# NOTE ON GROUPS OF AUTOMORPHISMS.

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In this paper, we shall denote by $e$ the identity element of a group, while we denote by $E$ the identity group. And we shall denote by $A(G)$ the group of automorphisms of a group $G$.

Definition 1. Let $G_0$ be a group whose center is $E$. Consider the chain

$$G_0 < G_1 < \cdots < G_n < \cdots$$

satisfying the following conditions;

(1) if there exists $G_i$ $(i \geq 1)$, $G_i = A(G_{i-1})$;

(2) when $G_i$ exists and $G_{i+1}$ does not exist, $A(G_i) = G_i$.

Then we shall call this chain the tower of $G_0$. If there exists a last term, say $G_n$, of this chain, we shall say that the length of this tower is $n$, and denote $G_n$ by $\overline{G}_0$. Otherwise we shall say that the length of this tower is infinite.

Remark: Wielandt has proved that a tower of any finite group has a finite length (cf. Math. Zeitschr. 45 (1939)).

Definition 2. We shall say that a group $G$ is complete if (i) the center of $G$ is $E$ and (ii) $\overline{G} = G$ (i.e. $A(G) = G$).

## §1. Groups of dihedral type.

Definition 3. We shall say that $G$ is of dihedral type, if (i) $G$ has a normal subgroup $U$ such that $(G : U) = 2$, (ii) $G$ has an element $x$ of order 2 such that $xax = a^{-1}$ for every $a \in U$ and (iii) $G = U.\{x\}$.

Remark: From this definition, it follows easily that $U$ is an abelian group.

Lemma 1. Let $G$ be a group of dihedral type. Then the center of $G$ is $E$ if and only if $U$ (with the notation introduced in definition 3) has no element of order even.

Proof. If $U$ has an element $a$ of order 2, $a$ is contained in the center of $G$. In the other case, it is clear that the center of $G$ is $E$.

Lemma 2. Let $G$ be a group of dihedral type with minimal condition on normal subgroups. Then the order of $G$ is finite.

Proof. This follows readily from that every subgroup of $U$ (in definition 3) is a normal subgroup of $G$.

From now on we shall consider only groups with minimal condition on normal subgroups if contrary is not expressed.

Proposition 1. Let $G$ be a group of dihedral type. Then $G$ is complete if and only if $G$ is the symmetric group of degree 3 (we shall denote this by $S_3$). (Cf. example 2, §4)

Proof. If $U$ (with notation in definition 3) has an element of order $k$ $(k > 3)$ then $G$ has an outer automorphism $\sigma$ such as $a^{\sigma} = a^2$ for every $a \in U$ and $x^{\sigma} = x$ (with notations in definition 3). Therefore, if $(U : E) > 3$, $U$ is of the type $(3, 3, \ldots, 3)$. Then $G$ has outer automorphisms; for instance, those which permute the basis of $U$ and leave $x$ fixed. So $G$ must be $S_3$. On the other hand, it is clear that $S_3$ is complete.

## §2. On $A(G \times G)$.

Proposition 2. Let $G$ be a complete, directly indecompisable group. Then $A(G \times G) = (G \times G)\{y\}$ where $y^2 = e$, $y(a, b)y = (b, a)$ for every $(a, b) \in G \times G$. Furthermore $A(G \times G)$ is directly indecomposable.

Proof. Let $\sigma$ be an automorphism of $G \times G$. We set $H_1 = \{(a, e); (ab, c) = (ba, c)$ for every $(b,c) \in (G \times E)^{\sigma}\}$ and $H_2 = \{(e, a); (b, ac) = (b, ca)$ for every $(b, c) \in (G \times E)^{\sigma}\}$. Then we have $(E \times G)^{\sigma} = H_1 \times H_2$. This implies that $(E \times G)^{\sigma} = E \times G$ or $G \times E$, because $G$ is directly indecomposable. If we observe that $G$ is complete, we have the first part of proposition 2.

Assume now that $A(G \times G) = M \times N$ where $M \neq E$ $N \neq E$, and set

$K_1 = M \cap (G \times G)$, $K_2 = N \cap (G \times G)$. Then we have $(M : K_1) = 2$ or $M = K_1$, $(N : K_2) = 2$ or $N = K_2$. If $K_2 = E$, then $N$ would be contained in the center of $A(G \times G)$. Therefore $K_2 \neq E$, and similarly $K_1 \neq E$. We can assume without loss of generality that $M \neq K_1$, whence $M$ contains at least one element of the type $(g_1, g_2)y$. Then we have

$$(g_1, g_2)\, y\, (h_1, h_2) = (h_1, h_2)(g_1, g_2)y \quad \text{for every } (h_1, h_2) \in K_2,$$

This shows that each pair $(h_1, h_2)$ in $K_2$ is already determined by one of its components. Let $(h_1, h_2) \neq e$, $(h_1, h_2) \in H_2$. Then we have $h_1 \neq e$. We choose an element $(1, e)$ such that $h_1 1 \neq 1 h_1$, then we have $(1, e)^{-1} \cdot (h_1, h_2) \cdot (1, e) = (1^{-1} h_1 1, h_2) \in K_2$. This contradicts with the fact just observed.

Proposition 3. Under the same assumptions in propositions 2, $A(G \times G)$ is not complete if and only if $G = S_3$.

Proof. If $A(G \times G)$ is not complete, $A(G \times G)$ has at least one outer automorphism $\sigma$, whence $(G \times G)^{\sigma} \neq G \times G$. We shall set $(G\times E)^{\sigma} \wedge (G\times G) = H_1$ and $(E\times G)^{\sigma} \wedge (G\times G) = H_2$ Then we have $((G\times E):H_1) = ((E\times G) : H_2) = 2$ because $(A(G \times G) : (G \times G)) = 2$ and at least one, whence both, of $G_i$'s contain some elements which are not contained in $G \times G$. And we have that each pair $(h_1, h_2)$ in $H_1$ is uniquely determined by one component and the same for each pair $(g_1, g_2)$ in $H_2$ (cf. the proof of proposition 2).

If we set

$$N_1 \times E^{*} = \{(a, e) \in G \times G \;;\; (a, b) \in H_1\},$$

$$N_2 \times E = \{(a, e) \in G \times G \;;\; (a, b) \in H_2\},$$

we have $(G : N_1) = (G : N_2) = 2$, and $U = N_1 \wedge N_2$ is abelian. If $N_1 \neq N_2$, we have $G = N_1 \cdot N_2$, whence $U$ is in the center of $G$. This means $(G : E) = 4$ and $G$ is abelian. So we have $U = N_1 = N_2$: Therefore, proposition 3 follows from proposition 1 if we observe that $A(S_3 \times S_3)$ is not complete.

## § 3. Groups of a special type.

We define $K_n$ $(n = 1, 2, \ldots\ldots)$ by induction on $n$. Let $K_1$ be a complete and directly indecomposable group other than $S_3$. If $K_n$ is already defined, we set $K_{n+1} = \overline{K_n \times K_n} = A(K_n \times K_n)$

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$$= \overline{\overline{\overline{K_1 \times K_1} \times K_2} \times \cdots\cdots \times K_n}^{\,(n)}$$

Lemma 3. Let $G = H \times K_n$, where $K_1 \times K_1 \times K_2 \times \ldots\ldots \times K_{n-1} \subseteq H \subset K_n$. Then both $H$ and $K_n$ are invariant by every automorphism of $G$.

Proof. Let $\sigma$ be an automorphism of $G$. We set

$$H_1 = \{h \in H;\ ha = ah \text{ for every } (a, b) \in H^{\sigma}\},$$

$$H_2 = \{g \in K_n;\ gb = bg \text{ for every } (a, b) \in H^{\sigma}\},$$

Then we have $K_n^{\sigma} = H_1 \times H_2$. If we observe that $K_n$ is directly indecomposable and $(H : E) < (K_n : E)$, we find that $K_n$ is invariant by $\sigma$. Therefore $H$ is also invariant by $\sigma$.

Proposition 4. The tower of $K_1 \times K_1 \times K_2 \times K_3 \times \ldots \times K_n$ is given by

$$K_1 \times K_1 \times K_2 \times \ldots \times K_n < K_2 \times K_2 \times K_3 \times \ldots \times K_n < \ldots < K_n \times K_n < K_{n+1}$$

-f is evident.

## § 4. Remarks and examples.

(1) In order to show that "for an arbitrary integer $n$, there exists a group whose length of tower is larger than $n$" we need not use our above result. Simpler methods exist.

(2) We can not conclude the directly indecomposability of $A(G)$ from that of $G$.

Example 1. Let $A_m$, $S_m$ be respectively the alternating and symmetric group of degree $m$.

For $m_1$ and $m_2$, where $m_1 \neq 6$, $m_2 \neq 6$, $m_1 > 3$, $m_2 > 3$ and $m_1 \neq m_2$, we take $x$, $y$ from $S_{m_1}$ and $S_{m_2}$, respectively, such as $x \notin A_{m_1}$, $y \notin A_{m_2}$ and $x^2 = e$, $y^2 = e$. If we set $G = (A_{m_1} \times A_{m_2})(x,y)$, then $G$ is directly indecomposable and $A(G) = S_{m_1} \times S_{m_2}$.

(3) A group with infinite tower length.

Remark: The group in this example is of dihedral type and satisfies maximal condition but not minimal condition on normal subgroups.

Example 2. Let $U$ be a cyclic group of infinite order and let $x$ be an element such that $x^2 = e$, $xax = a^{-1}$ where $U = \{a\}$. And we set $G = U\{x\}$. $A(G)$ is generated by $G$ and $w$ where $w^{-1}aw = a$ and $wxw^{-1} = ax$. Then we have $w^2 = a$, and $A(G) \cong G$.

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