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ON ALMOST COMPLEX SURFACES OF THE NEARLY KAEHLER 6-SPHERE II

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Abstract

Let M be a 2-dimensional almost complex submanifold with Gauss curvature K in the nearly Kaehler unit 6-sphere $S^6(1)$· Then, in case K is constant, either $K=1$, $K=1/6$ or $K=0$ [S]. In [D-0-V-V], we proved that for compact M, if $1/6 \leqq K \leqq 1$, then either $K=1$ or $K=1/6$. In the present paper we prove that for compact M, if $0 \leqq K \leqq 1/6$, then either $K=0$ or $K=1/6$.

1. Introduction.

On a 6-dimensional unit sphere $S^6(1)$, a *nearly Kaehler structure* J can be constructed in a natural way, making use of the *Cayley number system* [C]. We recall this construction in Section 3. In this paper we study (connected) *almost complex* (2-dimensional) *surfaces* M *of* $S^6(1)$. The basic formulas for such surfaces are given in Section 4. Let K denote the *Gaussian curvature* of M. In [S], Sekigawa proved that, if K is constant, then $K=1$, $K=1/6$ or $K=0$. In [D-O-V-V] we proved that, if M is compact and $1/6 \leqq K \leqq 1$, then either $K=1/6$ or $K=1$ (this result follows also from the papers [O] and [D], and the fact that an almost complex surface cannot lie in a totally geodesic $S^4(1) \subset S^6(1)$). In Section 5 we prove the following result, which solves a problem proposed in [D-O-V-V].

THEOREM. *Let* M *be a compact almost complex surface in the nearly Kaehler* $S^6(1)$. *If the Gaussian curvature* K *of* M *satisfies the inequality* $0 \leqq K \leqq 1/6$, *then either* $K=0$ *or* $K=1/6$.

Examples of almost complex surfaces of $S^6(1)$ with $K=0$ or $K=1/6$ are given in [S]. The proof of this Theorem essentially uses some integral formulas of Ros, which are stated in Section 2.

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2. Integral formulas.

Let M be a compact Riemannian manifold, UM its unit tangent bundle, and UM_p the fiber of UM over a point p of M. Let dp, du and du_p be respectively the canonical measures on M, UM and UM_p. For any continuous function $f: UM \to \boldsymbol{R}$, one has

$$\int_{UM} f\, du = \int_M \left(\int_{UM_p} f\, du_p \right) dp .$$

Let T be any k-covariant tensor field on M and let ∇ be the Levi Civita connection of M. Then the *integral formulas of Ros* [R] state that

$$\int_{UM} (\nabla T)(u, u, u, \cdots, u)\, du = 0 \tag{2.1}$$

and

$$\int_{UM} \sum_{i=1}^{n} (\nabla T)(e_i, e_i, u, \cdots, u)\, du = 0 , \tag{2.2}$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of TM, the tangent bundle over M.

3. The nearly Kaehler $S^6(1)$.

Let $e_0, e_1, \cdots, e_7$ be the standard basis of $\boldsymbol{R}^8$. Then each point α of $\boldsymbol{R}^8$ can be written in a unique way as

$$\alpha = A e_0 + x ,$$

where $A \in \boldsymbol{R}$ and x is a linear combination of $e_1, \cdots, e_7$. α can be viewed as a Cayley number, and is called purely imaginary when $A=0$. For any pair of purely imaginary x and y, we consider the multiplication $\cdot$ given by

$$x \cdot y = \langle x, y \rangle e_0 + x \times y ,$$

where $\langle\, ,\, \rangle$ is the standard scalar product on $\boldsymbol{R}^8$ and $x \times y$ is defined by the following multiplication table for $e_j \times e_k$,

j/k	1	2	3	4	5	6	7
1	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
2	$-e_3$	0	e_1	e_6	$-e_7$	$-e_4$	e_5
3	e_2	$-e_1$	0	$-e_7$	$-e_6$	e_5	e_4
4	$-e_5$	$-e_6$	e_7	0	e_1	e_2	$-e_3$
5	e_4	e_7	e_6	$-e_1$	0	$-e_3$	$-e_2$
6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	0.

For two Cayley numbers $\alpha=Ae_0+x$ and $\beta=Be_0+y$, the Cayley multiplication $\cdot$, which makes $\boldsymbol{R}^8$ the Cayley algebra $\mathcal{C}$, is defined by

$$\alpha\cdot\beta=ABe_0+Ay+Bx+x\cdot y\,.$$

We recall that the multiplication $\cdot$ of $\mathcal{C}$ is neither commutative nor associative. The set $\mathcal{C}_+$ of all purely imaginary Cayley numbers clearly can be viewed as a 7-dimensional linear subspace $\boldsymbol{R}^7$ of $\boldsymbol{R}^8$. In $\mathcal{C}_+$ we consider the unit hypersphere which is centered at the origin:

$$S^6(1)=\{x\in\mathcal{C}_+ \mid \langle x,\, x\rangle=1\}\,.$$

Then the tangent space T_xS^6 of $S^6(1)$ at a point x may be identified with the affine subspace of $\mathcal{C}_+$ which is orthogonal to x.

On $S^6(1)$ we now define a (1, 1)-tensor field J by putting

$$J_xU=x\times U\,,$$

where $x\in S^6(1)$ and $U\in T_xS^6$. This tensor field is well-defined (i.e., $J_xU\in T_xS^6$) and determines an almost complex structure on $S^6(1)$, i.e.

$$J^2=-Id\,,$$

where Id is the identity transformation ([F]). The compact simple Lie group G_2 is the group of automorphisms of $\mathcal{C}$ and acts transitively on $S^6(1)$ and preserves both J and the standard metric on $S^6(1)$ ([F-I]).

Further, let G be the (2, 1)-tensor field on $S^6(1)$ defined by

$$G(X,\,Y)=(\tilde{\nabla}_XJ)Y\,, \tag{3.1}$$

where $X,\,Y\in\mathfrak{X}(S^6)$ and where $\tilde{\nabla}$ is the Levi Civita connection on $S^6(1)$. This tensor field has the following properties:

$$G(X,\,X)=0\,, \tag{3.2}$$

$$G(X,\,Y)+G(Y,\,X)=0\,, \tag{3.3}$$

$$G(X,\,JY)+JG(X,\,Y)=0\,, \tag{3.4}$$

$$(\tilde{\nabla}_XG)(Y,\,Z)=\langle Y,\,JZ\rangle X+\langle X,\,Z\rangle JY-\langle X,\,Y\rangle JZ\,, \tag{3.5}$$

$$\langle G(X,\,Y),\,Z\rangle+\langle G(X,\,Z),\,Y\rangle=0\,, \tag{3.6}$$

$$\begin{aligned}\langle G(X,\,Y),\,G(Z,\,W)\rangle=&\langle X,\,Z\rangle\langle Y,\,W\rangle-\langle X,\,W\rangle\langle Z,\,Y\rangle\\ &+\langle JX,\,Z\rangle\langle Y,\,JW\rangle-\langle JX,\,W\rangle\langle Y,\,JZ\rangle\,,\end{aligned} \tag{3.7}$$

where $X,\,Y,\,Z,\,W\in\mathfrak{X}(S^6)$ ([S], [G]). We recall that (3.2) means that the structure J is *nearly Kaehler*, i.e. that $\forall X\in\mathfrak{X}(S^6)$: $(\tilde{\nabla}_XJ)X=0$.

4. Almost complex surfaces of $S^6(1)$.

A submanifold M of the nearly Kaehler $S^6(1)$ is called *almost complex* if $J(T_pM)\subseteq T_pM$ for every $p\in M$, where T_pM denotes the tangent space to M at p. On an almost complex submanifold the almost complex structure of $S^6(1)$ naturally induces an almost Kaehler structure, which we also denote by J. Therefore any almost complex submanifold must be even-dimensional. Gray [G2] showed that there are no 4-dimensional almost complex submanifolds in $S^6(1)$.

In the following, M always denotes a (2-dimensional) almost complex surface of $S^6(1)$. It is clear that the almost Kaehler structure J on M actually determines a Kaehler structure with respect to the induced metric. The Levi Civita connection of M will be denoted by ∇.

The *formulas of Gauss and Weingarten* for M in $S^6(1)$ are respectively given by

$$\tilde{\nabla}_X Y=\nabla_X Y+h(X, Y)$$

and

$$\tilde{\nabla}_X \xi=-A_\xi X+D_X\xi,$$

where ξ is a local normal vector field on M in $S^6(1)$ and $X, Y\in\mathfrak{X}(M)$. h is called the *second fundamental form*, A_ξ a *second fundamental tensor* and D the *normal connection* of M in $S^6(1)$. h and A_ξ are related by

$$\langle h(X, Y), \xi\rangle=\langle A_\xi X, Y\rangle,$$

whereby $\langle\,,\rangle$ denotes the metric on $S^6(1)$ as well as the induced metric on M. From these formulas, it follows easily that

$$h(X, JY)=Jh(X, Y), \tag{4.1}$$

$$A_{J\xi}=JA_\xi=-A_\xi J \tag{4.2}$$

and that

$$D_X(J\xi)=G(X, \xi)+JD_X\xi. \tag{4.3}$$

We recall that M is *minimal*, as follows from (4.1). The *equation of Gauss* is given by

$$K=1-2\|h(v, v)\|^2,$$

where K denotes the Gaussian curvature of M, and v is a unit vector tangent to M.

The *equation of Codazzi* states that

$$(\nabla h)(X, Y, Z)=(\nabla h)(Y, X, Z),$$

where $X, Y, Z\in\mathfrak{X}(M)$ and ∇h is defined by

$$(\nabla h)(X, Y, Z)=D_X h(Y, Z)-h(\nabla_X Y, Z)-h(Y, \nabla_X Z).$$

The *equation of Ricci* is given by

$$R^D(v, Jv)\xi=2(\langle h(v, Jv), \xi\rangle h(v, v)-\langle h(v, v), \xi\rangle h(v, Jv)),$$

where v is a unit vector tangent to M, ξ is a normal vector field, and R^D is the *normal curvature tensor* corresponding to the normal connection D, i.e. $R^D(X, Y)=[D_X, D_Y]-D_{[X,Y]}$.

The second and third derivative $\nabla^2 h$ and $\nabla^3 h$ of h are defined by

$$(\nabla^2 h)(X, Y, Z, W)=D_X(\nabla h)(Y, Z, W)-(\nabla h)(\nabla_X Y, Z, W)$$
$$-(\nabla h)(Y, \nabla_X Z, W)-(\nabla h)(Y, Z, \nabla_X W),$$

and

$$(\nabla^3 h)(X, Y, Z, V, W)=D_X(\nabla^2 h)(Y, Z, V, W)-(\nabla^2 h)(\nabla_X Y, Z, V, W)$$
$$-(\nabla^2 h)(Y, \nabla_X Z, V, W)-(\nabla^2 h)(Y, Z, \nabla_X V, W)$$
$$-(\nabla^2 h)(Y, Z, V, \nabla_X W).$$

Then we have the following Ricci identities:

$$(\nabla^2 h)(X, Y, Z, W)=(\nabla^2 h)(Y, X, Z, W)+R^D(X, Y)h(Z, W)$$
$$-h(R(X, Y)Z, W)-h(Z, R(X, Y)W)$$

and

$$(\nabla^3 h)(X, Y, Z, V, W)=(\nabla^3 h)(Y, X, Z, V, W)+R^D(X, Y)(\nabla h)(Z, V, W)$$
$$-(\nabla h)(R(X, Y)Z, V, W)-(\nabla h)(Z, R(X, Y)V, W)$$
$$-(\nabla h)(Z, V, R(X, Y)W).$$

5. Proof of the Theorem.

In the following v always denotes a unit tangent vector at some point p of M, and also a unit local vector field around p such that $\nabla_v v=0$ at p.

LEMMA 1. (a) $(\nabla h)(v, Jv, v)=J(\nabla h)(v, v, v)+G(v, h(v, v))$.

(b) $(\nabla h)(Jv, Jv, Jv)=-(\nabla h)(Jv, v, v)$.

LEMMA 2. (a) $(\nabla^2 h)(v, Jv, Jv, v)=-(\nabla^2 h)(v, v, v, v)$

(b) $(\nabla^2 h)(Jv, v, Jv, v)=-(\nabla^2 h)(v, v, v, v)+(3K-1)h(v, v)$.

(c) $(\nabla^2 h)(Jv, v, v, v)=(\nabla^2 h)(v, Jv, v, v)+(1-3K)h(Jv, v)$

(d) $(\nabla^2 h)(v, Jv, v, v)=J(\nabla^2 h)(v, v, v, v)+2G(v, (\nabla h)(v, v, v))$
$-Jh(v, v)$.

Proofs.
1(a). We know that

$$\begin{aligned}(\nabla h)(v, Jv, v) &= D_v h(Jv, v)\\ &= D_v Jh(v, v)\\ &= G(v, h(v, v)) + JD_v h(v, v),\end{aligned}$$

where we have used (4.1) and (4.3).

1(b) and 2(a) follow straightforwardly from the minimality of M.
2(b) and 2(c). From the equations of Gauss and Ricci and the Ricci identities it follows that

$$\begin{aligned}&(\nabla^2 h)(Jv, v, Jv, v)\\ &= (\nabla^2 h)(v, Jv, Jv, v) + R^D(Jv, v)h(Jv, v) - h(R(Jv, v)Jv, v) - h(Jv, R(Jv, v)v)\\ &= (\nabla^2 h)(v, Jv, Jv, v) - (1-K)h(v, v) + 2Kh(v, v)\\ &= -(\nabla^2 h)(v, v, v, v) + (3K-1)h(v, v)\end{aligned}$$

and

$$\begin{aligned}(\nabla^2 h)(Jv, v, v, v) &= (\nabla^2 h)(v, Jv, v, v) + R^D(Jv, v)h(v, v) - 2h(R(Jv, v)v, v)\\ &= (\nabla^2 h)(v, Jv, v, v) + (1-3K)h(Jv, v).\end{aligned}$$

2(d). From 1(a), (4.3), (3.1) and (3.5) it follows that

$$\begin{aligned}(\nabla^2 h)(v, Jv, v, v) &= D_v(\nabla h)(Jv, v, v)\\ &= D_v(J(\nabla h)(v, v, v) + G(v, h(v, v)))\\ &= G(v, (\nabla h)(v, v, v)) + J(\nabla^2 h)(v, v, v, v)\\ &\quad + (\tilde{\nabla}_v G)(v, h(v, v)) + A_{G(v, h(v, v))}v + G(v, (\nabla h)(v, v, v))\\ &= J(\nabla^2 h)(v, v, v, v) + 2G(v, (\nabla h)(v, v, v)) - Jh(v, v),\end{aligned}$$

since $A_{G(v, h(v, v))} = 0$ because $G(v, h(v, v))$ is perpendicular to $\mathrm{im}(h)$ (which is a consequence of (3.2), (3.4) and (3.6)). ■

LEMMA 3. $x \cdot K = -4\langle(\nabla h)(x, v, v), h(v, v)\rangle$.

Proof. This follows directly from the equation of Gauss. ■

Define covariant tensor fields T_1, T_2 and T_3 by

$$T_1(X_1, X_2, X_3, X_4) = \langle h(X_1, X_2), h(X_3, X_4)\rangle,$$

$$T_2(X_1, X_2, \cdots, X_7, X_8) = \langle h(X_1, X_2), h(X_3, X_4)\rangle\langle h(X_5, X_6), h(X_7, X_8)\rangle$$

and

$$T_3(X_1, X_2, \cdots, X_6, X_7) = \langle(\nabla^2 h)(X_1, X_2, X_3, X_4), (\nabla h)(X_5, X_6, X_7)\rangle.$$

Since the measure of UM is invariant with respect to J, the first integral formula of Section 2 implies that

$$\int_{UM}(\nabla^2 T_1)(v, v, v, v, v, v)+(\nabla^2 T_1)(Jv, Jv, Jv, Jv, Jv, Jv)=0 \tag{5.1}$$

and

$$\int_{UM}(\nabla^2 T_2)(v, v, \cdots, v)+(\nabla^2 T_2)(Jv, Jv, Jv, \cdots, Jv)=0 \tag{5.2}$$

and the second integral formula implies that

$$\int_{UM}(\nabla T_3)(v, v, v, \cdots, v)+(\nabla T_3)(Jv, Jv, v, \cdots, v)=0. \tag{5.3}$$

By Lemma 1 and 2 we know that

$$\begin{aligned}
&(\nabla^2 T_1)(v, v, \cdots, v)+(\nabla^2 T_1)(Jv, Jv, \cdots, Jv)\\
&=2\langle(\nabla^2 h)(v, v, v, v)+(\nabla^2 h)(Jv, Jv, v, v),\ h(v, v)\rangle\\
&\quad+2\|(\nabla h)(v, v, v)\|^2+2\|(\nabla h)(Jv, v, v)\|^2\\
&=2(3K-1)\|h(v, v)\|^2+\|J(\nabla h)(v, v, v)+(\nabla h)(Jv, v, v)\|^2\\
&\quad+\|(\nabla h)(Jv, v, v)-J(\nabla h)(v, v, v)\|^2\\
&=(3K-1)(1-K)+\|2J(\nabla h)(v, v, v)+G(v,\ h(v, v))\|^2+\|G(v,\ h(v, v))\|^2\\
&=(3K-1)(1-K)+s(v)+\|h(v, v)\|^2\\
&=(3K-1/2)(1-K)+s(v),
\end{aligned}$$

where we have used the Gauss equation, the parallelogram law, and (3.7), and where we have put

$$s(v)=\|2J(\nabla h)(v, v, v)+G(v,\ h(v, v))\|^2. \tag{5.4}$$

Then (5.1) yields

$$\int_{UM}(1-K)(3K-1/2)+\int_{UM}s(v)=0. \tag{5.5}$$

Note that, if $1/6\leqq K\leqq 1$, (5.5) implies $K=1$ or $K=1/6$.

Again by Lemma 1 and 2, we obtain similarly that

$$\begin{aligned}
&(\nabla^2 T_2)(v, v, \cdots, v)+(\nabla^2 T_2)(Jv, Jv, \cdots, Jv)\\
&=8(\langle(\nabla h)(v, v, v),\ h(v, v)\rangle^2+\langle(\nabla h)(Jv, v, v),\ h(v, v)\rangle^2)\\
&\quad+4\|h(v, v)\|^2(\|\nabla h(v, v, v)\|^2+\|\nabla h(Jv, v, v)\|^2)\\
&\quad+4\|h(v, v)\|^2(\langle(\nabla^2 h)(v, v, v, v)+(\nabla^2 h)(Jv, v, Jv, v),\ h(v, v)\rangle)
\end{aligned}$$

$$=8t(v)+(1-K)\Big(s(v)+\frac{1}{2}(1-K)\Big)+(3K-1)(1-K)^2$$
$$=8t(v)+(1-K)s(v)+(1-K)^2(3K-1/2),$$

where we have put

$$t(v)=\langle(\nabla h)(v, v, v), h(v, v)\rangle^2+\langle(\nabla h)(Jv, v, v), h(v, v)\rangle^2.$$

Then (5.2) yields

$$\int_{UM}(1-K)^2(3K-1/2)+\int_{UM}(1-K)s(v)+8\int_{UM}t(v)=0. \tag{5.6}$$

Subtracting (5.5) and (5.6) implies

$$3\int_{UM}(1-K)(1/6-K)K-\int_{UM}Ks(v)+8\int_{UM}t(v)=0. \tag{5.7}$$

Finally we know that

$$\begin{aligned}(5.8)\qquad &(\nabla T_3)(v, v, v, \cdots, v)+(\nabla T_3)(Jv, Jv, v, \cdots, v)\\ &=\langle(\nabla^3 h)(v, v, v, v, v)+(\nabla^3 h)(Jv, Jv, v, v, v), (\nabla h)(v, v, v)\rangle\\ &\quad+\|(\nabla^2 h)(v, v, v, v)\|^2+\|(\nabla^2 h)(Jv, v, v, v)\|^2.\\ &=\langle(\nabla^3 h)(v, v, v, v, v)+(\nabla^3 h)(Jv, Jv, v, v, v), (\nabla h)(v, v, v)\rangle\\ &\quad+\frac{1}{2}\|(\nabla^2 h)(Jv, v, v, v)+J(\nabla^2 h)(v, v, v, v)\|^2\\ &\quad+\frac{1}{2}\|(\nabla^2 h)(Jv, v, v, v)-J(\nabla^2 h(v, v, v, v)\|^2.\end{aligned}$$

Next, we need some more lemmata.

LEMMA 4. $(\nabla^3 h)(Jv, Jv, v, v, v)+(\nabla^3 h)(v, v, v, v, v)$
$=14\langle(\nabla h)(Jv, v, v), h(v, v)\rangle h(Jv, v)$
$-14\langle(\nabla h)(v, v, v), h(v, v)\rangle h(v, v)-(2-9K)(\nabla h)(v, v, v).$

Proof.
By Lemma 1, 2 and 3, we know that

$$\begin{aligned}(5.9)\qquad (\nabla^3 h)(Jv, Jv, v, v, v)&=D_{Jv}(\nabla^2 h)(Jv, v, v, v)\\ &=D_{Jv}((\nabla^2 h)(v, Jv, v, v)+(1-3K)h(Jv, v))\\ &=(\nabla^3 h)(Jv, v, Jv, v, v)-(1-3K)(\nabla h)(v, v, v)\\ &\quad+12\langle(\nabla h)(Jv, v, v), h(v, v)\rangle h(Jv, v).\end{aligned}$$

Using the Ricci identities, the equation of Ricci, and Lemma 1, 2 and 3 we also obtain that

$$
\begin{aligned}
(3.10)\qquad & (\nabla^3 h)(Jv, v, Jv, v, v) \\
&= (\nabla^3 h)(v, Jv, Jv, v, v) + R^D(Jv, v)(\nabla h)(Jv, v, v) \\
&\quad -(\nabla h)(R(Jv, v)Jv, v, v) - 2(\nabla h)(Jv, R(Jv, v)v, v) \\
&= D_v(\nabla^2 h)(Jv, Jv, v, v) + 2\langle h(v, v), (\nabla h)(Jv, v, v)\rangle h(Jv, v) \\
&\quad -2\langle h(Jv, v), (\nabla h)(Jv, v, v)\rangle h(v, v) + 3K(\nabla h)(v, v, v) \\
&= -(\nabla^3 h)(v, v, v, v, v) - 12\langle (\nabla h)(v, v, v), h(v, v)\rangle h(v, v) \\
&\quad +(3K-1)(\nabla h)(v, v, v) + 2\langle h(v, v), (\nabla h)(Jv, v, v)\rangle h(Jv, v) \\
&\quad -2\langle h(v, v), (\nabla h)(v, v, v)\rangle h(v, v) + 3K(\nabla h)(v, v, v) \\
&= -(\nabla^3 h)(v, v, v, v, v) - 14\langle h(v, v), (\nabla h)(v, v, v)\rangle h(v, v) \\
&\quad +2\langle h(v, v), (\nabla h)(Jv, v, v)\rangle h(Jv, v) + (6K-1)(\nabla h)(v, v, v).
\end{aligned}
$$

The combination of (5.9) and (5.10) yields the proof of this lemma. ■

LEMMA 5. $\|(\nabla^2 h)(Jv, v, v, v) - J(\nabla^2 h)(v, v, v, v)\|^2$

$$= 4\|(\nabla h)(v, v, v)\|^2 + \frac{9}{2}K^2(1-K) + 12K\langle G(v, h(v, v)), J(\nabla h)(v, v, v)\rangle.$$

Proof. $\|(\nabla^2 h)(Jv, v, v, v) - J(\nabla^2 h)(v, v, v, v)\|^2$

$$
\begin{aligned}
&= \|2G(v, (\nabla h)(v, v, v)) - 3Kh(Jv, v)\|^2 \\
&= 4\|(\nabla h)(v, v, v)\|^2 + \frac{9}{2}K^2(1-K) - 12K\langle G(v, (\nabla h)(v, v, v)), h(Jv, v)\rangle \\
&= 4\|(\nabla h)(v, v, v)\|^2 + \frac{9}{2}K^2(1-K) + 12K\langle G(v, h(v, v)), J(\nabla h)(v, v, v)\rangle,
\end{aligned}
$$

where we have used the equation of Gauss, Lemma 2, (3.4), (3.6), (3.7) and (4.1).

LEMMA 6. (a) $4\int_{UM} K\|(\nabla h)(v, v, v)\|^2 = \int_{UM} Ks(v) + \frac{1}{2}\int_{UM} K(1-K).$

(b) $4\int_{UM} K\langle G(v, h(v, v)), J(\nabla h)(v, v, v)\rangle + \int_{UM} K(1-K) = 0.$

Proof. Since J preserves the measure of UM, we know that

$$\int_{UM} K\|(\nabla h)(v, v, v)\|^2 = \int_{UM} K\|(\nabla h)(Jv, Jv, Jv)\|^2.$$

Consequently,

$$
\begin{aligned}
4\int_{UM} K\|(\nabla h)(v, v, v)\|^2 &= \int_{UM} K(2\|(\nabla h)(v, v, v)\|^2 + 2\|(\nabla h)(Jv, v, v)\|^2) \\
&= \int_{UM} Ks(v) + \frac{1}{2}\int_{UM} K(1-K),
\end{aligned}
$$

which proves (a).

On the other hand, we have that

$$\begin{aligned}
&\int_{UM} K\|(\nabla h)(v, v, v)\|^2 \\
&=\int_{UM} K\|(\nabla h)(Jv, v, v)\|^2 \\
&=\int_{UM} K\|J(\nabla h)(v, v, v)+G(v, h(v, v))\|^2 \\
&=\int_{UM} K\|(\nabla h)(v, v, v)\|^2+2\int_{UM} K\langle J(\nabla h)(v, v, v), G(v, h(v, v))\rangle \\
&\quad +\frac{1}{2}\int_{UM}(1-K)K,
\end{aligned}$$

which proves (b). ■

Integrating (5.8) and using (5.3), Lemma 4, 5 and 6, we obtain that

$$\begin{aligned}
0=&-14\int_{UM}[\langle(\nabla h)(Jv, v, v), h(v, v)\rangle^2+\langle(\nabla h)(v, v, v), h(v, v)\rangle^2] \\
&-\int_{UM}(2-9K)\|(\nabla h)(v, v, v)\|^2 \\
&+\frac{1}{2}\int_{UM}\|(\nabla^2 h)(Jv, v, v, v)+J(\nabla^2 h)(v, v, v, v)\|^2 \\
&+2\int_{UM}\|(\nabla h)(v, v, v)\|^2+\frac{9}{4}\int_{UM} K^2(1-K)-\frac{3}{2}\int_{UM} K(1-K) \\
=&-14\int_{UM} t(v)+\frac{9}{4}\int_{UM} Ks(v)+\frac{9}{4}\int_{UM} K(1-K)\left(K+\frac{1}{2}-\frac{2}{3}\right)+\frac{1}{2}\int_{UM} r(v).
\end{aligned}$$

Thus we find that

$$0=-\frac{56}{9}\int_{UM} t(v)+\int_{UM} Ks(v)+\int_{UM} K(1-K)\left(K-\frac{1}{6}\right)+\frac{2}{9}\int_{UM} r(v), \tag{5.11}$$

where we have put

$$r(v)=\|(\nabla^2 h)(Jv, v, v, v)+J(\nabla^2 h)(v, v, v, v)\|^2.$$

Adding (5.11) and (5.7) implies

$$2\int_{UM} K(1-K)\left(\frac{1}{6}-K\right)+\frac{16}{9}\int_{UM} t(v)+\frac{2}{9}\int_{UM} r(v)=0. \tag{5.12}$$

If we suppose that $0\leqq K\leqq\frac{1}{6}$, then all terms on the left hand side in (5.12) are non negative, and consequently zero.

In particular we obtain that

$$\int_{UM} K(1-K)\left(\frac{1}{6}-K\right)=0.$$

Since $K(1-K)\left(\frac{1}{6}-K\right)$ is a positive function under the assumption $0\leqq K\leqq\frac{1}{6}$, it follows that either $K=0$ or $K=\frac{1}{6}$.

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