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ITERATIVE FIXED POINTS OF NON-LIPSCHITZIAN SELF-MAPPINGS

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Abstract

In this paper, we shall establish iterative fixed points of non-Lipschitzian continuous self-mappings on Banach spaces with weak uniform normal structure.

1. Introduction

Let C be a nonempty subset of a real Banach space X and let $\boldsymbol{N}$ be the set of natural numbers. A mapping $T: C \to C$ is said to be Lipschitzian if for each $n \in \boldsymbol{N}$, there exists a real number $k(n)$ such that

$$\|T^n x - T^n y\| \leqq k(n)\|x-y\| \qquad \text{for all} \quad x, y \in C .$$

In particular, T is said to be asymptotically nonexpansive [7] if $\lim_{n\to\infty} k(n)=1$ and it is said to be nonexpansive if $k(n)=1$ for any $n \in \boldsymbol{N}$. We now consider a non-Lipschitzian self-mapping on C, that is to say, a mapping of weakly asymptotically nonexpansive type. We say that a mapping $T: C \to C$ is said to be weakly asymptotically nonexpansive type (simply, w.a.n.t.) on C (see [10]) if, for each $x \in C$ and each bounded subset D of C,

$$\limsup_{n\to\infty}(\sup\{[\|T^n x - T^n y\| - \|x-y\|] : y \in D\}) \leqq 0 .$$

Immediately, we can see that all mappings of w.a.n.t. include all mappings of asymptotically nonexpansive type (see [11]). In particular, if $T: C \to C$ is a Lipschitzian mapping with an additional condition, i.e., $\limsup_{n\to\infty} k(n) \leqq 1$ (see [12] and [14]), then it is obviously a continuous mapping of w.a.n.t. Further if C is bounded, then any mapping of w.a.n.t. is asymptotically nonexpansive type.

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In this paper, we first give new sharp expressions of the weak convergent sequence coefficient $WCS(X)$ of a non-Schur space X according to G. L. Zhang [16]. Finally, we shall present an iterative fixed point of a non-Lipschitzian continuous self-mapping on a Banach space X with weak uniform normal structure (see Theorem 3.3), which improves the result due to T.D. Benavides, G.L. Acedo and H.K. Xu [2].

2. Expressions of the $WCS(X)$

Let X be a Banach space which is not Schur and let $\{x_n\}$ be a sequence of X. For $x \in X$, set $r(x, \{x_n\}) = \limsup_{n\to\infty}\|x_n - x\|$. $A(\{x_n\}) = \limsup_{n\to\infty}\{\|x_i - x_j\| : i, j \geqq n\}$ and $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in \overline{co}(\{x_n\})\}$ are called *the asymptotic diameter of* $\{x_n\}$ and *the Chebyshev radius of* $\{x_n\}$ *relative to* $\overline{co}(\{x_n\})$, respectively, where $\overline{co}(\{x_n\})$ denotes the closed convex hull of $\{x_n\}$. The *weakly convergent sequence coefficient* of X, denoted by $WCS(X)$, is the supremum of the set of all numbers M with the property that for each weakly convergent sequence $\{x_n\}$ with asymptotic diameter $A(\{x_n\})$, there is some $y \in \overline{co}(\{x_n\})$ such that $M \cdot \limsup_{n\to\infty}\|x_n - y\| \leqq A(\{x_n\})$. Equivalently,

$$WCS(X) = \inf\{A(\{x_n\})/r(\{x_n\})\},$$

where the first infimum is taken over all sequences $\{x_n\}$ in X which are weakly (not strongly) convergent (see [3] and [13]). It is well-known (see [3]) that if $WCS(X) > 1$, then X has weak normal structure. This means that any weakly compact convex subset C of X with $\mathrm{diam}(C) > 0$ has a nondiametral point, i.e., an $x \in C$ such that

$$\sup\{\|x - y\| : y \in C\} < \mathrm{diam}(C).$$

The coefficients $WCS(X)$ play important roles in fixed point theory (cf. [4], [9], [15]). A space X such that $WCS(X) > 1$ is said to have *weak uniform normal structure*.

Recently, G. L. Zhang [16] has proved the following improvement of expression of the $WCS(X)$:

$$WCS(X) = \sup\{M : x_n \rightharpoonup u \Rightarrow M \cdot \limsup_n \|x_n - u\| \leqq A(\{x_n\})\}, \tag{2.1}$$

where " $\rightharpoonup$ " means the weak convergence.

For a sequence $\{x_n\}$ of a Banach space X, we set

$$D(\{x_n\}) := \limsup_{m\to\infty}(\limsup_{n\to\infty}\|x_n - x_m\|).$$

We easily see that $D(\{x_n\}) \leqq A(\{x_n\})$. However, $D(\{x_n\}) \neq A(\{x_n\})$ in general. For example, consider the James' quasi-reflexive space J consisting of all real sequences $x := \{x_n\} = \sum_{n=1}^{\infty} x_n e_n$ for which $\lim_{n\to\infty} x_n = 0$ and $\|x\|_J < \infty$, where

$$\|x\|_J=\sup\{[(x_{p_1}-x_{p_2})^2+\cdots+(x_{p_{m-1}}-x_{p_m})^2+(x_{p_m}-x_{p_1})^2]^{1/2}\}$$

and the supremum is taken over all choices of m and $p_1<p_2<\cdots<p_m$. Then J is a Banach space with the norm $\|\cdot\|_J$ and the sequence $\{e_n\}$ given by $e_n=(0, \cdots, 0, 1, 0, \cdots)$ where the 1 is in the nth position, is a Schauder basis for J (see [5]).

Take $x_n=e_n-e_{n+1}$ for each $n\in\boldsymbol{N}$. Since $\|x_n\|_J=\sqrt{6}$ for each $n\in\boldsymbol{N}$, we have $x_n\in J$ and we now show that

$$D(\{x_n\})=2\sqrt{3}<A(\{x_n\})=2\sqrt{5}\,.$$

Indeed, for each fixed $n\in\boldsymbol{N}$, it is obvious that $\|x_n-x_m\|_J=2\sqrt{3}$ for all $m\geqq n+3$ and so $D(\{x_n\})=\limsup_{m\to\infty}(\limsup_{n\to\infty}\|x_n-x_m\|_J)=2\sqrt{3}$. On the other hand, for each $k\in\boldsymbol{N}$, if we take $n=k$ and $m=n+1\geqq k$, then $\|x_n-x_m\|_J=2\sqrt{5}$ and so

$$\sup\{\|x_n-x_m\|_J:\ n,\ m\geqq k\}=2\sqrt{5}\qquad\text{for each } k\in\boldsymbol{N},$$

which gives $A(\{x_n\})=2\sqrt{5}$.

We now give a sharp expression of $WCS(X)$ which improves the result due to G. H. Zhang [16; Theorem 1].

We begin with the following easy lemma.

LEMMA 2.1. *If* $z_n=y_n/\|y_n\|$, $\alpha:=\lim_{n\to\infty}\|y_n\|\neq 0$, *then*

$$D(\{z_n\})=\frac{1}{\alpha}D(\{y_n\})\,.$$

Proof. For each n, $m\in\boldsymbol{N}$, we easily get

$$\begin{aligned}\|z_n-z_m\|&=\left\|\frac{1}{\alpha}(y_n-y_m)+\left(\frac{1}{\|y_n\|}-\frac{1}{\alpha}\right)y_n-\left(\frac{1}{\|y_m\|}-\frac{1}{\alpha}\right)y_m\right\|\\&\leqq\frac{1}{\alpha}\|y_n-y_m\|+\left|\frac{1}{\|y_n\|}-\frac{1}{\alpha}\right|\cdot\|y_n\|+\left|\frac{1}{\|y_m\|}-\frac{1}{\alpha}\right|\cdot\|y_m\|.\end{aligned}$$

Taking at first $\limsup_{m\to\infty}$ and next $\limsup_{n\to\infty}$ in both sides, we obtain

$$D(\{z_n\})\leqq\frac{1}{\alpha}D(\{y_n\})\,.$$

The converse inequality is similarly obtained.

LEMMA 2.2. *Let* $M>0$. *Then the following statements are equivalent*:

(a) $M\cdot\limsup_{n\to\infty}\|x_n-x\|\leqq A(\{x_n\})$ *for any* $x_n\rightharpoonup x$ (*not strongly convergent*).

(b) $M\cdot\limsup_{n\to\infty}\|x_n'-x'\|\leqq D(\{x_n'\})$ *for any* $x_n'\rightharpoonup x'$ (*not strongly convergent*).

Proof. Since $D(\{x_n\})\leqq A(\{x_n\})$, it is obvious that (b) $\Rightarrow$ (a).

To show (a)$\Rightarrow$(b), let $x'_n \rightharpoonup x'$ (not strongly convergent) and $\alpha := \limsup_{n\to\infty} \|x'_n - x'\| \neq 0$. Then we can choose a subsequence $\{x'_m\}$ of $\{x'_n\}$ such that $\alpha = \lim_{m\to\infty}\|x'_m - x'\|$. Setting $z_m := (x'_m - x')/\alpha$, we have $z_m \rightharpoonup 0$ and $\|z_m\| \to 1$, by using a diagonal method as in [1], we can choose a subsequence $\{z_{m_k}\}$ of $\{z_m\}$ such that $\lim_{k,l\to\infty\, k\neq l}\|z_{m_k} - z_{m_l}\|$ exists. Then (a) and Lemma 2.1 yield that

$$M = M\cdot\lim_{k\to\infty}\|z_{m_k}\| \leqq A(\{z_{m_k}\}) = D(\{z_{m_k}\})$$

$$\leqq D(\{z_m\}) = \frac{1}{\alpha}D(\{x'_m\}) \leqq \frac{1}{\alpha}D(\{x'_n\})$$

and hence $M\cdot\limsup_{m\to\infty}\|x'_m - x'\| = M\alpha \leqq D(\{x'_n\})$, which completes the proof.

As a direct consequence of Lemma 2.2 and (2.1), we can obtain the following

THEOREM 2.3.

$$WCS(X) = \sup\{M:\ x_n \rightharpoonup u \Rightarrow M\cdot\limsup_{n\to\infty}\|x_n - u\| \leqq D(\{x_n\})\}.$$

As a direct consequence of Theorem 2.3, by using a similar method as in G. H. Zhang [16], we can easily obtain the following expressions of $WCS(X)$.

$$\begin{aligned} WCS(X) &= \inf\left\{\frac{D(\{x_n\})}{r(u,\ \{x_n\})}:\ \{x_n\} \text{ weakly (not strongly) converges to } u\right\} \\ &= \inf\{D(\{x_n\}):\ \{x_n\} \subset S(X) \text{ and } x_n \rightharpoonup 0\} \\ &= \inf\{D(\{x_n\}:\ x_n \rightharpoonup 0 \text{ and } \lim_{n\to\infty}\|x_n\| = 1\}, \end{aligned}$$

where $S(X)$ denotes the unit sphere of X, i.e., $S(X) = \{x \in X:\ \|x\| = 1\}$.

3. Iterative fixed points of non-Lipschitzian mappings

Let C be a nonempty subset of a Banach space X. In this section, we recall that a mapping $T: C \to C$ is said to be *weakly asymptotically nonexpansive type* (simply, w.a.n.t.) on C (see [10]) if, for each $x \in C$ and each bounded subset D of C,

$$\limsup_{n\to\infty}(\sup\{[\|T^n x - T^n y\| - \|x - y\|]:\ y \in D\}) \leqq 0.$$

Here we give an example of a continuous mapping which is of w.a.n.t. and not Lipschitz.

Example. Let $X = \boldsymbol{R}$, $C = [0, \infty)$ and let $|k| < 1$. For each $n \in \boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$, we define

$$T^{n+1}x = \begin{cases} kT^n x\left|\sin\dfrac{1}{T^n x}\right|, & \text{if } x \in C - Z(T^n); \\ 0, & \text{if } x \in Z(T^n). \end{cases}$$

We denote by $Z(T^n)$ the set of all zeros of T^n and by T^0 the identy mapping I on C. We see that $Z(T)=\{1/m\pi : m\in N\}\cup\{0\}$, $Z(T^n)=Z(T^{n-1})\cup\{z\in C : T^{n-1}z=(1/m\pi)$ for some $m\in N\}$ for $n\in N$, and $\{Z(T^n): n\in \boldsymbol{N}\}$ is nondecreasing. Since $\{T^n x\}$ converges uniformly to zero on any closed bounded interval of C, $T\equiv T^1 : C\to C$ is really a continuous mapping of w.a.n.t. Indeed, for each $x\in C$ and each bounded subset D of C, there is a closed bounded interval I_D containing D. For each $n\in N$, the map $h_n(y)=|T^n x-T^n y|-|x-y|$ achieves its maximum on I_D, say $y_n\in I_D$. Then,

$$\begin{aligned}
&\limsup_{n\to\infty}\{|T^n x-T^n y|-|x-y| : y\in I_D\}\\
&=\limsup_{n\to\infty}(|T^n x-T^n y_n|-|x-y_n|)\\
&\leqq\lim_{n\to\infty}|T^n x-T^n y_n|-\liminf_{n\to\infty}|x-y_n|\\
&=-\liminf_{n\to\infty}|x-y_n|\leqq 0 .
\end{aligned}$$

Therefore $T : C\to C$ is of w.a.n.t. However, it is obviously not Lipschitz.

Recall that $T : C\to C$ is said to be *asymptotically regular* on C if

$$\lim_{n\to\infty}\|T^n x-T^{n+1}x\|=0 \qquad \forall x\in C .$$

DEFINITION 3.1. A mapping $T : C\to X$ is said to be *weakly demicompact* if whenever $\{x_n\}\subset C$ is a bounded sequence and $\{x_n-Tx_n\}$ is a convergent sequence, then there exists a subseqence $\{x_{n_k}\}$ of $\{x_n\}$ which is weakly convergent.

It is clear that every demicompact mappings are weakly demicompact, and also that if C is weakly compact, then $T : C\to X$ is weakly demicompact.

LEMMA 3.2. *Let C be a nonempty closed convex separable subset of a Banach space X and let $T : C\to C$ be a mapping of w.a.n.t. Let $x_0\in C$ such that $\{T^n x_0\}$ is bounded. If T is weakly demicompact and asymptotically regular on C, then there exists a subsequence $\{n_j\}$ of positive integers such that*

$$\{T^{n_j}x\} \quad \textit{converges weakly for every } x\in C .$$

Proof. Since T is a mapping of w.a.n.t., it is obvious that for each $x\in C$, $\{T^n x\}$ is bounded. For each $x\in C$, we set $x_n=T^n x$ for $n\in \boldsymbol{N}$. Since T is asymptotically regular on C, by the weak demicompactness of T, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly.

Let E be a countable dense subset of C. By using a usual diagonal method, we can choose a subsequence $\{n_j\}$ of $\boldsymbol{N}$ such that $\{T^{n_j}z\}$ converges weakly to $u_z\in C$ for every $z\in E$. Given $\varepsilon>0$, $x\in C$ and $x^*(\neq 0)\in X^*$, where X^* is a dual space of X, there is a $z\in E$ such that $\|x-z\|<\varepsilon/3\|x^*\|$. Since T is a mapping of w.a.n.t., it follows that, for each x, $z\in C$ and for any subsequence $\{n_i\}$ of $\boldsymbol{N}$,

$$\limsup_{i\to\infty}\|T^{n_i}x-T^{n_i}z\|\leqq\|x-z\|,$$

in particular, there exists $j_0'\in\boldsymbol{N}$ such that

$$\sup_{j\geqq j_0'}\|T^{n_j}x-T^{n_j}z\|\leqq\|x-z\|+\frac{\varepsilon}{3\|x^*\|}<\frac{2\varepsilon}{3\|x^*\|}.$$

On the other hand, since $\{T^{n_j}z\}$ converges weakly to $u_z\in C$, there also exists $j_0''\in\boldsymbol{N}$ such that

$$\|x^*(T^{n_j}z-u_z)\|<\frac{\varepsilon}{3}\qquad\text{for}\quad j\geqq j_0'',$$

By taking $j_0=\max\{j_0', j_0''\}$, we obtain that, for $j\geqq j_0$,

$$\|x^*(T^{n_j}x-u_z)\|\leqq\|x^*\|\|T^{n_j}x-T^{n_j}z\|+\|x^*(T^{n_j}z-u_z)\|<\varepsilon.$$

which completes the proof.

If C is a closed convex subset of a Banach space X and if $T: C\to C$ is a mapping, we can easily construct a separable subspace X_∞ of X and a closed convex subset C_∞ of X_∞ which is T-invariant. This shows that in many cases, i.e., when the other assumptions on C are inherited by C_∞, it suffices to formulate fixed point problems in a separable setting (see [7; pp. 35-36]).

By using an iterative method, we show a fixed point theorem of a continuous self-mapping of w.a.n.t. on a Banach space X with weak uniform normal structure, i.e., $WCS(X)>1$. We employ the method of the proof of [2].

THEOREM 3.3. *Suppose X is a Banach space such that $WCS(X)>1$, C is a nonempty closed convex subset of X, and a continuous mapping $T: C\to C$ of w.a.n.t. is weakly demicompact. Suppose in addition that T is asymptotically regular on C and $\{T^nx_0\}$ is bounded for some $x_0\in C$. Then T has an iterative fixed point in C.*

Proof. By above argument, we may assume that C is separable. By Lemma 3.2, we can choose a subsequence $\{n_j\}$ of positive integers such that

$$\{T^{n_j}x\}\qquad\text{converges weakly for every } x\in C.$$

Now we can construct a sequence $\{x_n\}$ in C in the following way:

$$x_0\in C\quad\text{arbitrary}$$

$$x_m=w-\lim_{j\to\infty}T^{n_j}x_{m-1},\qquad\forall m\geqq1.$$

Note that the asymptotic regularity of T on C ensures that

$$x_m=w-\lim_{j\to\infty}T^{n_j+p}x_{m-1},\qquad\forall p\geqq0.$$

We now show that $\{x_m\}$ converges strongly to a fixed point of T. To this end, for each integer $m\geqq 0$,

$$B_m:=\limsup_{j\to\infty}\|T^{n_j}x_m-x_{m+1}\|.$$

Then, by Theorem 2.3, we have

$$B_m\leqq\frac{1}{WCS(X)}D(\{T^{n_j}x_m\}). \tag{3.1}$$

Since T is a mapping of w.a.n.t., for each fixed $m\in\boldsymbol{N}$ and bounded subset $D_m=\{T^n x_m:\ n\in\boldsymbol{N}\}$ of C

$$\limsup_{j\to\infty}\{\sup[\|T^{n_j}x_m-T^{n_j}z\|-\|x_m-z\|]:\ z\in D_m\}\leqq 0, \tag{3.2}$$

and by asymptotic regularity of T on C, it follows that, for each $j\in\boldsymbol{N}$ and $p\geqq 0$,

$$\limsup_{i\to\infty}\|T^{n_i+p}u-T^{n_j}v\|=\limsup_{i\to\infty}\|T^{n_i}u-T^{n_j}v\|\qquad \forall u,\ v\in C. \tag{3.3}$$

Replacing p and u, v in (3.3) by n_j and x_m respectively, it follows that

$$\begin{aligned}D(\{T^{n_j}x_m\})&=\limsup_{j\to\infty}(\limsup_{i\to\infty}\|T^{n_i}x_m-T^{n_j}x_m\|)\\&=\limsup_{j\to\infty}(\limsup_{i\to\infty}\|T^{n_i+n_j}x_m-T^{n_i}x_m\|).\end{aligned}$$

Noting also that, for each $j\in\boldsymbol{N}$,

$$\begin{aligned}&\limsup_{i\to\infty}\|T^{n_i+n_j}x_m-T^{n_j}x_m\|\\&\leqq\sup_{i\in N}[\|T^{n_j}x_m-T^{n_j}(T^{n_i}x_m)\|-\|x_m-T^{n_i}x_m\|]\\&\quad+\limsup_{i\to\infty}\|x_m-T^{n_i}x_m\|\\&\leqq\sup_{z\in D}[\|T^{n_j}x_m-T^{n_j}z\|-\|x_m-z\|]\\&\quad+\limsup_{i\to\infty}\|x_m-T^{n_i}x_m\|,\end{aligned}$$

it follows from (3.2) that

$$D(\{T^{n_j}x_m\})\leqq\limsup_{i\to\infty}\|x_m-T^{n_i}x_m\|.$$

On the other hand, by the $w-l.s.c.$ of the norm of X and with the same method as before, we easily obtain

$$\begin{aligned}\limsup_{i\to\infty}\|x_m-T^{n_i}x_m\|&\leqq\limsup_{i\to\infty}(\liminf_{j\to\infty}\|T^{n_i}x_m-T^{n_j}x_{m-1}\|)\\&\leqq\limsup_{i\to\infty}(\limsup_{j\to\infty}\|T^{n_i}x_m-T^{n_j+n_i}x_{m-1}\|)\\&\leqq\limsup_{j\to\infty}\|x_m-T^{n_j}x_{m-1}\|=B_{m-1},\end{aligned} \tag{3.4}$$

which immediately gives that $D(\{T^{n_j}x_m\})\leqq B_{m-1}$. Hence, by (3.1),

$$B_m \leqq \frac{1}{WCS(X)} B_{m-1},$$

and since $WCS(X)>1$, we have $\lim_{m\to\infty} B_m=0$. Now using the $w-l.s.c.$ of the norm of X and (3.4) again, we deduce that

$$\begin{aligned}\|x_m - x_{m+1}\| &\leqq \limsup_{i\to\infty}\|x_m - T^{n_i}x_m\| + \limsup_{i\to\infty}\|T^{n_i}x_m - x_{m+1}\| \\ &\leqq B_{m-1}+B_m,\end{aligned}$$

which implies that $\{x_m\}$ is Cauchy. Let $v:=\lim_{m\to\infty} x_m$. Then, for each $j\in\boldsymbol{N}$,

$$\|v-T^{n_j}v\| \leqq \|v-x_{m+1}\| + \|x_{m+1}-T^{n_j}x_m\| + \|T^{n_j}x_m - T^{n_j}v\|.$$

Since T is of w.a.n.t., by taking the $\limsup_{j\to\infty}$ in both sides we obtain that

$$\limsup_{j\to\infty}\|v-T^{n_j}v\| \leqq \|v-x_{m+1}\| + B_m + \|x_m - v\| \to 0$$

as $m\to\infty$. Hence $T^{n_j}v\to v$ and $Tv=v$ by the continuity and asymptotic regularity of T at v. This completes the proof.

COROLLARY 3.4. *Suppose X is a Banach space such that $WCS(X)>1$, C is a nonempty weakly compact convex subset of X, and $T: C\to C$ is a continuous mapping of asymptotically nonexpansive type. Suppose in addition that T is asymptotically regular on C. Then T has an iterative fixed point in C.*

Finally, as a direct consequence of Corollary 3.4, we give an iterative fixed point for a nonexpansive mappng on a Banach space. Let $T: C\to C$ be a nonexpansive mapping. For a fixed $\lambda\in(0, 1)$, we set

$$S_\lambda := \lambda I + (1-\lambda)T,$$

where I is the identity operator of X. Then it is obvious that $S_\lambda: C\to C$ is also nonexpansive with the same fixed point set of T. Moreover, it is well-known (see [6]) that S_λ is asymptotically regular on C. Therefore, we obtain the following result due to T. D. Benavides, G. L. Acedo and H. K. Xu [2].

COROLLARY 3.5. *Suppose X is a Banach space such that $WCS(X)>1$, C is a nonempty weakly compact convex subset of X, and $T: C\to C$ is a nonexpansive mapping. Then T has an iterative fixed point in C.*

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