ON UNIFORMIZING FUNCTIONS

By Kihachiro ARIMA

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§ 1. Let S be a closed Riemann surface with genus $g \geqq 2$, whose equation is given by $S(x, y) = 0$, $S(x, y)$ being an irreducible polynomial of degree n in x and m in y. Let $f(t)$ and $g(t)$ be meromorphic functions in the circle $|t| < R$. If $S(f(t), g(t)) \equiv 0$, we say that $f(t)$ and $g(t)$ are uniformizing functions. In this note we will prove the following theorem:

$$\varlimsup_{r \to R} \frac{T(r, f)}{\log \frac{1}{R-r}} \leqq \frac{m}{2g-2},$$

$$\varlimsup_{r \to R} \frac{T(r, g)}{\log \frac{1}{R-r}} \leqq \frac{n}{2g-2},$$

where $T(r, f)$ and $T(r, g)$ are Nevanlinna's characteristic functions.

§ 2. The algebraic function can be uniformized by means of Fuchsian functions $x = x(z)$, $y = y(z)$, in such a manner that in a sufficiently small neighbourhood of a point z in the principal circle of the group the correspondence between the points of the plane and the points of S is one to one.

Putting $z = Z(x, y)$ and $u = \log \dfrac{\left|\frac{dz}{dx}\right|}{1 - |z|^2}$, then $\Delta_x u = 4 e^{2u}$, where $\Delta_x u = \dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial^2 u}{\partial \eta^2}$ and $x = \xi + i\eta$. Putting $Z(t) = Z(f(t), g(t))$ and $u(t) = \log \dfrac{\left|\frac{dz}{dx}\right|\left|\frac{dx}{dt}\right|}{1-|z|^2} = \log W$, then $u(t) = u + \log\left|\frac{dx}{dt}\right|$ and $\Delta u(t) = 4 e^{2u(t)}$.

In this note, by infinity points on S we mean points where $x = \infty$. We suppose that infinity points on S are not branch points.

(I). At a branch point $(x, y) = (a, b)$ of order $m-1$:

$$y - b = a_p (x-a)^{p/m} + a_{p+1} (x-a)^{(p+1)/m} + \cdots,$$

where $a_p \neq 0$;

$$(1) \qquad u = (1-m)/m \log |x-a| + v(x),$$

where $v(x)$ is bounded function in a neighborhood of the point (a, b). Since $f(t)$ and $g(t)$ are single-valued functions,

$$(2) \qquad f(t) - a = a_{km}(t-t_0)^{km} + a_{km+1}(t-t_0)^{km+1} \cdots,$$

$$g(t) - b = b_{kp}(t-t_0)^{kp} + b_{kp+1}(t-t_0)^{kp+1} \cdots,$$

where $f(t_0) = a$, $g(t_0) = b$, $a_{km} \neq 0$, $b_{kp} \neq 0$. From (1) and (2) we get $W = |t-t_0|^{k-1} S(t)$, where $S(t)$ is a bounded function in a neighbourhood of $t = t_0$.

(II). At an infinity point on S:

$$(3) \qquad u = -2 \log |x| + v(x),$$

$$(4) \qquad f(t) = \frac{c_{-k}}{(t-t_0)^k} + \frac{c_{-k+1}}{(t-t_0)^{k-1}} + \cdots,$$

where $c_k \neq 0$. From (3) and (4) we get

$$W = |t - t_0|^{k-1} S(t).$$

By Ahlfors' theorem (An extension of Schwarz's lemma, Trans. of Amer. Math. Soc. 43, 1938) we get the following theorem:

Theorem 1. We suppose that infinity points on S are not branch points. If we put $z(t) = Z(f(t), g(t))$, then

$$\frac{\left|\frac{dz}{dx}\right|\left|\frac{dx}{dt}\right|}{1 - |z|^2} \leqq \frac{R^2}{R^2 - |t|^2}.$$

§ 3. Let $G(x, y, \Gamma_1, \Gamma_2)$ be harmonic on S, except two logarithmic singular points at Γ_1 and Γ_2 and let $\Gamma_1(c_1, d_1)$ and $\Gamma_2(c_2, d_2)$ be branch points of order $k_1 - 1$ and $k_2 - 1$, respectively;

$$G(x, y; \Gamma_1, \Gamma_2) = \begin{cases} \dfrac{1}{k_1} \log \dfrac{1}{|x - c_1|} + \text{bounded harmonic function at } \Gamma_1, \\ \dfrac{1}{k_2} \log |x - c_2| + \text{bounded harmonic function at } \Gamma_2. \end{cases}$$

We put $G(t, \alpha, \Gamma) = G(f(t), g(t); \alpha, \Gamma)$ and $\bar{m}(r, \alpha, \Gamma) = \frac{1}{2\pi}\int_0^{2\pi} G^+(t, \alpha, \Gamma)\, d\theta$, where $t = re^{i\theta}$. Putting

$$G(t; \alpha_1, \alpha_2) = G^+(t; \alpha_1, \Gamma) - G^+(t; \alpha_2, \Gamma) + U(t, \alpha_1, \alpha_2, \Gamma),$$

$$G^+(t; \alpha, \Gamma_1) - G^+(t; \alpha, \Gamma_2) = U(t, \alpha, \Gamma_1, \Gamma_2),$$

then $U(t, \alpha_1, \alpha_2, \Gamma)$ and $U(t; \alpha, \Gamma_1, \Gamma_2)$ are bounded functions in the circle $|t| < R$. Hence we get

(1) $\frac{1}{2\pi}\int_0^{2\pi} G(t;\alpha_1,\alpha_2)\,d\theta = \bar{m}(r;\alpha_1,\Gamma) - \bar{m}(r,\alpha_2,\Gamma) + O(1)$,

and

(2) $\bar{m}(r;\alpha,\Gamma_1) = \bar{m}(r;\alpha,\Gamma_2) + O(1)$.

Let $\alpha(a,b)$ be a branch point of order $k-1$:

$$y-b = a_p(x-a)^{p/k} + a_{p+1}(x-a)^{(p+1)/k} + \cdots,$$

where $a_p \neq 0$,

$$f(t)-a = A(t-t_0)^{k\bar{n}} + \cdots,$$

$$g(t)-b = B(t-t_0)^{p\bar{n}} + \cdots,$$

where $A \neq 0$, $B \neq 0$. Then we say that $(f(t), g(t))$ takes $\alpha(a,b)$ at $t=t_0$ and $\bar{n}$ is the order at t_0, and we denote by $\bar{n}(r,\alpha)$ the total sum of orders which $(f(t), g(t))$ takes α in the circle $|t| \leq r$. We put

$$\bar{N}(r,\alpha) = \int_0^r \frac{\bar{n}(r,\alpha)-\bar{n}(0,\alpha)}{r}\,dr + \bar{n}(0,\alpha)\log r$$

and

$$T(r,\alpha,\Gamma) = T(r,f,g,\alpha,\Gamma) = \bar{m}(r,\alpha,\Gamma) + \bar{N}(r,\alpha).$$

Applying Green s formula to $G(t;\alpha_1,\alpha_2)$, we get

(3) $\frac{1}{2\pi}\int_0^{2\pi} G(t;\alpha_1,\alpha_2)\,d\theta + \bar{N}(r,\alpha_1) - \bar{N}(r,\alpha_2) = C$,

where C is a constant. From (1) and (3) we get

$$T(r,\alpha_1,\Gamma) = T(r,\alpha_2,\Gamma) + O(1).$$

From (2) we get

$$T(r,\alpha,\Gamma_1) = T(r,\alpha,\Gamma_2) + O(1).$$

Therefore we may use the notation $\bar{T}(r,\alpha) = T(r,f,g,\alpha)$ in the place of $T(r,f,g,\alpha,\Gamma)$. Then we have the following theorem:

Theorem 2.

$$\bar{T}(r,\alpha_1) = \bar{T}(r,\alpha_2) + O(1).$$

Let $\alpha_i(a,b_i)$ $(i=1,2,\cdots s)$ be points on S where $x=a$, and at α_i let k_i sheets hang together;

$$\sum_{i=1}^{s} k_i = m.$$

From Theorem 2

$$T(r,\frac{1}{f-a}) = \sum_{i=1}^{s} k_i \bar{T}(r,\alpha_i) + O(1)$$
$$= m\,\bar{T}(r,\alpha) + O(1).$$

Theorem 3.

$$\bar{T}(r,\alpha) = \frac{1}{m}T(r,f) + O(1)$$
$$= \frac{1}{n}T(r,g) + O(1).$$

§ 4. Let $X = \varphi(x,y)$, $Y = \psi(x,y)$ be a birational transformation where $\varphi(x,y)$ has μ simple poles on S and $\psi(x,y)$ has ν simple poles. Then we get $G(X,Y)=0$, where $G(X,Y)$ is an irreducible polynomial of degree ν in X and μ in Y. We suppose that the following conditions are satisfied:

(I) $X = \varphi(x,y)$ has μ simple poles which are not branch points on S;

(II) Infinity points on S are not branch points and the m values of X at infinity points are represented by m distinct expansions

$$X^i = a_0^i + a_1^i\frac{1}{x} + a_2^i\frac{1}{x^2} + \cdots + a_k^i\frac{1}{x^k} + \cdots,$$

where $i=1,2,3,\cdots,m$ and $a_1^i \neq 0$.

On the condition (I). We choose X_0 so that $\varphi(x,y) = X_0$ has simple roots α_i $(i=1,2,\cdots)$ on S. We put

$$X_1 = \frac{1}{X-X_0}, \quad Y_1 = Y;$$

$$F(t) = \varphi(f(t),g(t)), \quad G(t) = \psi(f(t),g(t));$$

$$F_1(t) = \frac{1}{F(t)-X_0}, \quad G_1(t) = G(t).$$

Let α on S be transformed into A by $X=\varphi(x,y), Y=\psi(x,y)$ and let A be transformed into A_1 by $X_1 = \frac{1}{X-X_0}$, $Y_1 = Y$. If we consider (X_1, Y_1) in place of (X, Y), then the condition (I) is satisfied and $T(r,F,G,A) = T(r,F_1,G_1,A_1) + O(1)$.

On the condition (II). We choose x_0 so that $S(x_0,y)=0$ has m simple roots y_i $(i=1,2,\cdots,m)$ and $X_i = \varphi(x_0,y_i) \neq X_j = \varphi(x_0,y_j)$ $(i\neq j)$ and

$$X_j = a_0^i + a_1^i(x-x_0) + \cdots + a_k^i(x-x_0)^k + \cdots,$$

where $a_1^i \neq 0$,

$$x_1 = \frac{1}{x-x_0}, \quad y_1 = y;$$

$$f_1(t) = \frac{1}{f(t)-x_0}, \quad g_1(t) = g(t).$$

Let α be transformed into α_1 by $x_1 = \frac{1}{x-x_0}$, $y_1 = y$. If we consider (x_1, y_1) in place of (x, y), then the condition (II) is satisfied and $T(r,f,g,\alpha) = T(r,f_1,g_1,\alpha_1) + O(1)$.

Under the conditions (I) and (II) we classify points on S in the following four cases:

(a) poles of $X(x, y)$;
(b) infinity points;
(c) branch points;
(d) points which are neither branch points nor infinity points.

Now,

$$F(t) = \varphi(f(t), g(t)), \quad G(t) = \psi(f(t), g(t)).$$

Let α be transformed into A by the birational transformation, then

$$T(r, f, g, \alpha) = T(r, F, G, A) + O(1).$$

In all cases this proposition can be easily proved. We will give the proof only in the case (c).

Let $\alpha(a, b)$ be a branch point of order $k-1$. Let

$$X = X_0 + c_p(x-a)^{p/k} + c_{p+1}(x-a)^{(p+1)/k} + \cdots$$

where $c_p \neq 0$ and by the condition (I) p is a positive integer.

(1) $(X - X_0)^{1/p} = (x-a)^{1/k}(c_p + c_{p+1}(x-a)^{1/k} + \cdots)^{1/p}$

By inversion $(x-a)^{1/k} = P((X-X_0)^{1/p})$.
As $y-b$ is a regular function of $(x-a)^{1/k}$, we have

$$y - b = Q((X-X_0)^{1/p})$$

(2) $$Y = \psi(x, y) = R((X-X_0)^{1/p}) = Y_0 + d_s(X-X_0)^{s/p} + \cdots,$$

where $P(t)$, $Q(t)$, and $R(t)$ are regular at $t=0$ and by the property of the birational transformation s/p is an irreducible rational number. Let α be transformed into $A(X_0, Y_0)$, then A is a branch point of order $p-1$. We have

(3) $f(t) - a = a_{k\bar{n}}(t-t_0)^{k\bar{n}} + a_{k\bar{n}+1}(t-t_0)^{k\bar{n}+1} + \cdots,$

(4) $F(t) - X_0 = A_{p\bar{n}}(t-t_0)^{p\bar{n}} + A_{p\bar{n}+1}(t-t_0)^{p\bar{n}+1} + \cdots.$

From (1), (2), (3) and (4), we get, by the definition of $T(r, f, g, \alpha)$,

$$T(r, f, g, \alpha) = T(r, F, G, A) + O(1).$$

5. We suppose that infinity points are not branch points. Let α_i $(i = 1, 2, \cdots, k)$ be a branch point of order $m_i - 1$ and β_i $(i = 1, 2, \cdots, m)$ be an infinity point. Applying Green's formula to $u(t)$ stated in § 2, we have

$$\sum_{i=1}^{k}(m_i - 1)\bar{T}(r, \alpha_i) - 2\sum_{i=1}^{m}\bar{T}(r, \beta_i) = \int_0^r \frac{A(r)}{r}\,dr + O(1),$$

where

$$A(r) = \frac{2}{\pi}\int_0^r\int_0^{2\pi} \frac{\left(\left|\frac{dz}{dx}\right|\left|\frac{dx}{dt}\right|\right)^2}{(1-|z|^2)^2}\, r\,dr\,d\theta,$$

$$t = re^{i\theta}.$$

By Theorem 2

(1) $$\left(\sum_{i=1}^{k}(m_i - 1) - 2m\right)\bar{T}(r, \alpha) = \int_0^r \frac{A(r)}{r}\,dr + O(1),$$

(2) $$\sum_{i=1}^{k}(m_i - 1) - 2m = 2g - 2,$$

where g is genus. From Theorem 1, we have

$$A(r) \leqq \frac{2}{\pi}\int_0^r\int_0^{2\pi}\frac{R^2 r}{(R^2-r^2)^2}\,dr\,d\theta = \frac{2r^2}{R^2-r^2},$$

(3) $$\int_0^r \frac{A(r)}{r}\,dr \leqq \int_0^r \frac{2r}{R^2-r^2}\,dr = \log\frac{1}{R-r} + O(1).$$

From (1), (2) and (3), we have

$$\bar{T}(r, \alpha) \leqq \frac{1}{2g-2}\log\frac{1}{R-r} + O(1).$$

Applying the birational transformation, from the result of § 4, we have the following theorem.

Theorem 4.

$$\bar{T}(r, \alpha) \leqq \frac{1}{2g-2}\log\frac{1}{R-r} + O(1).$$

From Theorems 3 and 4 we have the following theorem

Theorem 5.

$$T(r, f) \leqq \frac{m}{2g-2}\log\frac{1}{R-r} + O(1).$$

Corollary. If $\overline{\lim\limits_{r\to R}}\dfrac{T(r, f)}{\log\frac{1}{R-r}} = \infty$ then genus $g < 2$.

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Kagoshima University.