) \geqq 0$, we have $\varphi(x) = 0$ almost everywhere in $[a, b]$, q.e.d.

II. Carleman's theorem on integral equations.

Carleman[4] extended Fredholm's theorem on integral equations with continuous kernel $K(x, y)$ to the case, where $K(x, y)$ are square integrable as follows, where ~ 0 means $= 0$ almost everywhere in $[a, b]$.

Theorem 3. Let $K(x, y)$ be square integrable in $a \leqq x \leqq b$, $a \leqq y \leqq b$, then the integral equation:

$$\varphi(x) - \lambda \int_a^b K(x, y)\, \varphi(y)\, dy - f(x) \sim 0 \quad \text{(I)}$$

either (i) has one and only one solution $\varphi(x) \subset L^2$ for any $f(x) \subset L^2$, or (ii) the homogeneous equation:

$$\varphi(x) - \lambda \int_a^b K(x, y)\, \varphi(y)\, dy \sim 0 \quad \text{(I')}$$

has r $(1 \leqq r < \infty)$ linearly independent solutions $\psi_1, \psi_2, \cdots, \psi_r$. In the first case, the conjugate equation:

$$\varphi(x)-\lambda\int_a^b K(y,x)\,\varphi(y)\,dy-g(x)\sim 0 \quad (\mathrm{II})$$

has one and only one solution for any $g(x)\subset L^2$. In the second case, the conjugate homogeneous equation:

$$\varphi(x)-\lambda\int_a^b K(y,x)\,\varphi(y)\,dy\sim 0 \quad (\mathrm{II}')$$

has r linearly independent solutions $\chi_1, \chi_2, \dots, \chi_r$ and (I) has solutions when and only when $f(x)$ satisfies r conditions:

$$(f,\chi_i)=\int_a^b f(x)\,\chi_i(x)\,dx=0 \quad (i=1,2,\cdots,r).$$

We assume that Fredholm's theorem holds for continuous degenerated kernel $K(x,y)=\sum_i A_i(x)\,B_i(y)$, where $A_i(x)$, $B_i(y)$ are continuous and by means of Theorem 1, we will prove our theorem. If we specialize that $f(x)$ and $K(x,y)$ are continuous, or more generally, $K(x,y)$ are of the form $K(x,y)=\dfrac{H(x,y)}{|x-y|^\alpha}$ $(0<\alpha<\frac{1}{2})$, where

$H(x,y)$ are continuous, then we can prove easily that the solutions $\varphi(x)$ are continuous and ~ 0 becomes $=0$, hence we have Fredholm's theorem for such kernels.

Proof. We approximate $K(x,y)$ by $K_n(x,y)$, where $K_n(x,y)$ are polynomials in x and y, so that are degenerated kernels, such that

$$\lim_n\int_a^b\int_a^b |K_n(x,y)-K(x,y)|^2\,dx\,dy=0,$$

$$\int_a^b\int_a^b K^2(x,y)\,dx\,dy=\lim_n\int_a^b\int_a^b K_n^2(x,y)\,dx\,dy \tag{1}$$

and we approximate $f(x)$ by continuous $f_n(x)$, such that

$$\lim_n f_n(x)=f(x) \tag{2}$$

almost everywhere in $[a,b]$, and

$$\lim_n\int_a^b |f_n(x)-f(x)|^2\,dx=0 \tag{3}$$

and for a fixed λ, we consider integral equations:

$$\varphi(x)-\lambda\int_a^b K_n(x,y)\,\varphi(y)\,dy=f_n(x) \tag{4}$$

Then the following two cases occur.

Case I. (4) has solutions for infitely many n, so that we may assume that (4) has solution $\varphi_n(x)$ for all n,

$$\varphi_n(x)-\lambda\int_a^b K_n(x,y)\,\varphi_n(y)\,dy=f_n(x) \tag{5}$$

such that

$$c_n=(\varphi_n,\varphi_n)=\int_a^b \varphi_n^2(x)\,dx\leqq M \quad (n=1,2,\cdots). \tag{6}$$

Case II. Either II(a): (4) has solution $\varphi_n(x)$ for all n, but $c_n\to\infty$ $(n\to\infty)$ or II(b): (4) has no solutions for all n, so that there exists $\varphi_n(x)$, $(\varphi_n,\varphi_n)=1$, such that

$$\varphi_n(x)-\lambda\int_a^b K_n(x,y)\,\varphi_n(y)\,dy=0 \quad (n=1,2,\cdots). \tag{7}$$

In case I, we will prove that

$$F_n(x)=\int_a^b K_n(x,y)\,\varphi_n(y)\,dy \tag{8}$$

satisfy the condition (A) of Theorem 1. Since

$$(F_n(x))^2\leqq\int_a^b\varphi_n^2(y)\,dy\int_a^b K_n^2(x,y)\,dy\leqq M\int_a^b K_n^2(x,y)\,dy,$$

$$(F_n(x+t)-F_n(x))^2\leqq M\int_a^b (K_n(x+t,y)-K_n(x,y))^2\,dy,$$

we have by (1),

$$\int_a^b F_n^2(x)\,dx\leqq M\int_a^b\int_a^b K_n^2(x,y)\,dx\,dy\leqq K \quad (n=1,2,\cdots), \tag{9}$$

$$\begin{aligned}\int_a^b (F_n(x+t)-F_n(x))^2\,dx &\leqq M\int_a^b\int_a^b (K_n(x+t,t)-K_n(x,y))^2\,dx\,dy\\
&\leqq 3^2M\Big[\int_a^b\int_a^b (K_n(x+t,y)-K(x+t,y))^2\,dx\,dy\\
&\quad+\int_a^b\int_a^b (K(x+t,y)-K(x,y))^2\,dx\,dy\\
&\quad+\int_a^b\int_a^b (K_n(x,y)-K(x,y))^2\,dx\,dy\Big]\\
&\leqq 3^2M\Big[\varepsilon+\int_a^b\int_a^b (K(x+t,y)-K(x,y))^2\,dx\,dy\Big]\\
&\qquad (n\geqq n_0).\end{aligned}$$

We choose $\delta_1>0$, such that if $|t|<\delta_1$

$$\int_a^b\int_a^b (K(x+t,y)-K(x,y))^2\,dx\,dy<\varepsilon,$$

then

$$\int_a^b (F_n(x+t)-F_n(x))^2\,dx\leqq 18M\varepsilon \quad (n\geqq n_0).$$

Hence we can choose $0<\delta<\delta_1$, such that

$$\int_a^b (F_n(x+t) - F_n(x))^2 dx < 18 M \varepsilon, \text{ if } |t|<\delta, \quad (n = 1, 2, \ldots). \tag{10}$$

Hence $F_n(x)$ satisfy the condition (A). Similarly $f_n(x)$ satisfy the condition (A), so that $\varphi_n(x) = f_n(x) + \lambda F_n(x)$ satisfy the condition (A). Hence by Theorem 1, we can find a partial sequence (which we denote $\varphi_n(x)$), such that

$$\lim_n \varphi_n(x) = \varphi(x) \subset L^2 \tag{11}$$

almost everywhere in $[a, b]$ and

$$\lim_n \int_a^b |\varphi_n(x) - \varphi(x)|^2 dx = 0 \tag{12}$$

We will prove

$$\lim_n \int_a^b K_n(x,y)\varphi_n(y)dy = \int_a^b K(x,y)\varphi(y)dy \tag{13}$$

almost everywhere in $[a, b]$.

Now

$$\int_a^b\int_a^b |K_n(x,y)\varphi_n(y) - K(x,y)\varphi(y)|dxdy$$
$$\leqq \int_a^b\int_a^b |K_n(x,y) - K(x,y)||\varphi_n(y)|dxdy$$
$$+ \int_a^b\int_a^b |K(x,y)||\varphi_n(y) - \varphi(y)| dxdy . \tag{14}$$

By (1), (6), (12).

$$\left(\int_a^b\int_a^b |K_n(x,y) - K(x,y)||\varphi_n(y)|dxdy\right)^2$$
$$\leqq (b-a)M\int_a^b\int_a^b (K_n(x,y) - K(x,y))^2 dxdy, \quad (n \geqq n_0),$$

$$\left(\int_a^b\int_a^b |K(x,y)||\varphi_n(y) - \varphi(y)| dxdy\right)^2$$
$$\leqq (b-a)M\int_a^b (\varphi_n(y) - \varphi(y))^2 dy \int_a^b\int_a^b K^2(x,y)dxdy$$
$$< \varepsilon, \quad (n \geqq n_0),$$

so that

$$\lim_n \int_a^b\int_a^b |K_n(x,y)\varphi_n(y) - K(x,y)\varphi(y)| dxdy = 0. \tag{15}$$

Hence by Theorem 2, we can find a partial sequence (which we denote $\varphi_n(x)$), such that

$$\lim_n \int_a^b |K_n(x,y)\varphi_n(y) - K(x,y)\varphi(y)| dy = 0$$

almost everywhere in $[a, b]$, so that

$$\lim_n \int_a^b K_n(x,y)\varphi_n(y)dy = \int_a^b K(x,y)\varphi(y)dy \tag{13}$$

almost everywhere in $[a, b]$. Hence from (2), (5), (11), (13), we have

$$\varphi(x) - \lambda\int_a^b K(x,y)\varphi(y)dy - f(x) \sim 0 \tag{16}$$

In case II (a), we put

$$\sigma_n(x) = \varphi_n(x)/c_n \quad ((\sigma_n, \sigma_n) = 1), \tag{17}$$

then

$$\sigma_n(x) - \lambda\int_a^b K_n(x,y)\sigma_n(y)dy = f_n(x)/c_n. \tag{18}$$

We can prove that $\sigma_n(x)$ satisfy the condition (A), so that we can find a partial sequence, which converges to $\varphi(x) \subset L^2$ almost everywhere in $[a, b]$, such that

$$\varphi(x) - \lambda\int_a^b K(x,y)\varphi(y)dy \sim 0 \tag{19}$$

In case II (b), we can prove similarly that there exists $\varphi(x) \subset L^2$ which satisfies (19). The other part of the theorem can be proved similarly as Courant-Hilbert's book.[5] Hence our theorem is proved.

(*) Received March 27, 1951.

(1) M.Riesz: Sur les ensembles compacts des fonctions sommables. Acta de Szeged. 16 (1932-34).
M.Riesz' theorem can be deduced easily from Kolmogoroff-Tulajkov's theorem:
A.Kolmogoroff: Über die Kompaktheit der Funktionenmengen bei Konvergenz im Mittel. Göttinger Nachr. 1931:
A.Tulajkov: Zur Kompaktheit im Raum L_p für $p=1$. Göttinger Nachr. 1933.

(2) E.W.Hobson: Generalization of a theorem of F.Riesz. Journ. London Math. Soc. 1 (1926).

(3) T.Carleman: Zur Theorie der Integralgleichungen. Math. Zeits. 9 (1921).

(4) T.Carleman: l.c.3).

(5) Courant-Hilbert: Methoden der mathematischen Physik, I. Berlin (1931).

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