REPRESENTATION OF FUNCTIONS ANALYTIC IN A MULTIPLY-CONNECTED DOMAIN

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1. We may and do use, as a canonical domain of multiplicity $n\ (>2)$, a concentric annular ring slit along concentric circular arcs. Let the boundary components of such a domain D, laid on z-plane, be

$$C_1:\ |z|=1; \qquad C_2:\ |z|=Q\ (<1);$$

$$C_j:\ |z|=m_j, \quad \theta_j \leqq \arg z \leqq \theta_j+\gamma_j \quad (3 \leqq j \leqq n),$$

and the interior and the exterior sides of the slits $C_j\ (3 \leqq j \leqq n)$ be

$$C_j^{(i)}:\ |z|=m_j-0, \quad \theta_j \leqq \arg z \leqq \theta_j+\gamma_j,$$

$$C_j^{(e)}:\ |z|=m_j+0, \quad \theta_j+\gamma_j \geqq \arg z \geqq \theta_j,$$

respectively. The total boundary of D be denoted by

$$C=\sum_{j=1}^{n} C_j .$$

Any function $U(z)$ regular harmonic in the domain D and continuous on the closed domain $D+C$ is represented by Green's formula in the form

$$U(z)=\frac{1}{2\pi}\int_C U(\zeta)\frac{\partial g(\zeta,z)}{\partial \nu_\zeta}\,ds_\zeta,$$

$g(\zeta, z)$ being, as usual, Green function (with variable ζ) of D with singularity at z, ν_ζ and s_ζ denoting inward normal and arc-length parameter at a boundary point ζ.

If we denote the equation of the boundary C by $z=\zeta(s)$ and the harmonic measure of a part of C from a fixed point to the point $\zeta(s)$ by $\omega(z, \zeta(s))$, then we have

$$\frac{1}{2\pi}\frac{\partial g(\zeta,z)}{\partial \nu_\zeta}ds_\zeta = d\omega(z,\zeta(s)) \equiv \omega(z, d\zeta(s)).$$

But, we use here an another aggregation, namely the one corresponding to Herglotz type. Let $\Phi(z)$ be an analytic function one-valued and regular in D and continuous on $D+C$. We denote by $G(\zeta, z)$ an analytic function of z whose real part coincides with $g(\zeta, z)$; $G(\zeta, z)$ being uniquely determined except an additive purely imaginary quantity depending possibly on ζ and possessing multivaluedness due to periodicity moduli with respect to the boundary components. We have then, by the formula mentioned above,

$$\Phi(z)=\frac{1}{2\pi}\int_C \mathcal{R}\Phi(\zeta)\frac{\partial G(\zeta,z)}{\partial \nu_\zeta}\,ds_\zeta + ic,$$

c being a real constant.

We now assume that $\mathcal{R}\Phi(z)$ is of bounded variation along C. Then, so is also the function $(\zeta \in C_j)$

$$\rho_j(\varphi)=\int^{\varphi} \mathcal{R}\Phi(\zeta)\,ds_\zeta \qquad (\varphi=\arg\zeta);$$

in fact,

$$\int_{C_j}|d\rho_j(\varphi)| = \int_{C_j}|\mathcal{R}\Phi(\zeta)|\,ds_\zeta.$$

In this case, we may write the expression as in the Herglotz type which states

$$\Phi(z)=\frac{1}{2\pi}\sum_{j=1}^{n}\int_{C_j}\frac{\partial G(\zeta,z)}{\partial \nu_\zeta}\,d\rho_j(\varphi)+ic.$$

Now, considering residue at point z, we have particularly

$$\frac{1}{2\pi}\int_C \frac{\partial G(\zeta,z)}{\partial \nu_\zeta}\,ds_\zeta = 1,$$

and hence

$$1=\frac{1}{2\pi}\sum_{j=1}^{n}\int_{C_j}\frac{\partial G(\zeta,z)}{\partial \nu_\zeta}\,d\sigma_j(\varphi),$$

where $\sigma_j(\varphi)$ is defined by

$$\sigma_j(\varphi)=\begin{cases} \varphi & \text{on } C_1, \\ -Q\varphi & \text{on } C_2; \\ m_j(\varphi-\theta_j) & \text{on } C_j^{(i)}, \\ -m_j(\varphi-\theta_j-\gamma_j) & \text{on } C_j^{(e)} \end{cases} \quad (3 \leqq j \leqq n).$$

The last equation shows that an additive purely imaginary constant ic contained in the general representation vanishes out for the particular function $\Phi(z) \equiv 1$.

2. Consider now an analytic function $f(z)$ one-valued and regular in D and piecewise regular on $D+C$.

The boundary points, finite in number, where the regularity of $f(z)$ is broken down, be

$$z_{j\mu} \qquad \left(\begin{array}{l} \mu = 1, 2, \cdots, n_j, \\ j = 1, \quad , n \end{array} \right).$$

The existence of limits of $f'(z)$ from both sides along C will be assumed at each of such points.

We assume further that $f(z)$ vanishes nowhere on $D + C$ except at these exceptional points $z_{j\mu}$. The image of D by mapping $w = f(z)$ then possesses, on Riemann surface, a piecewise analytic boundary and the function $f(z)$ can be prolonged analytically over every boundary arc containing no exceptional point. Denoting generally by $'\zeta$ any exceptional point, then the image-curve of C possesses at $f('\zeta)$ an angular point. Denoting by $\alpha\pi$ the exterior angle at such an angular point with respect to the image-domain, the jump of $\arg f'(z)$ at $'\zeta$ along C is given by

$$\arg \frac{f'('\zeta_+)}{f'('\zeta_-)} = (\alpha - 1)\pi,$$

$'\zeta_\pm$ being infinitely adjacent points at both sides of $'\zeta$.

The image-curve of C will moreover have angular points, in general, also at the image-points of end-points of the slits. If $f'(z)$ is regular at such an end-point p and does not vanish there, then the exterior angle of the image-curve at $f(p)$ is 0 and the jump of $\arg f'(z)$ there vanishes out. But, if p coincides with an exceptional point $'\zeta$ for which the image-curve possesses an angular point with exterior angle $\alpha\pi$, then the jump of $\arg f'(z)$ there becomes $\alpha\pi$ since $\arg dz$ jumps there by $-\pi$.

Let $'\zeta$ be an exceptional point coinciding with none of end-points of the slits and the corresponding angle $\alpha\pi$ be different from 2π. Then $(f(z) - f('\zeta))^{1/(2-\alpha)}$ is regular at a vicinity of $'\zeta$ and has $'\zeta$ as a simple pole; namely, the function $f(z) - f('\zeta)$ is uniformized by a local parameter $(z - '\zeta)^{2-\alpha}$. Therefore, $f(z) - f('\zeta)$, as a function of $(z - '\zeta)^{2-\alpha}$, possesses a simple pole at $'\zeta$. In case $\alpha = 2$, instead of $(z - '\zeta)^{2-\alpha}$, $\lg(z - '\zeta)$ may be taken as a local uniformizing parameter. In any case, the function

$$(z - '\zeta)^{\alpha - 1} f'(z)$$

is regular and non-vanishing around $'\zeta$. If an exceptional point $'\zeta$ coincides with an end-point p of a slit, then the power $2 - \alpha$ in local parameter has to be replaced by $(2-\alpha)/2$.

In the following, we suppose none of exceptional points coincide with any one of end-points of the slits, i. e., $'\zeta \neq p$. But, if it happens $'\zeta = p$, the only modification must be made, according to the fact stated just above, that α has to be replaced by $\alpha/2+1$.

Now, the function defined by

$$\Phi(z) = z \frac{d}{dz} \lg \left(f'(z) \prod_{j=1}^{n} \prod_{\mu=1}^{n_j} (z - z_{j\mu})^{\alpha_{j\mu}-1} \right)$$
$$= \frac{z f''(z)}{f'(z)} + \sum_{j=1}^{n} \sum_{\mu=1}^{n_j} \frac{(\alpha_{j\mu}-1) z}{z - z_{j\mu}}$$

is evidently one-valued and regular throughout $D + C$. Hence, it is expressible in the form

$$\Phi(z) = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{C_j} \frac{\partial G(\zeta, z)}{\partial \nu_\zeta} d\rho_j(\varphi) + i c,$$

c being a real constant and $\rho_j(\varphi)$ being a real function of $\varphi = \arg \zeta$ given by

$$\rho_j(\varphi) = \int^{\varphi} \mathcal{R}\Phi(\zeta) ds_\zeta \quad \text{for } \zeta \in C_j.$$

The linear function $z/(z - z_{j\mu})$ behaves regularly everywhere except only at a simple pole $z_{j\mu}$ and its real part is identically equal to 1/2 along C_j. It will be easily seen that a representation of the same type as given above for $\Phi(z)$ holds good also for such a function.[1] Hence, we obtain the following representation formula with respect to $f(z)$:

$$\frac{z f''(z)}{f'(z)} = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{C_j} \frac{\partial G(\zeta, z)}{\partial \nu_\zeta} m_j \, d\arg f'(\zeta) + i c^*,$$

c^* being a real constant and $m_1 = 1$, $m_2 = Q$.

On the other hand, we have seen that, for particular case $f'(z) \equiv z$, the corresponding representation reduces to

$$1 = \frac{1}{2\pi} \sum_{j=1}^{n} \int_{C_j} \frac{\partial G(\zeta, z)}{\partial \nu_\zeta} m_j \, d\arg \zeta,$$

an additive constant vanishing out. Hence, remembering that the relation

$$d\arg df(\zeta) = d\arg(\zeta f'(\zeta) i d\varphi) \qquad (\varphi = \arg \zeta)$$
$$= d\arg(\zeta f'(\zeta))$$

is valid along C, we have

$$1+\frac{zf''(z)}{f'(z)}$$
$$=\frac{1}{2\pi}\sum_{j=1}^{n}\int_{C_j}\frac{\partial G(\zeta,z)}{\partial \nu_\zeta}m_j\,d\arg df(\zeta)+ic^*.$$

The real constant c^* can be determined as follows. For any fixed point z_0 in D, we put

$$L(z,\zeta)=\frac{1}{2\pi}\int_{z_0}^{z}\frac{\partial G(\zeta,z)}{\partial \nu_\zeta}\frac{dz}{z}.$$

This function has a periodicity modulus around each boundary component C_j. Hence, if, introducing the uniformizing parameter $\lg z$, we put

$$M(\lg z,\zeta)=L(z,\zeta),$$

then the difference

$$M(\lg z+2\pi i,\zeta)-M(\lg z,\zeta)$$

remains constant, for fixed ζ, along each C_j, i.e., z is contained in this expression only apparently. Hence, we may put

$$M_j(\zeta)=M(\lg z+2\pi i,\zeta)-M(\lg z,\zeta) \qquad (z\in C_j)$$

Integrating the above obtained expression for $f''(z)/f'(z)$ with respect to z, we get

$$\lg\frac{f'(z)}{f'(z_0)}=\sum_{j=1}^{n}\int_{C_j}M(\lg z,\zeta)m_j\,d\arg f'(\zeta)+ic^*\lg\frac{z}{z_0}.$$

Now, $f'(z)$ being one-valued, the left-hand member of the last relation increases by an integral multiple of $2\pi i$ for substitution $\lg z \mid \lg z+2\pi i$. Accordingly, the real part of this increase calculated from the right-hand member must vanish. Hence we get

$$2\pi c^*=\sum_{j=1}^{n}m_j\int_{C_j}\mathcal{R}M_j(\zeta)\,d\arg f'(\zeta),$$

which is the relation determining c^*. Since, in particular case $f'(z)\equiv z$, the corresponding constant becomes 0, we may write also

$$c^*=\frac{1}{2\pi}\sum_{j=1}^{n}m_j\int_{C_j}\mathcal{R}M_j(\zeta)\,d\arg df(\zeta).$$

The constant c^* having been determined, we obtain the desired representation formula

$$1+\frac{zf''(z)}{f'(z)}$$
$$=\frac{1}{2\pi}\sum_{j=1}^{n}m_j\int_{C_j}\left(\frac{\partial G(\zeta,z)}{\partial\nu_\zeta}+\mathcal{R}M_j(\zeta)\right)d\arg df(\zeta),$$

which, by integration, yields a representation for $f(z)$ itself.(2)

3. As an application of the above general formula, we consider here the case where $w=f(z)$ maps the basic domain D onto a domain bounded by n rectilinear polygons. Then, the exceptional points ζ are the points corresponding to vertices of the image-curve of C, and $\arg df$ becomes a step function having jump with hight $(\alpha-1)\pi$ at each ζ. Hence, the general formula reduces here to a simple form without integration sign which states

$$1+\frac{zf''(z)}{f'(z)}$$
$$=\frac{1}{2}\sum_{j=1}^{n}m_j\sum_{\mu=1}^{n_j}(\alpha_{j\mu}-1)\frac{\partial G(z_{j\mu},z)}{\partial z_{j\mu}}+ic^*$$

c^* being given by

$$c^*=\frac{1}{2}\sum_{j=1}^{n}m_j\sum_{\mu=1}^{n_j}(\alpha_{j\mu}-1)\mathcal{R}M_j(z_{j\mu}).$$

The successive integration yields then

$$\lg\frac{zf'(z)}{z_0f'(z_0)}$$
$$=\pi\sum_{j=1}^{n}m_j\sum_{\mu=1}^{n_j}(\alpha_{j\mu}-1)L(z,z_{j\mu})+ic^*\lg\frac{z}{z_0}+A_1,$$

A_1 being an integration constant, and

$$f'(z)$$
$$=Az^{ic^*-1}\exp\left(\pi\sum_{j=1}^{n}m_j\sum_{\mu=1}^{n_j}(\alpha_{j\mu}-1)L(z,z_{j\mu})\right),$$

$$f(z)$$
$$=A\int^{z}z^{ic^*-1}\exp\left(\pi\sum_{j=1}^{n}m_j\sum_{\mu=1}^{n_j}(\alpha_{j\mu}-1)L(z,z_{j\mu})\right)dz$$
$$+A',$$

A and A' denoting integration constants which depend only on position and magnitude of the polygonal image domain. The last formula may be regarded as a generalization of Schwarz-Christoffel's one for simply-connected case and of a formula for doubly-connected case previously given by the present author.(3), (4)

(*) Received October 9, 1950.

(1) Cf. Y.Komatu, Darstellungen der in einem Kreisringe analytischen Funktionen nebst den Anwendungen auf konforme Abbildung über Polygonalringgebiete. Jap. Journ. Math. 19 (1945), 203-215.

(2) The corresponding formulae for simply- and doubly-connected cases have previously been given in Y.Komatu, Einige Darstellungen analytischer Funktionen und ihre Anwendungen auf konforme Abbildung. Proc. Imp. Acad. Tokyo 20(1944), 536-541 and in the paper cited[1], respectively.

(3) Loc. cit(1) and(2).

(4) For generalization of Schwarz-Christoffel formula, see also Y.Komatu, Conformal mapping of polygonal domains, Journ. Math. Soc. Japan 2(1950), a preliminary note of which has been reported under the same title in these Reports Nos. 3-4 (1949), 47-50.

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