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Analytic semigroups for the subelliptic oblique derivative problem

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Dedicated to Professor Hikosaburo Komatsu on the occasion of his 80th birthday

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Abstract. This paper is devoted to a functional analytic approach to the *subelliptic* oblique derivative problem for second-order, elliptic differential operators with a complex parameter λ. We prove an existence and uniqueness theorem of the homogeneous oblique derivative problem in the framework of L^p Sobolev spaces when $|\lambda|$ tends to ∞. As an application of the main theorem, we prove generation theorems of *analytic semigroups* for this subelliptic oblique derivative problem in the L^p topology and in the topology of uniform convergence. Moreover, we solve the long-standing open problem of the *asymptotic eigenvalue distribution* for the subelliptic oblique derivative problem. In this paper we make use of Agmon's technique of treating a spectral parameter λ as a second-order elliptic differential operator of an *extra variable* on the unit circle and relating the old problem to a new one with the additional variable.

1. Motivation and formulation of the oblique derivative problem.

In physical geodesy, investigations of the Earth's gravity field based on surface gravity data are usually associated with a simultaneous determination of the figure of the Earth. The precise 3D positioning by the Global Navigation Satellite Systems (GNSS) has brought new possibilities in gravity field modelling. Terrestrial gravimetric measurements located by precise satellite positioning yield oblique derivative boundary conditions in the form of surface gravity disturbances. Now the shape of the Earth can be obtained by geometric satellite triangulation and satellite altimetry over the oceans. In this way, the (linearized) fixed gravimetric boundary value problem in physical geodesy is an *oblique derivative problem* for the Laplace equation in the Earth's *exterior*, where the physical surface of the Earth is assumed to be known (see [**23**], [**7**]).

In this paper we will deal with an *interior* oblique derivative problem in a bounded domain. It should be noticed that the analysis of harmonic functions in an exterior domain can be reduced to that of harmonic functions in a bounded domain by using the Kelvin transform, called the inverse radii transform (see [**5**, Chapter 4]).

Now let Ω be a bounded domain of Euclidian space $\mathbf{R}^n$, $n \geq 3$, with smooth boundary $\Gamma = \partial\Omega$; its closure $\overline{\Omega}$ is an n-dimensional, compact smooth manifold with boundary Γ.

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Let A be a second-order, *uniformly elliptic* differential operator with real coefficients on the closure $\overline{\Omega} = \Omega \cup \Gamma$ such that

$$A = \sum_{i=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} + c(x).$$

Here:

(1) $a^{ij} \in C^\infty(\overline{\Omega})$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{\Omega}$ and $1 \leq i, j \leq n$, and there exists a constant $c_0 > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2 \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbf{R}^n.$$

(2) $b^i \in C^\infty(\overline{\Omega})$ for all $1 \leq i \leq n$.

(3) $c \in C^\infty(\overline{\Omega})$.

Let $B\boldsymbol{\gamma}$ be an *oblique derivative boundary condition* such that

$$B\boldsymbol{\gamma} u = \frac{\partial u}{\partial \boldsymbol{\nu}} = a(x') \frac{\partial u}{\partial \boldsymbol{n}} + \alpha(x') \cdot u.$$

Here:

(4) $a \in C^\infty(\Gamma)$.

(5) $\alpha(x')$ is a real smooth tangential vector field on Γ. More precisely, in terms of a local coordinate system $(x_1, x_2, \ldots, x_{n-1})$ of Γ, the vector field $\alpha(x')$ has the local expression

$$\alpha(x') = \sum_{k=1}^{n-1} \alpha_k(x') \frac{\partial}{\partial x_k}.$$

(6) $\boldsymbol{\nu} = a(x')\,\boldsymbol{n} + \alpha(x')$ is a smooth, nowhere vanishing vector field on Γ where $\boldsymbol{n} = (n_1, n_2, \ldots, n_n)$ is the unit outward normal to Γ.

We consider the following *homogeneous* oblique derivative problem: Given a function $f(x)$ defined in Ω, find a function $u(x)$ in Ω such that

$$\begin{cases} (A - \lambda)u = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = a(x') \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha(x') \cdot u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where λ is a *complex parameter*.

We remark that the oblique derivative problem (1.1) is non-degenerate (or coercive) if and only if $a(x') \neq 0$ on Γ, that is, the vector field $\boldsymbol{\nu} = a(x')\,\boldsymbol{n} + \alpha(x')$ is nowhere tangent to Γ.

In the near future, we would like to apply our main results (Theorem 2.2 and Corollary 2.3 below) to provide numerical solutions of the linearized fixed gravimetric boundary value problem on the real Earth surface topography in the *degenerate* (non-coercive) case, generalizing Holota [**19**] and Čunderlík–Mikula–Mojzeš [**11**].

2. Statement of main results.

The purpose of this section is to formulate an existence and uniqueness theorem of the *subelliptic* oblique derivative problem (1.1) in the framework of L^2 Sobolev spaces when $|\lambda|$ tends to ∞. As an application of the main theorem, we state generation theorems of *analytic semigroups* for this subelliptic oblique derivative problem in the L^p topology and in the topology of uniform convergence.

2.1. Existence and uniqueness theorem for the subelliptic oblique derivative problem.

Our starting point is to state a necessary and sufficient condition in order that the *non-homogeneous* oblique derivative problem is *subelliptic* in the framework of L^2 Sobolev spaces, due to [**35**, Théorème 11]:

THEOREM 2.1. *The following two assertions* (A) *and* (B) *are equivalent*:

(A) *The hypothesis* (H) *is satisfied*:

(H) *The vector field $\alpha(x')$ is non-zero on the set $\Gamma_0 = \{x' \in \Gamma : a(x') = 0\}$ of tangency and, along the integral curve $x(t, x_0')$ of $\alpha(x')$ passing through $x_0' \in \Gamma_0$ at $t = 0$, the function: $t \mapsto a(x(t, x_0'))$ has zeros of* even *order $\leq 2k$ for some non-negative integer k.*

(B) *For every $\theta \in (-\pi, \pi)$, there exists a constant $R_2(\theta) > 0$ depending on θ such that, for all $\lambda = r^2 e^{i\theta}$ satisfying $|\lambda| = r^2 \geq R_2(\theta)$ the* non-homogeneous *oblique derivative problem*

$$\begin{cases} (A - \lambda)u = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = a(x')\dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha(x') \cdot u = \varphi & \text{on } \Gamma \end{cases}$$

has a unique solution u in $W^{2-\delta,2}(\Omega)$ for any $f \in L^2(\Omega)$ and any $\varphi \in B^{1/2,2}(\Gamma)$, where $2k/(2k+1) \leq \delta < 1$.

Moreover, we have the a priori *estimate*

$$\begin{aligned} &\|u\|^2_{W^{2-\delta,2}(\Omega)} + |\lambda|^{2-\delta} \|u\|^2_{L^2(\Omega)} \\ &\quad \leq C_2(\theta) \left(\|f\|^2_{L^2(\Omega)} + |\varphi|^2_{B^{1/2,2}(\Gamma)} + |\lambda|^{1/2} \, |\varphi|^2_{L^2(\Gamma)} \right), \end{aligned}$$

with a constant $C_2(\theta) > 0$ depending only on θ.

Here $W^{s,2}(\Omega)$ and $B^{s,2}(\Gamma)$ denote the L^2 Sobolev space on Ω and the L^2 Besov space on Γ, respectively.

Remark 2.1. The hypothesis (H) implies that the function $a(x')$ does not change sign on the boundary Γ. This is called *Case I* in Guan–Sawyer [**18**]. Moreover, it is easy to see that the hypothesis (H) is equivalent to saying that the vector field $\boldsymbol{\nu} = a(x')\,\boldsymbol{n} + \alpha(x')$ is *of finite type* on Γ defined in Smith [**31**] and Guan–Sawyer [**18**].

In the *homogeneous* boundary condition case, we can prove that every solution u of the subelliptic oblique derivative problem (1.1) has the elliptic gain of 2 derivatives from f in the framework of L^2 Sobolev spaces (see [**31**], [**18**]). More precisely, the first purpose of this paper is to prove the following theorem:

Theorem 2.2. *Assume that the hypothesis* (H) *is satisfied. Then we have the following assertion*:

(C) *For every $\theta \in (-\pi, \pi)$, there exists a constant $R(\theta) > 0$ depending on θ such that, for all $\lambda = r^2\, e^{i\theta}$ satisfying $|\lambda| = r^2 \geq R(\theta)$ the homogeneous oblique derivative problem* (1.1) *has a unique solution u in $W^{2,2}(\Omega)$ for any $f \in L^2(\Omega)$, and further that we have the* a priori *estimate*

$$\|u\|^2_{W^{2,2}(\Omega)} + |\lambda|^2\, \|u\|^2_{L^2(\Omega)} \leq C(\theta)\, \|f\|^2_{L^2(\Omega)}\,, \tag{2.1}$$

with a constant $C(\theta) > 0$ depending only on θ.

This rather surprising result (C) (elliptic estimates for a degenerate problem) works, since we are considering the homogeneous boundary condition.

We associate with the homogeneous oblique derivative problem (1.1) a densely defined, closed linear operator

$$\mathfrak{A}_2 : L^2(\Omega) \longrightarrow L^2(\Omega)$$

in the Hilbert space $L^2(\Omega)$ as follows:

(a) The domain $\mathcal{D}(\mathfrak{A}_2)$ of definition of $\mathfrak{A}_2$ is the space

$$\mathcal{D}(\mathfrak{A}_2) = \left\{u \in W^{2,2}(\Omega) : B\boldsymbol{\gamma} u = 0 \text{ on } \Gamma\right\}. \tag{2.2}$$

(b) $\mathfrak{A}_2 u = Au$ for every $u \in \mathcal{D}(\mathfrak{A}_2)$.

Here Au and $B\boldsymbol{\gamma} u$ are taken in the sense of distributions.

Then, by combining Agmon [**2**, Theorems 14.4 and 15.1] with Theorem 2.2 we can obtain the following spectral properties of the operator $\mathfrak{A}_2$ similar to the non-degenerate case (cf. [**34**, Theorem 2]):

Corollary 2.3. *Assume that the hypothesis* (H) *is satisfied. Then the operator $\mathfrak{A}_2$ enjoys the following five spectral properties*:

(i) *The spectrum of $\mathfrak{A}_2$ is discrete and the eigenvalues λ_j of $\mathfrak{A}_2$ have finite multiplicities.*

(ii) *All rays* $\arg\lambda = \theta$ *different from the negative axis are rays of minimal growth of the resolvent* $(\mathfrak{A}_2 - \lambda I)^{-1}$*. In particular, there are only a finite number of eigenvalues outside the angle:* $-\pi + \varepsilon < \theta < \pi - \varepsilon$*, for any* $\varepsilon > 0$.

(iii) *The negative axis is a direction of condensation of eigenvalues of* $\mathfrak{A}_2$.

(iv) *Let*

$$N(t) := \sum_{\operatorname{Re}\lambda_j \geq -t} 1$$

be the number of eigenvalues λ_j *such that* $\operatorname{Re}\lambda_j \geq -t$*, where each* λ_j *is repeated according to its multiplicity. Then the* asymptotic eigenvalue distribution *formula*

$$N(t) = \frac{1}{(2\pi)^n} \int_\Omega |A(x)|\, dx \cdot t^{n/2} + o(t^{n/2}) \quad \text{as } t \to +\infty$$

holds true. Here $|A(x)|$ *denotes the volume of the subset* $A(x) = \{\xi \in \mathbf{R}^n : \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j < 1\}$.

(v) *The generalized eigenfunctions are complete in the Hilbert space* $L^2(\Omega)$*; they are also complete in the domain* $\mathcal{D}(\mathfrak{A}_2)$ *in the* $W^{2,2}(\Omega)$*-norm.*

The detailed proof of Corollary 2.3 will be given in the forthcoming paper [**40**].

2.2. Generation of analytic semigroups for the subelliptic oblique derivative problem.

The second purpose of this paper is to study the subelliptic oblique derivative problem (1.1) from the point of view of analytic semigroup theory in functional analysis. The generation theorem for analytic semigroups is well established in the non-degenerate case in the L^p topology for $1 < p < \infty$ (cf. [**4**], [**14**], [**27**], [**33**]). We shall generalize this generation theorem for analytic semigroups to the *subelliptic* case (Theorem 2.4).

To do so, we associate with the homogeneous oblique derivative problem (1.1) a densely defined, closed linear operator

$$\mathfrak{A}_p : L^p(\Omega) \longrightarrow L^p(\Omega)$$

in the Banach space $L^p(\Omega)$ as follows:

(a) The domain $\mathcal{D}(\mathfrak{A}_p)$ of definition of $\mathfrak{A}_p$ is the space

$$\mathcal{D}(\mathfrak{A}_p) = \left\{u \in W^{2,p}(\Omega) : B\gamma u = 0\right\}. \tag{2.3}$$

(b) $\mathfrak{A}_p u = Au$ for every $u \in \mathcal{D}(\mathfrak{A}_p)$.

Here Au and $B\gamma u$ are taken in the sense of *distributions*.

Then we can obtain the generation theorem of analytic semigroups for the subelliptic oblique derivative problem (1.1) in the L^p topology (cf. [**37**, Theorem 1.2]):

THEOREM 2.4. *Let $1 < p < \infty$. If the condition* (H) *is satisfied, then we have the following two assertions* (i) *and* (ii):

(i) *For every $0 < \varepsilon < \pi/2$, there exists a constant $r_p(\varepsilon) > 0$ such that the resolvent set of $\mathfrak{A}_p$ contains the set*

$$\Sigma_p(\varepsilon) = \left\{\lambda = r^2\, e^{i\theta} : r \geq r_p(\varepsilon), -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon\right\},$$

and that the resolvent $(\mathfrak{A}_p - \lambda I)^{-1}$ satisfies the estimate

$$\left\|(\mathfrak{A}_p - \lambda I)^{-1}\right\| \leq \frac{c_p(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma_p(\varepsilon), \tag{2.4}$$

where $c_p(\varepsilon) > 0$ is a constant depending on ε.

(ii) *The operator $\mathfrak{A}_p$ generates a semigroup $U(z)$ on the space $L^p(\Omega)$ which is* analytic *in the sector*

$$\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$$

for any $0 < \varepsilon < \pi/2$.

Finally, we formulate a generation theorem of analytic semigroups in the topology of uniform convergence (Theorem 2.5).

Let $C(\overline{\Omega})$ be the space of complex-valued, continuous functions $f(x)$ on $\overline{\Omega}$. We equip the space $C(\overline{\Omega})$ with the topology of uniform convergence on the whole $\overline{\Omega}$. Hence it is a Banach space with the maximum norm

$$\|f\|_\infty = \max_{x \in \overline{\Omega}} |f(x)|.$$

We introduce a densely defined, closed linear operator $\mathfrak{A}$ from $C(\overline{\Omega})$ into itself as follows:

(a) The domain $\mathcal{D}(\mathfrak{A})$ of definition of $\mathfrak{A}$ is the set

$$\mathcal{D}(\mathfrak{A}) = \left\{u \in C(\overline{\Omega}) : Au \in C(\overline{\Omega}),\ B\gamma u = 0\right\}. \tag{2.5}$$

(b) $\mathfrak{A}u = Au$ for every $u \in \mathcal{D}(\mathfrak{A})$.

Here Au and $B\gamma u$ are taken in the sense of *distributions*.

Our localization argument in the proof of Theorem 2.5 is inspired by geometric arguments due to Egorov–Kondratev [**13**]. To do so, we impose the following geometric condition on the set Γ_0 of tangency:

(G) The set $\Gamma_0 = \{x' \in \Gamma : a(x') = 0\}$ of tangency is an $(n-2)$-dimensional *submanifold* of the boundary Γ.

Then Theorem 2.4 remains valid with $L^p(\Omega)$ and $\mathfrak{A}_p$ replaced by $C(\overline{\Omega})$ and $\mathfrak{A}$, respectively. More precisely, we can prove the following theorem (cf. [**37**, Theorem 1.3]):

THEOREM 2.5. *If the conditions* (G) *and* (H) *are satisfied, then we have the following two assertions* (i) *and* (ii):

(i) *For every* $\varepsilon > 0$*, there exists a constant* $r(\varepsilon) > 0$ *such that the resolvent set of* $\mathfrak{A}$ *contains the set*

$$\Sigma(\varepsilon) = \left\{\lambda = r^2 e^{i\theta} : r \geq r(\varepsilon),\ -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon\right\},$$

and that the resolvent $(\mathfrak{A} - \lambda I)^{-1}$ *satisfies the estimate*

$$\|(\mathfrak{A} - \lambda I)^{-1}\| \leq \frac{c(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma(\varepsilon), \tag{2.6}$$

where $c(\varepsilon) > 0$ *is a constant depending on* ε*.*

(ii) *The operator* $\mathfrak{A}$ *generates a semigroup* T_z *on the space* $C(\overline{\Omega})$ *which is* analytic *in the sector*

$$\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$$

for any $0 < \varepsilon < \pi/2$*.*

Remark 2.2. Moreover, we can prove that the operators $\{T_t\}_{t\geq 0}$ form a *Feller semigroup* on the space $C(\overline{\Omega})$, that is, they are non-negative and contractive on $C(\overline{\Omega})$ (cf. [**38**]):

$$f \in C(\overline{\Omega}),\ 0 \leq f(x) \leq 1 \text{ on } \overline{\Omega} \quad \Longrightarrow \quad 0 \leq T_t f(x) \leq 1 \text{ on } \overline{\Omega}.$$

2.3. Outline of the contents.

The contents of this paper are organized as follows.

In Section 3 we present a brief description of the basic concepts and results of L^p Sobolev spaces such as Sobolev's imbedding theorem, the Gagliardo–Nirenberg inequality, the Rellich–Kondrachov compactness theorem, Seeley's extension theorem and the trace theorem which will be used in the study of the oblique derivative problem in the framework of function spaces of L^p type.

In Section 4, we formulate a characterization of classical subelliptic pseudo-differential operators due to Egorov [**12**] and Hörmander [**22**] (Theorem 4.1) which plays a crucial role in this paper.

In Section 5, by using the L^p theory of pseudo-differential operators we consider the Dirichlet problem in the framework of L^p Sobolev spaces. The pseudo-differential operator approach to elliptic boundary value problems can be traced back to the pioneering work of Calderón [**9**] in early 1960s ([**20**], [**30**]).

In Section 6, by using the Dirichlet problem we consider the *homogeneous* oblique derivative problem $(*)$ for second-order, uniformly elliptic differential operators in the framework of L^p Sobolev spaces.

In Subsection 6.1, by using the oblique boundary operator $B\boldsymbol{\gamma}$ (Proposition 6.1) we formulate the following homogeneous oblique derivative problem:

$$\begin{cases} Au = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = 0 & \text{on } \Gamma. \end{cases} \tag{$*$}$$

In Subsection 6.2 we show that the oblique derivative problem (∗) can be reduced to the study of a pseudo-differential operator $\mathcal{T}_p$ on the boundary Γ. The virtue of this reduction is that there is no difficulty in taking adjoints or transposes after restricting the attention to the boundary, whereas boundary value problems in general do not have adjoints or transposes. This allows us to discuss the existence theory more easily (see [**10**], [**24**], [**28**], [**41**]). In Subsection 6.3 we prove that if the condition (H) is satisfied, then the operator $\mathcal{T}_p$ is a Fredholm operator for every $1 < p < \infty$ (Proposition 6.9). Finally, in Subsection 6.4 we prove that if the condition (H) is satisfied, then the operator $\mathfrak{A}_p$ is a Fredholm operator for every $1 < p < \infty$ and its index $\operatorname{ind} \mathfrak{A}_p (= \operatorname{ind} \mathcal{T}_p)$ is *independent* of p (Theorem 6.11).

In Section 7, in order to prove an existence and uniqueness theorem for problem (1.1) in the framework of L^p Sobolev spaces when $|\lambda| \to \infty$, we make use of a method essentially due to Agmon ([**2**], [**25**]). This is a technique of treating a spectral parameter λ as a second-order elliptic differential operator of an extra variable y on the unit circle S, and relating the old problem to a new one with the additional variable. Our presentation of this technique is due to Fujiwara [**15**]. More precisely, if we express the complex parameter λ in the form

$$\lambda = r^2 e^{i\theta}, \quad r \geq 0, \ -\pi < \theta < \pi,$$

then we replace the differential operator $A - \lambda$ defined in Ω by the second-order differential operator

$$\widetilde{\Lambda}(\theta) = A + e^{i\theta} \frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi,$$

defined in $\Omega \times S$. We consider the homogeneous oblique derivative problem in the product domain $\Omega \times S$

$$\begin{cases} \widetilde{\Lambda}(\theta)\widetilde{u} = \left(A + e^{i\theta} \dfrac{\partial^2}{\partial y^2} \right) \widetilde{u} = \widetilde{f} & \text{in } \Omega \times S, \\ B\gamma\widetilde{u} = \dfrac{\partial \widetilde{u}}{\partial \boldsymbol{\nu}} = a(x') \dfrac{\partial \widetilde{u}}{\partial \boldsymbol{n}} + \alpha(x') \cdot \widetilde{u} = 0 & \text{on } \Gamma \times S. \end{cases}$$

We prove that this oblique derivative problem in $\Omega \times S$ has a *finite index* if the condition (H) is satisfied (Theorem 7.1). Theorem 7.1 is an essential step in the proof of Theorem 2.4 (and Theorem 2.2) and its proof will be given in Section 11, due to its length. The idea of proof of Theorem 7.1 can be visualized as in Diagram 1 below.

In Section 8, in order to apply Agmon's method we consider the Dirichlet problem for the second-order, strongly uniform elliptic differential operator $\widetilde{\Lambda}(\theta)$, $-\pi < \theta < \pi$, in the framework of L^p Sobolev spaces on the product domain $\Omega \times S$.

In Section 9, we reduce the homogeneous oblique derivative problem (7.1) to the study of a first-order, pseudo-differential operator $\widetilde{T}(\theta)$ on the boundary $\Gamma \times S$ (Proposition 9.1), just as in Smith [**31**] and Guan–Sawyer [**18**].

The purpose of Section 10 is to prove that the pseudo-differential operators $\widetilde{T}(\theta)$ and $\widetilde{T}(\theta)'$ are both *subelliptic* on $\Gamma \times S$ if the condition (H) is satisfied (Proposition 10.3). To

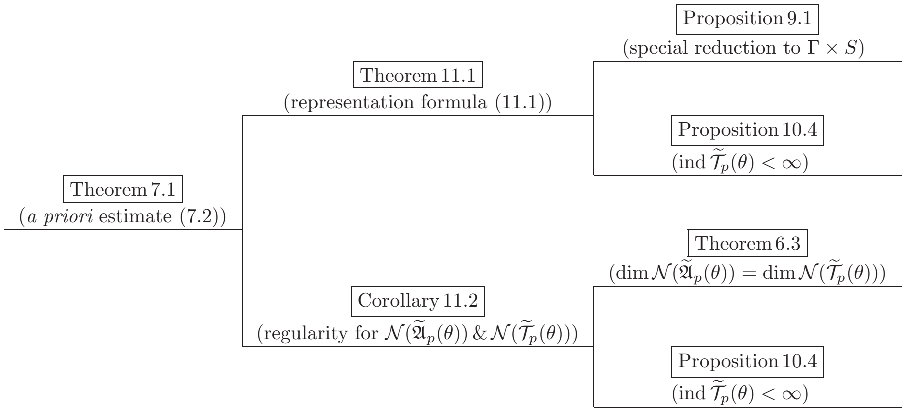

Diagram 1.

do so, we calculate the principal symbol $\widetilde{t}_1(x'', t, \xi'', \tau, y, \eta; \theta)$ of the pseudo-differential operator $\widetilde{T}(\theta)$. The essential point is to reduce the study of the general case $\widetilde{T}(\theta)$, $-\pi < \theta < \pi$, to the case where $\theta = 0$ (Proposition 10.2). In this way, we can prove that if the condition (H) is satisfied, then the pseudo-differential operator $\widetilde{\mathcal{T}}_p(\theta)$ is a *Fredholm operator* in the framework of L^p Sobolev spaces on the boundary $\Gamma \times S$ (Proposition 10.4). Our proof here is based on a characterization of subelliptic pseudo-differential operators due to Egorov [**12**] and Hörmander [**22**].

In Section 11 we show how Theorem 7.1 follows from Propositions 10.2 and 10.4. This section is the heart of the subject. Our proof of Theorem 7.1 is based on Smith [**31**, Main Theorem] and Guan–Sawyer [**18**, Theorem 2, part (i)] (Theorem 11.1). By Smith [**31**, Section 4] and Guan–Sawyer [**18**, Section 4], we can prove that the pseudo-differential operator $\widetilde{T}(\theta)$ has a unique right inverse $\widetilde{S}(\theta)$. Moreover, by using [**18**, Theorem 1, part (i)] we find that the *a priori* estimate (7.2) holds true for all functions $\widetilde{u} \in W^{2,p}(\Omega \times S)$ satisfying $B\boldsymbol{\gamma}\widetilde{u} = 0$ on $\Gamma \times S$ if the pseudo-differential operator $\widetilde{T}(\theta)$ on $\Gamma \times S$ satisfies the condition (H).

In Section 12 we prove the desired *a priori* estimate (2.1) in Theorem 2.4 by using Theorem 2.1, Theorem 7.1 and Theorem 12.2 when $p = 2$. Theorem 2.4 for $1 < p < \infty$ follows by combining Theorem 12.1 and Corollary 12.3. Remark that Theorem 2.2 for $p = 2$ is a special case of Theorem 2.4. The proof of Theorem 2.4 can be visualized as in Diagram 2 below.

Section 13 is devoted to the proof of Theorem 2.5. The essential resolvent estimate (2.6) is proved in Proposition 13.2. We make use of a λ-dependent localization argument in order to adjust the term $\|(A - \lambda)u\|_p$ in inequality (12.6) to obtain inequality (2.6), just as in [**37**]. Theorem 2.5 follows by combining Proposition 13.2, Theorem 2.4 and Sobolev's imbedding theorem (Theorem 3.1). The proof of Theorem 2.5 can be visualized as in Diagram 3 below.

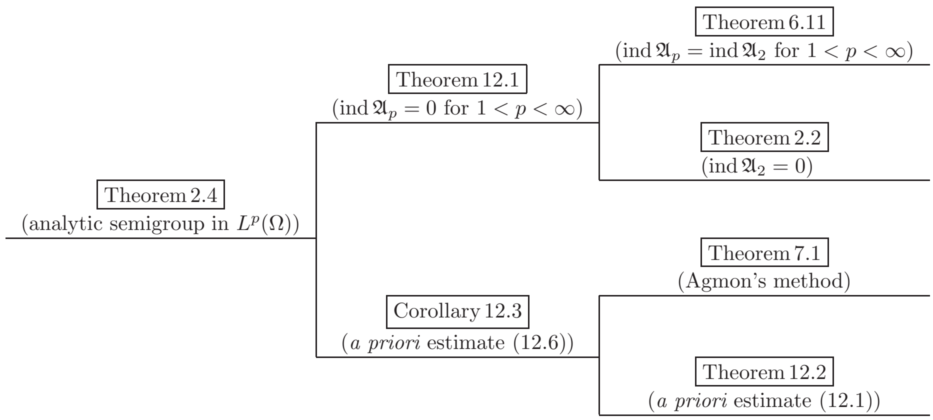

Diagram 2.

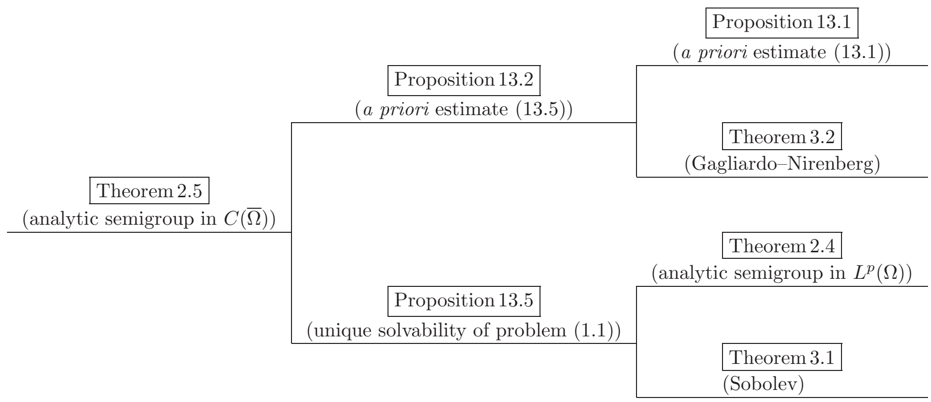

Diagram 3.

3. Function spaces.

In this section we present a brief description of the basic concepts and results of L^p Sobolev spaces which will be used in subsequent sections (see [**1**], [**6**], [**14**], [**32**], [**43**]).

(I) Let Ω be an open subset of $\mathbf{R}^n$. If $1 < p < \infty$ and if $s = m + \theta$ with a non-negative integer m and $0 < \theta < 1$, then the *Sobolev space* $W^{s,p}(\Omega)$ is defined to be the space of those functions $u \in W^{m,p}(\Omega)$ such that, for $|\alpha| = m$ the integral

$$\iint_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+p\theta}}\, dx\, dy$$

is finite. The norm $\|u\|_{W^{s,p}(\Omega)}$ of $W^{s,p}(\Omega)$ is defined by the formula

$$\|u\|_{W^{s,p}(\Omega)} = \left(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u(x)|^p\, dx + \sum_{|\alpha|=m} \iint_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+p\theta}}\, dx\, dy \right)^{1/p}.$$

If $1 < p < \infty$ and if $s = m + \theta$ with a non-negative integer m and $0 < \theta < 1$, then the *Besov space* $B^{s,p}(\mathbf{R}^{n-1})$ is defined to be the space of those functions $\varphi \in W^{m,p}(\mathbf{R}^{n-1})$ such that, for $|\alpha| = m$ the integral

$$\iint_{\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}} \frac{|D^\alpha\varphi(x') - D^\alpha\varphi(y')|^p}{|x'-y'|^{(n-1)+p\theta}}\, dx'\, dy'$$

is finite. We equip the space $B^{s,p}(\mathbf{R}^{n-1})$ with the norm

$$\Bigg(\sum_{|\alpha|\le m} \int_{\mathbf{R}^{n-1}} |D^\alpha\varphi(x')|^p\, dx' + \sum_{|\alpha|=m} \iint_{\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}} \frac{|D^\alpha\varphi(x') - D^\alpha\varphi(y')|^p}{|x'-y'|^{(n-1)+p\theta}}\, dx'\, dy'\Bigg)^{1/p}.$$

If Γ is the boundary of the bounded smooth domain Ω, then the *Besov spaces* $B^{s,p}(\Gamma)$ are defined to be locally the Besov spaces $B^{s,p}(\mathbf{R}^{n-1})$, upon using local coordinate systems flattening out Γ, together with a partition of unity. The norm of $B^{s,p}(\Gamma)$ will be denoted by $|\cdot|_{s,p}$.

(II) In the proof of Theorem 2.5 we need the following *imbedding properties* of L^p Sobolev spaces (see [**1**, Theorem 5.4], [**14**, Part I, Theorem 10.1]):

THEOREM 3.1 (Sobolev). *Let Ω be a bounded domain in the Euclidean space $\mathbf{R}^n$ with boundary Γ of class C^2. Then we have the following two assertions*:

(i) *If $1 \le p < n$, we have the continuous injection*

$$W^{2,p}(\Omega) \subset W^{1,q}(\Omega) \quad \textit{for } \frac{1}{p} - \frac{1}{n} \le \frac{1}{q} \le \frac{1}{p}.$$

(ii) *If $n/2 < p < \infty$, $p \ne n$, we have the continuous injection*

$$W^{2,p}(\Omega) \subset C^\nu(\overline{\Omega}) \quad \textit{for } 0 < \nu \le 2 - \frac{n}{p}.$$

THEOREM 3.2 (Gagliardo–Nirenberg). *Let Ω be a bounded domain in $\mathbf{R}^n$ with boundary of class C^2, and $1 \le p, r \le \infty$. Then we have the following assertions*:

(i) *If $p \ne n$ and if*

$$\frac{1}{q} = \frac{1}{n} + \theta\left(\frac{1}{p} - \frac{2}{n}\right) + (1-\theta)\frac{1}{r} \quad \textit{for } \frac{1}{2} \le \theta \le 1,$$

then we have, for all functions $u \in W^{2,p}(\Omega) \cap L^r(\Omega)$,

$$\|u\|_{1,q} \le C_1 \|u\|_{2,p}^{\theta} \|u\|_r^{1-\theta},$$

with a constant $C_1 = C_1(\Omega, p, r, \theta) > 0$.

(ii) *If $n/2 < p < \infty$, $p \ne n$ and if*

$$0 \le \nu < \theta \left(2 - \frac{n}{p}\right) - (1-\theta)\frac{n}{r},$$

then we have, for all functions $u \in W^{2,p}(\Omega) \cap L^r(\Omega)$,

$$\|u\|_{C^\nu(\overline{\Omega})} \le C_2 \|u\|_{2,p}^{\theta} \|u\|_r^{1-\theta}, \tag{3.1}$$

with a constant $C_2 = C_2(\Omega, p, r, \theta) > 0$.

(III) The next *compactness theorem* for function spaces of L^p Sobolev spaces will play an essential role in the study of boundary value problems (see [**1**, Theorem 6.3 and Paragraph 7.32], [**17**, Theorem 7.22]):

THEOREM 3.3 (Rellich–Kondrachov). *Let* Ω *be a bounded domain in Euclidian space* $\mathbf{R}^n$ *with smooth boundary* Γ. *If* $1 < p < \infty$ *and* $s > t$, *then the injections*

$$W^{s,p}(\Omega) \longrightarrow W^{t,p}(\Omega),$$
$$B^{s,p}(\Gamma) \longrightarrow B^{t,p}(\Gamma)$$

are both compact (*or completely continuous*).

(IV) The next theorem, due to Seeley [**29**], asserts that the functions in $C^\infty(\overline{\Omega})$ are the restrictions to Ω of functions in $C^\infty(\mathbf{R}^n)$ (see [**1**, Theorems 5.21 and 5.22]):

THEOREM 3.4 (Seeley). *Let* Ω *be either the half space* $\mathbf{R}^n_+$ *or a smooth domain in* $\mathbf{R}^n$ *with bounded boundary* Γ. *Then there exists a continuous linear extension operator*

$$E : C^\infty(\overline{\Omega}) \longrightarrow C^\infty(\mathbf{R}^n).$$

Furthermore, for every $1 \le p < \infty$ *and every integer* $m \ge 0$ *the extension operator*

$$E : W^{m,p}(\Omega) \longrightarrow W^{m,p}(\mathbf{R}^n)$$

is continuous.

(V) Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary Γ. Without loss of generality, we may assume the following:

(a) The domain Ω is a relatively compact open subset of an n-dimensional, compact smooth manifold M without boundary.

(b) In a neighborhood W of Γ in M a normal coordinate t is chosen so that the points of W are represented as (x', t), $x' \in \Gamma$, $-1 < t < 1$; $t > 0$ in Ω, $t < 0$ in $M \setminus \overline{\Omega}$ and $t = 0$ only on Γ.

(c) The manifold M is equipped with a strictly positive density μ which, on W, is the product of a strictly positive density ω on Γ and the Lebesgue measure dt on $(-1, 1)$. This manifold $M = \widehat{\Omega}$ is called the *double* of Ω.

If j is a non-negative integer, we can define the trace map

$$\gamma_j : C^\infty(\overline{\Omega}) \longrightarrow C^\infty(\Gamma)$$

by the formula

$$\gamma_j u(x') = \lim_{t \downarrow 0} D_t^j u(x', t) \quad \text{for all } u \in C^\infty(\overline{\Omega}).$$

Then we have the following theorem (see [**1**, Theorem 7.39]):

THEOREM 3.5 (the trace theorem). *Let $1 < p < \infty$. If $0 \leq j < s - 1/p$, then the trace map*

$$\begin{aligned} \boldsymbol{\gamma} : W^{s,p}(\Omega) &\longrightarrow \prod_{0 \leq j < s - 1/p} B^{s-j-1/p,p}(\Gamma) \\ u &\longmapsto (\gamma_j u)_{0 \leq j < s - 1/2} \end{aligned}$$

is continuous and surjective.

4. Subelliptic pseudo-differential operators.

Let Ω be an open subset of $\mathbf{R}^n$. A properly supported, pseudo-differential operator A in the Hörmander class $L^m_{1,0}(\Omega)$ of order $m \in \mathbf{R}$ is said to be *subelliptic* with loss of some $\delta \in [0,1)$ if, for every compact $K \subset \Omega$, $s \in \mathbf{R}$ and $t < s + m - \delta$ there exists a constant $C_{K,s,t} > 0$ such that we have the inequality

$$\|u\|_{W^{s+m-\delta,2}(\Omega)} \leq C_{K,s,t} \left(\|Au\|_{W^{s,2}(\Omega)} + \|u\|_{W^{t,2}(\Omega)} \right) \\ \text{for all functions } u \in C^\infty_K(\Omega).$$

Here $W^{\sigma,2}(\Omega)$ is the L^2 Sobolev space of order σ on Ω and

$$C^\infty_K(\Omega) = \text{the space of functions in } C^\infty(\Omega) \text{ with support in } K.$$

It is known (see Hörmander [**21**]) that subelliptic operators are hypoelliptic, with loss of δ-derivatives.

Egorov [**12**] and Hörmander [**22**] have obtained necessary and sufficient conditions in order that a properly supported, classical pseudo-differential operator $A \in L^m_{\mathrm{cl}}(\Omega)$ of order m is subelliptic. More precisely, we have the following theorem (see [**22**, Theorem 3.4], [**42**, Theorem I]):

THEOREM 4.1 (Egorov–Hörmander). *Let A be a properly supported, pseudo-differential operator in the class $L^m_{\mathrm{cl}}(\Omega)$ having the principal symbol $a_m(x,\xi)$. Then A is* subelliptic *with loss of some $\delta \in [0,1)$ if and only if, at every point x_0 of Ω there exists a neighborhood V of x_0 such that the following two conditions* (i) *and* (ii) *are satisfied*:

(i) *For any point $(x,\xi) \in V \times (\mathbf{R}^n \setminus \{0\})$, the function*

$$\left(H_{\mathrm{Re}\, z a_m}\right)^j \left(\mathrm{Im}\, z a_m\right)(x,\xi) \tag{4.1}$$

is different from zero for some complex number z and some non-negative integer

$j \leq \delta/(1-\delta)$. *Here* H_f *is the* Hamilton vector field *defined by the formula*

$$H_f = \sum_{i=1}^{n} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i}.$$

(ii) *If j is an odd integer and is the smallest integer such that the function* (4.1) *is not identically equal to zero, then the function* (4.1) *is non-negative for all* $(x, \xi) \in V \times (\mathbf{R}^n \setminus \{0\})$.

5. The Dirichlet problem.

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary Γ. Let A be a second-order, uniformly elliptic differential operator with real coefficients on the *double* $M = \widehat{\Omega}$ of Ω such that

$$A = \sum_{i=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} + c(x).$$

Here:

(1) $a^{ij} \in C^\infty(M)$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in M$ and $1 \leq i, j \leq n$, and there exists a constant $c_0 > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x) \xi_i \xi_j \geq c_0 |\xi|^2 \quad \text{for all } (x, \xi) \in T^*(M),$$

where $T^*(M)$ is the cotangent bundle of M.

(2) $b^i \in C^\infty(M)$ for all $1 \leq i \leq n$.

(3) $c \in C^\infty(M)$.

For simplicity, we assume that

$$\text{The function } c(x) \text{ does not vanish } \textit{identically} \text{ on } M. \tag{5.1}$$

Then we can prove the following *existence and uniqueness theorem* for the Dirichlet problem (see [**30**, Theorem], [**36**, Theorem 8.2.5]):

THEOREM 5.1. *Let $1 < p < \infty$ and $s > -2 + 1/p$. Assume that condition* (5.1) *is satisfied. Then the Dirichlet problem*

$$\begin{cases} Au = f & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma \end{cases} \tag{5.2}$$

has a unique solution u in the space $W^{s+2,p}(\Omega)$ for any $f \in W^{s,p}(\Omega)$ and any $\varphi \in B^{s+2-1/p,p}(\Gamma)$.

Indeed, it suffices to note that the unique solution u of the Dirichlet problem (5.2) is given by the following formula:

$$u = QEf|_{\Omega} + P\left(\varphi - (QEf)|_{\Gamma}\right) \quad \text{in } \Omega. \tag{5.3}$$

Here:

(1) $Q : W^{s,p}(M) \to W^{s+2,p}(M)$ is the *fundamental solution* of A.

(2) $P : B^{s-1/p,p}(\Gamma) \to W^{s,p}(M)$ is the *Poisson kernel* for A.

(3) $E : W^{s,p}(\Omega) \to W^{s,p}(M)$ is the Seeley extension operator (see Theorem 3.4).

6. Homogeneous oblique derivative problem.

In this section, by using the Dirichlet problem we consider the *homogeneous* oblique derivative problem for second-order, uniformly elliptic differential operators in the framework of L^p Sobolev spaces. We prove that if the condition (H) is satisfied, then the operator $\mathfrak{A}_p$ is a Fredholm operator and its index $\operatorname{ind}\mathfrak{A}_p$ is *independent* of p for all $1 < p < \infty$ (Theorem 6.11). The proof of Theorem 6.11 can be visualized as in Diagram 4 below.

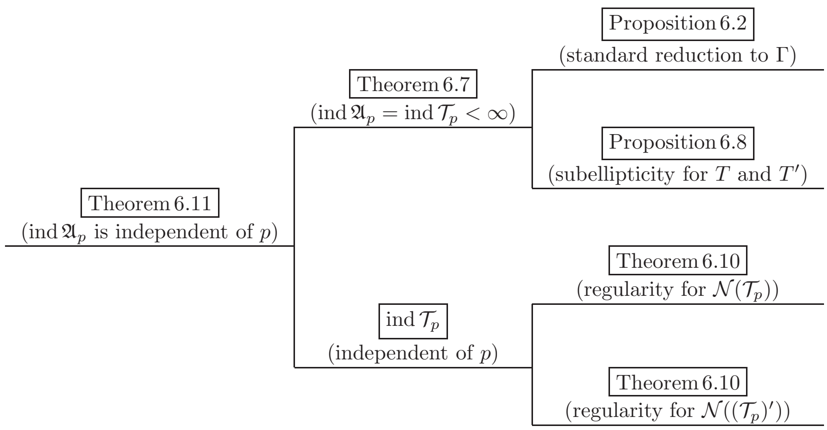

Diagram 4.

6.1. Formulation of the oblique derivative problem.

If $1 < p < \infty$, we introduce a *maximal domain* $H_A(\Omega)$ for the operator A in the Banach space $L^p(\Omega)$ as follows:

$$H_A(\Omega) = \{u \in L^p(\Omega) : Au \in L^p(\Omega)\}.$$

We equip the space $H_A(\Omega)$ with the graph norm

$$\|u\|_{H_A(\Omega)} = \left(\|u\|^2_{L^p(\Omega)} + \|Au\|^2_{L^p(\Omega)}\right)^{1/2}.$$

The maximal domain $H_A(\Omega)$ is a Banach space.

Then we have the following proposition (see [**36**, Proposition 8.3.2]):

Proposition 6.1. *The oblique boundary operator*

$$B\boldsymbol{\gamma} : H_A(\Omega) \longrightarrow B^{-1-1/p,p}(\Gamma)$$

is continuous.

Now we can formulate the homogeneous oblique derivative problem as follows: Given a function $f \in L^p(\Omega)$, find a function $u \in L^p(\Omega)$ such that

$$\begin{cases} Au = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = 0 & \text{on } \Gamma. \end{cases} \tag{$*$}$$

6.2. Standard reduction to the boundary Γ.

Let $1 < p < \infty$ and $s > -1 + 1/p$. Given a function $f \in W^{s,p}(\Omega)$, assume that a function $u \in W^{\sigma,p}(\Omega)$ with $\sigma \leq s + 2$ is a solution of problem $(*)$. Then, by using Proposition 6.1 we can reduce the study of problem $(*)$ to that of a pseudo-differential equation on the boundary Γ, just as in the classical Fredholm integral equation. In fact, we can prove the following proposition (see [**36**, Theorem 8.3.3]):

Proposition 6.2. *Let $1 < p < \infty$, $s > -2 + 1/p$ and $\sigma \leq s + 2$. For a given function $f \in W^{s,p}(\Omega)$, there exists a solution $u \in W^{\sigma,p}(\Omega)$ of problem $(*)$ if and only if there exists a solution $\psi \in B^{\sigma-1/p,p}(\Gamma)$ of the equation*

$$B\boldsymbol{\gamma}\,(P\psi) = -B\boldsymbol{\gamma}\,(QEf) \quad \text{on } \Gamma. \tag{$**$}$$

Moreover, the solutions u and ψ are related as follows:

$$u = QEf|_{\Omega} + P\psi.$$

If we let

$$\begin{aligned} T : C^\infty(\Gamma) &\longrightarrow C^\infty(\Gamma) \\ \varphi &\longmapsto B\boldsymbol{\gamma}\,(P\varphi)\,, \end{aligned}$$

then we have the formula

$$T = a(x')\,\Pi + \alpha(x'), \tag{6.1}$$

where Π is the Dirichlet–Neumann operator defined as follows:

$$\Pi\varphi = \frac{\partial}{\partial \boldsymbol{n}}\,(P\varphi)\bigg|_{\Gamma} \quad \text{for all } \varphi \in C^\infty(\Gamma).$$

It is well known (cf. [**10**], [**20**], [**22**], [**24**], [**28**], [**30**], [**41**]) that the operator Π is a classical, *elliptic* pseudo-differential operator of first order on the boundary Γ.

However, there is a *homotopy* through elliptic symbols between the two elliptic differential operators A and Δ if we take

$$A_t := t\,A + (1-t)\,\Delta \quad \text{for } 0 \le t \le 1.$$

Therefore, we have only to calculate concretely the symbol of the pseudo-differential operator Π in the case where $A = \Delta$. In this case, we can write down the complete symbol $p(x', \xi')$ of Π as follows (cf. [**16**], [**38**, Section 10.7]):

$$p(x', \xi') = |\xi'| + \frac{1}{2}\left(\frac{\omega_{x'}(\widehat{\xi'}, \widehat{\xi'})}{|\xi'|^2} - (n-1)M(x')\right) - \sqrt{-1}\,\frac{1}{2}\operatorname{div}\delta_{(\xi')}(x') + \text{ terms of order } \le -1. \tag{6.2}$$

Here:

(a) $|\xi'|$ is the length of $\xi' \in T^*_{x'}(\Gamma)$ with respect to the Riemannian metric of Γ induced by the natural metric of $\mathbf{R}^n$.

(b) $M(x')$ is the *mean curvature* of the boundary Γ at x'.

(c) $\omega_{x'}(\widehat{\xi'}, \widehat{\xi'})$ is the *second fundamental form* of Γ at x', while $\widehat{\xi'} \in T_{x'}(\Gamma)$ is the tangent vector corresponding to the cotangent vector $\xi' \in T^*_{x'}(\Gamma)$ by the duality between $T_{x'}(\Gamma)$ and $T^*_{x'}(\Gamma)$ with respect to the Riemannian metric $(g_{ij}(x'))$ of Γ.

(d) $\operatorname{div}\delta_{(\xi')}$ is the *divergence* of a real smooth vector field $\delta_{(\xi')}$ on Γ defined (in terms of local coordinates) by the formula

$$\delta_{(\xi')} = \sum_{j=1}^{n-1} \frac{\partial |\xi'|}{\partial \xi_j}\frac{\partial}{\partial x_j} \quad \text{for } \xi' \ne 0.$$

Hence, we find from formula (6.2) that the principal symbol $t_1(x', \xi')$ of the pseudo-differential operator T, defined by formula (6.1), is equal to the following:

$$t_1(x', \xi') = a(x')\,|\xi'| + \sqrt{-1}\left[\sum_{k=1}^{n-1} \alpha^k(x')\xi_k\right]. \tag{6.3}$$

By virtue of Proposition 6.2, we can reduce problem $(*)$ to the study of the pseudo-differential operator T on the boundary Γ. We shall formulate this fact more precisely in terms of functional analysis (cf. [**39**, Chapter 6]).

First, we associate with the homogeneous problem $(*)$ a densely defined, closed linear operator (see formula (2.3))

$$\mathfrak{A}_p : L^p(\Omega) \longrightarrow L^p(\Omega)$$

in the Banach space $L^p(\Omega)$ as follows.

(a) The domain $\mathcal{D}(\mathfrak{A}_p)$ of definition of $\mathfrak{A}_p$ is the space

$$\mathcal{D}(\mathfrak{A}_p) = \{u \in W^{2,p}(\Omega) : \, B\gamma u = 0 \quad \text{on } \Gamma\}.$$

(b) $\mathfrak{A}_p u = Au$ for every $u \in \mathcal{D}(\mathfrak{A}_p)$.

Indeed, since $A : L^p(\Omega) \to \mathcal{D}'(\Omega)$ and $B\boldsymbol{\gamma} : H_A(\Omega) \to B^{-1-1/p,p}(\Gamma)$ are both continuous, it follows that $\mathfrak{A}_p$ is a closed operator. Furthermore, the operator $\mathfrak{A}_p$ is densely defined, since the domain $\mathcal{D}(\mathfrak{A}_p)$ contains the space $C_0^\infty(\Omega)$ which is dense in $L^p(\Omega)$.

Similarly, by taking Proposition 6.2 with $s := 0$ and $\sigma := 2$ we associate with equation $(**)$ a densely defined, closed linear operator

$$\mathcal{T}_p : B^{2-1/p,p}(\Gamma) \longrightarrow B^{2-1/p,p}(\Gamma)$$

in the Banach space $B^{2-1/p,p}(\Gamma)$ as follows.

(α) The domain $\mathcal{D}(\mathcal{T}_p)$ of definition of $\mathcal{T}_p$ is the space

$$\mathcal{D}(\mathcal{T}_p) = \left\{ \varphi \in B^{2-1/p,p}(\Gamma) : T\varphi \in B^{2-1/p,p}(\Gamma) \right\}. \tag{6.4}$$

(β) $\mathcal{T}_p\varphi = T\varphi = B\boldsymbol{\gamma}\,(P\varphi)$ for every $\varphi \in \mathcal{D}(\mathcal{T}_p)$.

First, we have the following theorem (see [**36**, Theorem 8.3.4]):

Theorem 6.3 (Null Spaces). *The null spaces $\mathcal{N}(\mathfrak{A}_p)$ and $\mathcal{N}(\mathcal{T}_p)$ are* isomorphic. *Hence we have the formula*

$$\dim \mathcal{N}(\mathfrak{A}_p) = \dim \mathcal{N}(\mathcal{T}_p).$$

For the ranges $\mathcal{R}(\mathfrak{A}_p)$ and $\mathcal{R}(\mathcal{T}_p)$, we have the following theorem (see [**36**, Theorem 8.3.5]):

Theorem 6.4 (Ranges). *If the range $\mathcal{R}(\mathcal{T}_p)$ is closed in $B^{2-1/p,p}(\Gamma)$, then the range $\mathcal{R}(\mathfrak{A}_p)$ is closed in $L^p(\Omega)$.*

In order to study a relationship between $\operatorname{codim} \mathcal{R}(\mathfrak{A}_p)$ and $\operatorname{codim} \mathcal{R}(\mathcal{T}_p)$, we consider the transposes $(\mathfrak{A}_p)'$ and $(\mathcal{T}_p)'$. Here the transpose $(\mathfrak{A}_p)'$ of $\mathfrak{A}_p$ is a densely defined, closed linear operator

$$(\mathfrak{A}_p)' : L^{p'}(\Omega) \longrightarrow L^{p'}(\Omega), \quad p' = \frac{p}{p-1},$$

such that

$$\langle \mathfrak{A}_p u, v \rangle = \langle u, (\mathfrak{A}_p)' v \rangle\,, \quad u \in \mathcal{D}(\mathfrak{A}_p),\ v \in \mathcal{D}((\mathfrak{A}_p)'),$$

where $\langle \cdot, \cdot \rangle$ is the duality between the spaces $L^p(\Omega)$ and $L^{p'}(\Omega)$.

Similarly, the transpose $(\mathcal{T}_p)'$ of $\mathcal{T}_p$ is a densely defined, closed linear operator

$$(\mathcal{T}_p)' : B^{-2+1/p,p'}(\Gamma) \longrightarrow B^{-2+1/p,p'}(\Gamma), \quad p' = \frac{p}{p-1},$$

such that

$$\langle \mathcal{T}_p \varphi, \psi \rangle = \langle \varphi, (\mathcal{T}_p)' \psi \rangle\,, \quad \varphi \in \mathcal{D}(\mathcal{T}_p),\ \psi \in \mathcal{D}((\mathcal{T}_p)'),$$

where $\langle \cdot, \cdot \rangle$ is the duality between the spaces $B^{2-1/p,p}(\Gamma)$ and $B^{-2+1/p,p'}(\Gamma)$.

Then we have the following theorem (see [**36**, Theorem 8.3.6]):

THEOREM 6.5. *Assume that the ranges $\mathcal{R}(\mathfrak{A}_p)$ and $\mathcal{R}(\mathcal{T}_p)$ are closed. If the null space $\mathcal{N}((\mathcal{T}_p)')$ has finite dimension, then the null space $\mathcal{N}((\mathfrak{A}_p)')$ has finite dimension. Moreover, in this case, we have the formula*

$$\dim \mathcal{N}((\mathfrak{A}_p)') = \dim \mathcal{N}((\mathcal{T}_p)').$$

COROLLARY 6.6 (Cokernels). *Assume that the ranges $\mathcal{R}(\mathfrak{A}_p)$ and $\mathcal{R}(\mathcal{T}_p)$ are closed. If the range $\mathcal{R}(\mathcal{T}_p)$ has finite codimension, then the range $\mathcal{R}(\mathfrak{A}_p)$ has finite codimension. Moreover, in this case, we have the formulas*

$$\operatorname{codim} \mathcal{R}(\mathfrak{A}_p) = \dim \mathcal{N}((\mathfrak{A}_p)') = \dim \mathcal{N}((\mathcal{T}_p)') = \operatorname{codim} \mathcal{R}(\mathcal{T}_p).$$

Corollary 6.6 is an immediate consequence of the closed range theorem (see [**45**, Chapter VII, Section 5, Theorem]) and Theorem 6.5.

By combining Theorems 6.3 through 6.5 and Corollary 6.6, we obtain the following formula for the indices of the operators $\mathfrak{A}_p$ and $\mathcal{T}_p$ (see [**39**, Theorem 6.11]):

THEOREM 6.7 (Indices). *If the operator $\mathcal{T}_p$ is a Fredholm operator, then the operator $\mathfrak{A}_p$ is a Fredholm operator. Moreover, in this case, we have the formula*

$$\operatorname{ind} \mathfrak{A}_p = \operatorname{ind} \mathcal{T}_p.$$

6.3. Subellipticity for T and T'.

The purpose of this subsection is to prove the following:

PROPOSITION 6.8. *Assume that the hypothesis* (H) *is satisfied. Then the pseudo-differential operators T and T' are both* subelliptic *with loss of some δ on Γ where $2k/(2k+1) \le \delta < 1$.*

PROOF. We apply Theorem 4.1 (Egorov–Hörmander) to the pseudo-differential operators T and T'. The proof of Proposition 6.8 is divided into three steps.

Step 1: First, we straighten out the vector field $\alpha(x')$. For each initial point

$$(y'', 0) = (y_1, y_2, \dots, y_{n-2}, 0) \in \mathbf{R}^{n-1},$$

we consider the following initial-value problem for ordinary differential equations:

$$\begin{cases} \dfrac{d\gamma_1}{dt} = \alpha_1(\gamma(y'', t)), & \gamma_1(y'', 0) = y_1, \\ \quad\vdots & \quad\vdots \\ \dfrac{d\gamma_{n-2}}{dt} = \alpha_{n-2}(\gamma(y'', t)), & \gamma_{n-2}(y'', 0) = y_{n-2}, \\ \dfrac{d\gamma_{n-1}}{dt} = \alpha_{n-1}(\gamma(y'', t)), & \gamma_{n-1}(y'', 0) = 0. \end{cases} \tag{6.5}$$

We remark that the initial-value problem (6.5) has a unique local solution

$$\gamma(y'', t) = (\gamma_1(y'', t), \gamma_2(y'', t), \ldots, \gamma_{n-1}(y'', t)),$$

since the vector filed $\alpha(x')$ is Lipschitz continuous. Hence we can introduce a new change of variables $x' = \gamma(y')$ by the formula

$$\begin{gathered}(x_1, x_2, \ldots, x_{n-1}) = (\gamma_1(y'', y_{n-1}), \gamma_2(y'', y_{n-1}), \ldots, \gamma_{n-1}(y'', y_{n-1})), \\ y' = (y'', y_{n-1}) \in \mathbf{R}^{n-1}.\end{gathered}$$

Then we have the formula

$$\frac{d}{dt} = \sum_{k=1}^{n-1} \alpha^k(x') \frac{\partial}{\partial x_k} = \alpha(x').$$

Therefore, in view of formula (6.3) we may assume that the principal symbol $t_1(x'', t, \xi'', \tau)$ of $T = a(x')\mathit{\Pi} + \alpha(x')$ is equal to the following:

$$\begin{gathered}t_1(x'', t, \xi'', \tau) = \sqrt{-1}\ \tau + a(x'', t) \sqrt{|\xi''|^2 + \tau^2}, \\ (x', \xi') = ((x'', t), (\xi'', \tau)) .\end{gathered} \tag{6.6}$$

Step 2: In view of formula (6.6), we find that the characteristic set Σ of T is given by the formula

$$\Sigma = \{((x'', t), (\xi'', \tau)) \in T^*(\Gamma) \setminus \{0\} : a(x'', t) = 0,\ \tau = 0\} .$$

(a) The case where $a(x''_0, 0) \neq 0$. Then we can take $j = 0$ in condition (i) of Theorem 4.1, since we have, by formula (6.6),

$$\begin{gathered}\mathrm{Im}\ \left(\sqrt{-1}\ t_1(x''_0, t, \xi'', \tau)\right) = a(x''_0, t)\, p_1(x''_0, t, \xi'', \tau) \neq 0 \\ \text{for } |\xi''|^2 + \tau^2 = 1.\end{gathered}$$

(b) The case where $a(x''_0, 0) = 0$. If we consider instead of T the operator $\sqrt{-1}\,T$, then we have, by formula (6.6),

$$\sqrt{-1}\ t_1(x'', t, \xi'', \tau) = -\tau + \sqrt{-1}\ a(x'', t) \sqrt{|\xi''|^2 + \tau^2}.$$

Then we remark that

$$\mathrm{Im}\ \left(\sqrt{-1}\ t_1(x''_0, 0, \xi'', \tau)\right) = a(x''_0, 0)\, p_1(x''_0, 0, \xi'', \tau) = 0.$$

If we take $j = 1$ in condition (ii) of Theorem 4.1, we may assume that

$$\frac{\partial a}{\partial t}(x''_0, 0) \neq 0. \tag{6.7}$$

This proves that condition (1) of Theorem 4.1 for $j = 1$ is equivalent to condition (6.7). Namely, we have the assertion

$$H_{\mathrm{Re}\,(\sqrt{-1}\ t_1)}\left(\mathrm{Im}\,\left(\sqrt{-1}\ t_1(x_0'', t, \xi'', \tau)\right)\right) \neq 0 \Longleftrightarrow \frac{\partial a}{\partial t}(x_0'', 0) \neq 0.$$

Here it should be noticed that condition (6.7) implies that the vector field $\alpha(x')$ is non-zero on the set $\Gamma_0 = \{x' \in \Gamma : a(x') = 0\}$.

Therefore, we find that condition (ii) of Theorem 4.1 for the principal symbol

$$\sqrt{-1}\ t_1(x'', t, \xi'', \tau)$$

implies the following assertion:

$$\begin{aligned} &H_{\mathrm{Re}\,(\sqrt{-1}\ t_1)}\left(\mathrm{Im}\,\left(\sqrt{-1}\ t_1(x_0'', t, \xi'', \tau)\right)\right) = -\frac{\partial a}{\partial t}(x'', t)\sqrt{|\xi''|^2 + \tau^2} \geq 0 \\ &\Longleftrightarrow \\ &\frac{\partial a}{\partial t}(x_0'', t) \leq 0. \end{aligned} \tag{6.8}$$

On the other hand, it follows from formula (6.6) that the principal symbol $t_1'(x'', t, \xi'', \tau)$ of the *transpose* T' is given by the formula

$$t_1'(x'', t, \xi'', \tau) = a(x'', t)\sqrt{|\xi''|^2 + \tau^2} - \sqrt{-1}\ \tau.$$

Since we have the formula (for $z = \sqrt{-1}$)

$$\sqrt{-1}\ t_1'(x'', t, \xi'', \tau) = \tau + \sqrt{-1}\ a(x'', t)\sqrt{|\xi''|^2 + \tau^2 + \eta^2},$$

we find that condition (ii) of Theorem 4.1 for $\sqrt{-1}\ t_1'(x', \xi')$ implies the following:

$$\begin{aligned} &H_{\mathrm{Re}\,(\sqrt{-1}\ t_1')}\left(\mathrm{Im}\,\left(\sqrt{-1}\ t_1'(x_0', \xi')\right)\right) = \frac{\partial a}{\partial t}(x'', t)\sqrt{|\xi''|^2 + \tau^2} \geq 0 \\ &\Longleftrightarrow \\ &\frac{\partial a}{\partial t}(x_0'', t) \geq 0. \end{aligned} \tag{6.9}$$

By combining inequalities (6.8) and (6.9), we obtain that

$$\frac{\partial a}{\partial t}(x_0'', 0) = 0.$$

This contradicts assumption (6.7).

Therefore, we have proved that $j = 1$ is excluded.

In this way, we can prove that j is an *even* integer. More precisely, we have, for some even integer j,

$$\begin{aligned} &a(x_0'', t) = \frac{\partial a}{\partial t}(x_0'', 0) = \cdots = \frac{\partial^{j-1} a}{\partial t^{j-1}}(x_0'', 0) = 0, \\ &\frac{\partial^j a}{\partial t^j}(x_0'', 0) \neq 0. \end{aligned}$$

Step 3: Therefore, we find that T and T' are both subelliptic with loss of some

$\delta \in [0,1)$ if and only if, along the integral curve $x(t, x_0')$ of the vector field $\alpha(x')$ passing through $x_0' \in \Gamma_0$ at $t = 0$ the function: $t \mapsto a(x(t, x_0'))$ has zeros of *even* order $\leq 2k$, and $2k \leq \delta/(1-\delta)$.

The proof of Proposition 6.8 is complete. $\square$

In light of Proposition 6.8, we can derive from Smith [**31**] and Guan–Sawyer [**18**] that if condition (H) is satisfied, then the pseudo-differential operator T has a *parametrix* (see Remark 2.1). Therefore, we obtain the following fundamental result:

PROPOSITION 6.9. *If the condition* (H) *is satisfied, then the operator* $\mathcal{T}_p$, *defined by formula* (6.4), *is a* Fredholm operator *for every* $1 < p < \infty$.

Indeed, the proof of Proposition 6.9 can be carried out as in Diagram 5 below.

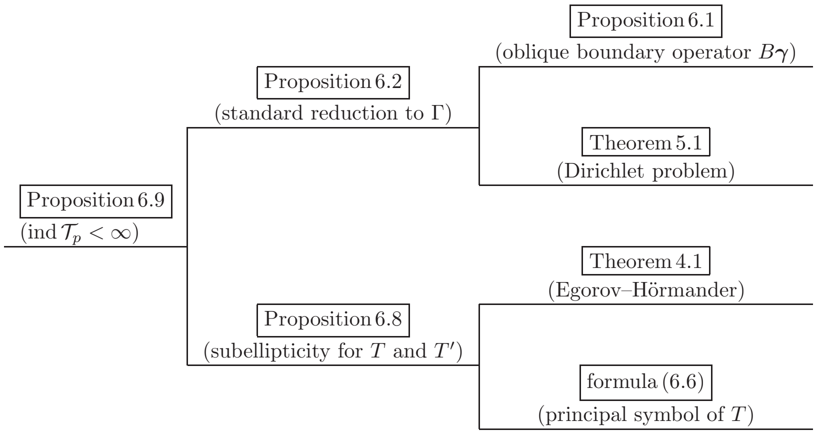

Diagram 5.

6.4. Index of the operator $\mathfrak{A}_p$.

The next theorem is an immediate consequence of Proposition 6.8 (cf. [**31**, Theorem 4.5], [**18**, Proposition 5.38]):

THEOREM 6.10 (Regularity). *Let* $1 < p < \infty$. *Assume that condition* (H) *is satisfied. Then we have the following two assertions*:

(i) *If* $\varphi \in \mathcal{D}'(\Gamma)$, $T\varphi \in B^{\sigma,p}(\Gamma)$ *for* $\sigma \in \mathbf{R}$, *then it follows that* $\varphi \in B^{\sigma,p}(\Gamma)$. *In particular, we have the assertion*

$$\mathcal{N}(\mathcal{T}_p) \subset C^\infty(\Gamma).$$

(ii) *If* $\psi \in \mathcal{D}'(\Gamma)$, $T'\psi \in B^{\sigma,p'}(\Gamma)$ *for* $\sigma \in \mathbf{R}$, *then it follows that* $\psi \in B^{\sigma,p'}(\Gamma)$. *In particular, we have the assertion*

$$\mathcal{N}\left((\mathcal{T}_p)'\right) \subset C^\infty(\Gamma).$$

Indeed, it suffices to note that the null space $\mathcal{N}((\mathcal{T}_p)')$ of the transpose $(\mathcal{T}_p)'$ can be characterized as follows (cf. [**36**, Lemma 8.4.8]):

$$\begin{aligned}\mathcal{N}\left((\mathcal{T}_p)'\right) &= \left\{\psi \in B^{-2+1/p,p'}(\Gamma) : \langle \psi, T\varphi\rangle = 0 \quad \text{for all } \varphi \in \mathcal{D}\left(\mathcal{T}_p\right)\right\} \\ &\subset \left\{\psi \in B^{-2+1/p,p'}(\Gamma) : \langle T'\psi, \varphi\rangle = \langle \psi, T\varphi\rangle = 0 \quad \text{for all } \varphi \in C^\infty(\Gamma)\right\}.\end{aligned}$$

This proves that

$$\mathcal{N}\left((\mathcal{T}_p)'\right) \subset \{\psi \in \mathcal{D}'(\Gamma) : T'\psi = 0\} \subset C^\infty(\Gamma).$$

Finally, by combining Proposition 6.9, Theorems 6.7 and 6.10 and Corollary 6.6 we obtain the following fundamental result of the indices of the operators $\mathfrak{A}_p$:

THEOREM 6.11 (Indices). *If the condition* (H) *is satisfied, then the index* $\operatorname{ind}\mathfrak{A}_p = \operatorname{ind}\mathcal{T}_p$ *is* independent *of* p *for all* $1 < p < \infty$.

7. Agmon's method.

First, we introduce an auxiliary variable y of the unit circle

$$S = \mathbf{R}/2\pi\mathbf{Z},$$

and replace the complex parameter λ by the second-order differential operator

$$-e^{i\theta}\frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi.$$

More precisely, if we express the complex parameter λ in the form

$$\lambda = r^2\, e^{i\theta}, \quad r \geq 0,\ -\pi < \theta < \pi,$$

then we replace the differential operator $A - \lambda = A - r^2\, e^{i\theta}$ defined in Ω by the second-order differential operator

$$\widetilde{\Lambda}(\theta) := A + e^{i\theta}\frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi,$$

defined in the product domain $\Omega \times S$. We remark that the operator $\widetilde{\Lambda}(\theta)$ is *strongly uniform elliptic* for $-\pi < \theta < \pi$ in $\Omega \times S$. Moreover, up to an appropriate constant of proportionality, the fundamental solution $\widetilde{E}(x, y; \theta)$ of $\widetilde{\Lambda}(\theta)$ is explicitly given by the following formula (due to K. Uchiyama):

$$\widetilde{E}(x, y; \theta) = \left(\frac{n + e^{i\theta/2}}{n+1}\,|x|^2 + \frac{n\,e^{-i\theta/2} + 1}{n+1}\,y^2\right)^{(1-n)/2}, \quad i = \sqrt{-1}.$$

Now we consider instead of the original oblique derivative problem with spectral parameter

$$\begin{cases} (A-\lambda)u = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = a(x') \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha(x') \cdot u = 0 & \text{on } \Gamma \end{cases} \tag{1.1}$$

the following oblique derivative problem in the product domain $\Omega \times S$: Given a function $\widetilde{f}(x,y)$ defined in $\Omega \times S$, find a function $\widetilde{u}(x,y)$ in $\Omega \times S$ such that

$$\begin{cases} \widetilde{\Lambda}(\theta)\widetilde{u} = \left(A + e^{i\theta} \dfrac{\partial^2}{\partial y^2} \right) \widetilde{u} = \widetilde{f} & \text{in } \Omega \times S, \\ B\boldsymbol{\gamma}\widetilde{u} = a(x') \dfrac{\partial \widetilde{u}}{\partial \boldsymbol{n}} + \alpha(x') \cdot \widetilde{u} = 0 & \text{on } \Gamma \times S. \end{cases} \tag{7.1}$$

In order to prove Theorem 2.4, we associate with the oblique derivative problem (7.1) a densely defined, closed linear operator

$$\widetilde{\mathfrak{A}}_p(\theta) : L^p(\Omega \times S) \longrightarrow L^p(\Omega \times S)$$

in the Banach space $L^p(\Omega \times S)$ as follows:

(a) The domain $\mathcal{D}(\widetilde{\mathfrak{A}}_p(\theta))$ of definition of $\widetilde{\mathfrak{A}}_p(\theta)$ is the space

$$\mathcal{D}(\widetilde{\mathfrak{A}}_p(\theta) = \left\{ \widetilde{u} \in W^{2,p}(\Omega \times S) : \ B\boldsymbol{\gamma}\widetilde{u} = 0 \right\}.$$

(b) $\widetilde{\mathfrak{A}}_p(\theta)\widetilde{u} = \widetilde{\Lambda}(\theta)\widetilde{u}$ for every $\widetilde{u} \in \mathcal{D}(\widetilde{\mathfrak{A}}_p(\theta))$.

Here $\widetilde{\Lambda}(\theta)\widetilde{u}$ and $B\widetilde{u}$ are taken in the sense of distributions.

The next theorem asserts that if the condition (H) is satisfied, then the operator $\widetilde{\mathfrak{A}}_p(\theta)$ is a Fredholm operator for every $1 < p < \infty$:

THEOREM 7.1. *Let $1 < p < \infty$ and $\theta \in (-\pi, \pi)$. Assume that the condition* (H) *is satisfied. Then the operator $\widetilde{\mathfrak{A}}_p(\theta) : L^p(\Omega \times S) \to L^p(\Omega \times S)$ is a* Fredholm operator*. Moreover, there exists a constant $\widetilde{C}(\theta) > 0$ depending on θ such that the* a priori *estimate*

$$\|\widetilde{u}\|_{W^{2,p}(\Omega \times S)} \leq \widetilde{C}(\theta) \left(\left\| \widetilde{\Lambda}(\theta)\widetilde{u} \right\|_{L^p(\Omega \times S)} + \|\widetilde{u}\|_{L^p(\Omega \times S)} \right) \tag{7.2}$$

holds true for all functions $\widetilde{u} \in \mathcal{D}(\widetilde{\mathfrak{A}}_p(\theta))$.

The proof of Theorem 7.1 will be given in Section 11, due to its length.

8. The Dirichlet problem for Agmon's method.

In this section, by using the theory of pseudo-differential operators we consider the Dirichlet problem for the second-order, *strongly uniform elliptic* differential operator

$$\widetilde{\Lambda}(\theta) = A + e^{i\theta} \frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi,$$

in the framework of L^p Sobolev spaces on the product domain $\Omega \times S$: Given functions $\widetilde{f}$ and $\widetilde{\varphi}$ defined in $\Omega \times S$ and on $\Gamma \times S$, respectively, find a function $\widetilde{u}$ in $\Omega \times S$ such that

$$\begin{cases} \widetilde{\Lambda}(\theta)\widetilde{u} = \widetilde{f} & \text{in } \Omega \times S, \\ \widetilde{u} = \widetilde{\varphi} & \text{on } \Gamma \times S. \end{cases} \tag{8.1}$$

8.1. Symbol of the operator $\widetilde{\Lambda}(\theta)$.

In this subsection, we calculate explicitly the symbol of the elliptic differential operator

$$\widetilde{\Lambda}(\theta) = A + e^{i\theta} \frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi.$$

However, it is easy to see that there is a *homotopy* through elliptic symbols between the two elliptic differential operators

$$\widetilde{\Lambda}_1(\theta) = \widetilde{\Lambda}(\theta) = A + e^{i\theta} \frac{\partial^2}{\partial y^2}, \qquad \widetilde{\Lambda}_0(\theta) = \Delta + e^{i\theta} \frac{\partial^2}{\partial y^2}.$$

For example, we may take

$$\widetilde{\Lambda}_t(\theta) := t\, A + (1-t)\, \Delta + e^{i\theta} \frac{\partial^2}{\partial y^2} \quad \text{for } 0 \leq t \leq 1.$$

Therefore, we have only to calculate explicitly the symbol of the differential operator $\widetilde{\Lambda}_0(\theta)$ for the Laplacian $A = \Delta$:

$$\widetilde{\Lambda}_0(\theta) = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + e^{i\theta} \frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi.$$

To do so, let

$$(x, \xi, y, \eta) = (x_1, \dots, x_n, \xi_1, \dots, \xi_n, y, \eta)$$

be a local coordinate system of the cotangent bundle $T^*(\Omega) \times T^*(S) = T^*(\Omega \times S)$. Then the complete symbol of $\widetilde{\Lambda}_0(\theta)$ is equal to the following:

$$-\left(|\xi|^2 + \cos\theta \cdot \eta^2\right) - \sqrt{-1}\, \sin\theta \cdot \eta^2.$$

Moreover, we remark that

$$\begin{aligned} &\left(|\xi|^2 + \cos\theta \cdot \eta^2\right) + \sqrt{-1}\, \sin\theta \cdot \eta^2 \\ &\quad = \left(\frac{|\xi|^2 + \cos\theta \cdot \eta^2 + \sqrt{-1}\, \sin\theta \cdot \eta^2}{|\xi|^2 + \eta^2} \right) \left(|\xi|^2 + \eta^2\right), \end{aligned}$$

and further that the middle term is estimated as follows:

$$\left| \frac{|\xi|^2 + \cos\theta \cdot \eta^2 + \sqrt{-1}\, \sin\theta \cdot \eta^2}{|\xi|^2 + \eta^2} \right| \geq \sqrt{\frac{1 + \cos\theta}{2}} \quad \text{for all } \theta \in (-\pi, \pi).$$

In this way, we are reduced to the study of the Dirichlet problem for the usual Laplacian ($\theta := 0$)

$$\widetilde{\Lambda}_0(0) = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y^2} \quad \text{in } \Omega \times S.$$

8.2. Unique solvability of the Dirichlet problem for Agmon's method.
We can prove the following *existence and uniqueness theorem* for the Dirichlet problem (8.1) in the framework of L^p Sobolev spaces (cf. [**3**], [**17**], [**25**], [**44**]):

THEOREM 8.1. *Let $1 < p < \infty$ and $s > -2 + 1/p$. Then the Dirichlet problem* (8.1) *has a unique solution $\widetilde{u}$ in the space $W^{s+2,p}(\Omega \times S)$ for any $\widetilde{f} \in W^{s,p}(\Omega \times S)$ and any $\widetilde{\varphi} \in B^{s+2-1/p,p}(\Gamma \times S)$. Moreover, the unique solution $\widetilde{u}$ of the Dirichlet problem* (8.1) *can be expressed as follows* (*cf. formula* (5.3)):

$$\widetilde{u} = \widetilde{Q}(\theta)\widetilde{E}\widetilde{f}\Big|_{\Omega \times S} + \widetilde{P}(\theta)\left(\widetilde{\varphi} - \left(\widetilde{Q}(\theta)\widetilde{E}\widetilde{f}\right)\Big|_{\Gamma \times S}\right) \quad \text{in } \Omega \times S.$$

Here:

(1) $\widetilde{Q}(\theta)$ *is the* fundamental solution *of* $\widetilde{\Lambda}(\theta)$.

(2) $\widetilde{P}(\theta)$ *is the* Poisson kernel *for* $\widetilde{\Lambda}(\theta)$.

(3) $\widetilde{E} : W^{s,p}(\Omega \times S) \to W^{s,p}(M \times S)$ *is the Seeley extension operator* (*see Theorem* 3.4).

9. Special reduction to the boundary $\Gamma \times S$.

In this section, we reduce the homogeneous oblique derivative problem (7.1) to the study of a first-order, pseudo-differential operator $\widetilde{T}(\theta)$ on the boundary $\Gamma \times S$ (Proposition 9.1), just as in Smith [**31**] and Guan–Sawyer [**18**].

Step 1: Let $\widetilde{f} \in W^{s,p}(\Omega \times S)$ with $1 < p < \infty$ and $s > -1 + 1/p$. We denote by $\widetilde{f}_0$ the extension of $\widetilde{f}$ to Euclidean space $\mathbf{R}^{n+1}$ with $\widetilde{f}_0 \equiv 0$ outside $\Omega \times S$:

$$\widetilde{f}_0(x,y) = \begin{cases} \widetilde{f}(x,y) & \text{for } (x,y) \in \Omega \times S, \\ 0 & \text{for } (x,y) \in \mathbf{R}^{n+1} \setminus (\Omega \times S). \end{cases}$$

If $\widetilde{Q}(\theta)$ is the fundamental solution of the second-order, strongly uniform elliptic differential operator

$$\widetilde{\Lambda}(\theta) = A + e^{i\theta}\,\frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi,$$

then it follows from the *transmission property* of the fundamental solution $\widetilde{Q}(\theta)$ (see Boutet de Monvel [**8**], Rempel–Schulze [**28**, p. 161, Theorem 2]) that

$$\left(\widetilde{Q}(\theta)\widetilde{f}_0\right)\Big|_{\Omega \times S} \in W^{s+2,p}(\Omega \times S) \quad \text{for } s > -1 + 1/p,$$

so that

$$B\boldsymbol{\gamma}\left(\widetilde{Q}(\theta)\widetilde{f}_0\right) \in B^{s+1-1/p,p}(\Gamma \times S) \quad \text{for } s > -1 + 1/p.$$

If a function $\widetilde{u}$ satisfies the equation

$$\widetilde{\Lambda}(\theta)\widetilde{u} = \widetilde{f} \quad \text{in } \Omega \times S,$$

then it follows that

$$\widetilde{\Lambda}(\theta)\left(\widetilde{Q}(\theta)\widetilde{f}_0 - \widetilde{u}\right) = \widetilde{f}_0 - \widetilde{f} = 0 \quad \text{in } \Omega \times S.$$

We let

$$\widetilde{v} = \left(\widetilde{Q}(\theta)\widetilde{f}_0 - \widetilde{u}\right)\Big|_{\Gamma \times S}.$$

If $\widetilde{P}(\theta)$ is the Poisson kernel of the elliptic differential operator $\widetilde{\Lambda}(\theta)$ in the domain $\Omega \times S$, then we have the formula

$$\widetilde{Q}(\theta)\widetilde{f}_0 - \widetilde{u} = \widetilde{P}(\theta)\widetilde{v} \quad \text{in } \Omega \times S,$$

or equivalently,

$$\widetilde{u} = \widetilde{Q}(\theta)\widetilde{f}_0 - \widetilde{P}(\theta)\widetilde{v} \quad \text{in } \Omega \times S.$$

Then we find that the boundary condition

$$B\boldsymbol{\gamma}\widetilde{u} = \frac{\partial \widetilde{u}}{\partial \boldsymbol{\nu}} = \alpha(x') \cdot \widetilde{u} + a(x') \left.\frac{\partial \widetilde{u}}{\partial \boldsymbol{n}}\right|_{\Gamma \times S} = 0 \quad \text{on } \Gamma \times S$$

is equivalent to the following condition:

$$\begin{aligned}
&B\boldsymbol{\gamma}\left(\widetilde{Q}(\theta)\widetilde{f}_0\right) - B\boldsymbol{\gamma}\left(\widetilde{P}(\theta)\widetilde{v}\right) \\
&= B\boldsymbol{\gamma}\left(\widetilde{Q}(\theta)\widetilde{f}_0\right) - \alpha(x') \cdot (\widetilde{P}(\theta)\widetilde{v})|_{\Gamma \times S} - a(x')\frac{\partial}{\partial \boldsymbol{n}}(\widetilde{P}(\theta)\widetilde{v})\Big|_{\Gamma \times S} \\
&= B\boldsymbol{\gamma}\left(\widetilde{Q}(\theta)\widetilde{f}_0\right) - \left(\alpha(x') \cdot \widetilde{v} + a(x')\frac{\partial}{\partial \boldsymbol{n}}(\widetilde{P}(\theta)\widetilde{v})\Big|_{\Gamma \times S}\right) \\
&= 0 \quad \text{on } \Gamma \times S.
\end{aligned} \tag{9.1}$$

Now we let

$$\begin{aligned}
\widetilde{T}(\theta) : C^\infty(\Gamma \times S) &\longrightarrow C^\infty(\Gamma \times S) \\
\widetilde{\varphi} &\longmapsto B\boldsymbol{\gamma}\left(\widetilde{P}(\theta)\widetilde{\varphi}\right),
\end{aligned}$$

then we have the formula

$$\widetilde{T}(\theta) = a(x')\widetilde{\Pi}(\theta) + \alpha(x'), \tag{9.2}$$

where $\widetilde{\Pi}(\theta)$ is the Dirichlet–Neumann operator defined as follows:

$$\widetilde{\Pi}(\theta)\widetilde{\varphi} := \frac{\partial}{\partial \boldsymbol{n}} \left(\widetilde{P}(\theta)\widetilde{\varphi} \right)\Big|_{\Gamma \times S} \quad \text{for all } \widetilde{\varphi} \in C^\infty(\Gamma \times S).$$

Therefore, we obtain from formulas (9.1) and (9.2) that

$$\widetilde{T}(\theta)\widetilde{v} = \alpha(x') \cdot \widetilde{v} + a(x')\widetilde{\Pi}(\theta)\widetilde{v} = B\boldsymbol{\gamma}\left(\widetilde{Q}(\theta)\widetilde{f}_0\right) \quad \text{on } \Gamma \times S. \tag{9.3}$$

Step 2: On the other hand, since the function $\widetilde{f}_0$ is compactly supported in $\mathbf{R}^{n+1}$, it follows that the function $\widetilde{Q}(\theta)\widetilde{f}_0$ satisfies the homogeneous equation

$$\widetilde{\Lambda}(\theta)\widetilde{Q}(\theta)\widetilde{f}_0 = \widetilde{f}_0 = 0$$

in the *exterior domain*

$$\overline{\Omega \times S}^c = \mathbf{R}^{n+1} \setminus (\overline{\Omega} \times S),$$

and vanishes at infinity.

If $\widetilde{P}^{\mathrm{ext}}(\theta)$ is the *Poisson kernel* of the elliptic differential operator $\widetilde{\Lambda}(\theta)$ in the exterior domain $\overline{\Omega \times S}^c$, then we have the formula

$$\widetilde{Q}(\theta)\widetilde{f}_0 = \widetilde{P}^{\mathrm{ext}}(\theta)\left(\widetilde{Q}(\theta)\widetilde{f}_0|_{\Gamma\times S}\right) \quad \text{in } \overline{\Omega \times S}^c. \tag{9.4}$$

We recall that the analysis of the Poisson kernel $\widetilde{P}^{\mathrm{ext}}(\theta)$ can be reduced to that of compact domains by using the Kelvin transform (see [**5**, Chapter 4]).

Hence we have, by formula (9.4),

$$B\boldsymbol{\gamma}\left(\widetilde{Q}(\theta)\widetilde{f}_0\right) = B\boldsymbol{\gamma}\left(\widetilde{P}^{\mathrm{ext}}(\theta)\left(\widetilde{Q}(\theta)\widetilde{f}_0|_{\Gamma\times S}\right)\right).$$

However, it should be noticed that the outward normal field $\boldsymbol{n}$ to $\Gamma \times S$ in the interior domain $\Omega \times S$ is the *inward* normal for the exterior domain $\overline{\Omega \times S}^c$.

Therefore, if we define the Dirichlet–Neumann operator $\widetilde{\Pi}^{\mathrm{ext}}(\theta)$ by the formula

$$\widetilde{\Pi}^{\mathrm{ext}}(\theta)\widetilde{\varphi} := \frac{\partial}{\partial(-\boldsymbol{n})}\left(\widetilde{P}^{\mathrm{ext}}(\theta)\widetilde{\varphi}\right)\Big|_{\Gamma\times S} \quad \text{for all } \widetilde{\varphi} \in C^\infty(\Gamma \times S),$$

then we have the formula

$$\begin{aligned} B\boldsymbol{\gamma}\left(\widetilde{Q}(\theta)\widetilde{f}_0\right) &= B\boldsymbol{\gamma}\left(\widetilde{P}^{\mathrm{ext}}(\theta)\left(\widetilde{Q}(\theta)\widetilde{f}_0|_{\Gamma\times S}\right)\right) \\ &= \left(\alpha(x') - a(x')\frac{\partial}{\partial(-\boldsymbol{n})}\left(\widetilde{P}^{\mathrm{ext}}(\theta)\left(\widetilde{Q}(\theta)\widetilde{f}_0|_{\Gamma\times S}\right)\right)\right)\Big|_{\Gamma\times S} \\ &= \left(\alpha(x') - a(x')\widetilde{\Pi}^{\mathrm{ext}}(\theta)\right)\left(\widetilde{Q}(\theta)\widetilde{f}_0|_{\Gamma\times S}\right) \quad \text{on } \Gamma \times S. \end{aligned} \tag{9.5}$$

Step 3: By combining formulas (9.3) and (9.5), we have proved the following fundamental proposition (cf. Proposition 6.2):

PROPOSITION 9.1. *Let $\widetilde{f} \in W^{s,p}(\Omega \times S)$ with $1 < p < \infty$ and $s > -1 + 1/p$. Then the homogeneous oblique derivative problem* (7.1)

$$\begin{cases} \widetilde{A}(\theta)\widetilde{u} = \left(A + e^{i\theta}\,\dfrac{\partial^2}{\partial y^2}\right)\widetilde{u} = \widetilde{f} & \text{in } \Omega \times S, \\ B\boldsymbol{\gamma}\widetilde{u} = a(x')\dfrac{\partial \widetilde{u}}{\partial \boldsymbol{n}} + \alpha(x') \cdot \widetilde{u} = 0 & \text{on } \Gamma \times S \end{cases}$$

can be reduced to the study of the pseudo-differential equation

$$\widetilde{T}(\theta)\widetilde{v} = \left(\alpha(x') - a(x')\widetilde{\Pi}^{\mathrm{ext}}(\theta)\right)\left(\widetilde{Q}(\theta)\widetilde{f}_0|_{\Gamma\times S}\right) \quad \text{on } \Gamma \times S, \tag{9.6}$$

where $\widetilde{T}(\theta) = \alpha(x') + a(x')\widetilde{\Pi}(\theta)$, $\widetilde{v} = (\widetilde{Q}(\theta)\widetilde{f}_0 - \widetilde{u})|_{\Gamma\times S}$ *and*

$$\widetilde{Q}(\theta)\widetilde{f}_0\Big|_{\Gamma\times S} \in B^{s+2-1/p,p}(\Gamma \times S).$$

10. Symbolic calculus.

The purpose of this section is to prove that if the condition (H) is satisfied, then the operator $\widetilde{\mathcal{T}_p}(\theta)$ is a Fredholm operator for every $1 < p < \infty$ (Proposition 10.4). First, we show that the pseudo-differential operators $\widetilde{T}(\theta)$ and $\widetilde{T}(\theta)'$ are both subelliptic with loss of some δ if the condition (H) is satisfied (Proposition 10.3). To do so, we have only to calculate the principal symbol $\widetilde{t_1}(x'', t, \xi'', \tau, y, \eta; \theta)$ of the pseudo-differential operator $\widetilde{T}(\theta) = a(x')\widetilde{\Pi}(\theta) + \alpha(x')$ in the case where $A = \Delta$:

$$\widetilde{A}_0(\theta) = \Delta + e^{i\theta}\,\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + e^{i\theta}\,\frac{\partial^2}{\partial y^2}, \quad -\pi < \theta < \pi.$$

The essential point is how to reduce the study of the general case $\widetilde{T}(\theta)$, $-\pi < \theta < \pi$, to the simplest case $\widetilde{T}(0)$ when $\theta = 0$ (Proposition 10.2). The proof of Proposition 10.4 can be visualized as in Diagram 6 below.

10.1. Principal symbol of $\widetilde{T}(\theta)$.

In this subsection, we calculate the principal symbols of the pseudo-differential operators $\widetilde{\Pi}(\theta)$ and $\widetilde{T}(\theta)$.

Step 1: First, we calculate the symbol of the pseudo-differential operator $\widetilde{\Pi}(\theta)$. To do this, let

$$(x', \xi', y, \eta) = (x_1, \dots, x_{n-1}, \xi_1, \dots, \xi_{n-1}, y, \eta)$$

be a local coordinate system of the cotangent bundle $T^*(\Gamma) \times T^*(S) = T^*(\Gamma \times S)$. Then it is known that the complete symbol of $\widetilde{\Pi}(\theta)$ is given by the following formula (cf. [**36**, Section 10.2]):

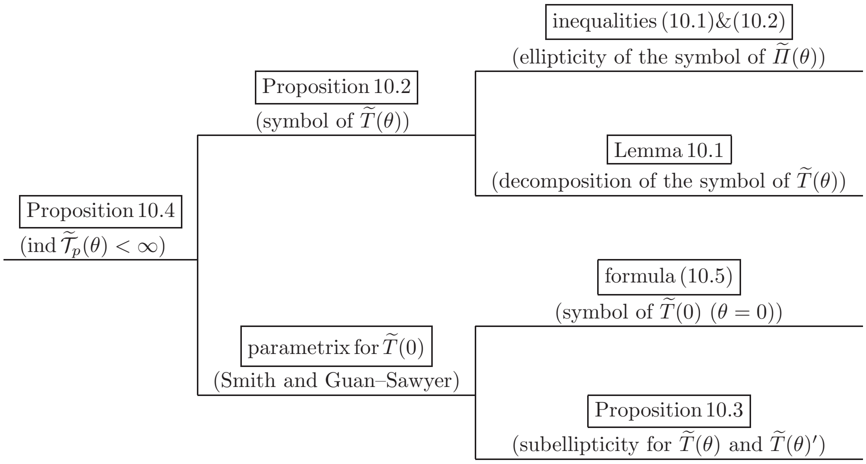

Diagram 6.

$$\begin{aligned}
&\left(\widetilde{p}_1(x', \xi', y, \eta; \theta) + \sqrt{-1}\, \widetilde{q}_1(x', \xi', y, \eta; \theta)\right) \\
&\quad + \left(\widetilde{p}_0(x', \xi', y, \eta; \theta) + \sqrt{-1}\, \widetilde{q}_0(x', \xi', y, \eta; \theta)\right) + \text{terms of order} \leq -1,
\end{aligned}$$

where $\widetilde{p}_1(x', \xi', y, \eta; \theta) > 0$ on the bundle $T^*(\Gamma \times S) \setminus \{0\}$ of non-zero cotangent vectors, for $-\pi < \theta < \pi$. More precisely, we have the formula

$$\begin{aligned}
&\widetilde{p}_1(x', \xi', y, \eta; \theta) \\
&\quad = \frac{1}{\sqrt{2}} \left[\left[\left(|\xi'|^2 + \cos\theta \cdot \eta^2\right)^2 + \sin^2\theta \cdot \eta^4 \right]^{1/2} + \left(|\xi'|^2 + \cos\theta \cdot \eta^2\right) \right]^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{q}_1(x', \xi', y, \eta; \theta) \\
&\quad = \frac{1}{\sqrt{2}} \left[\left[\left(|\xi'|^2 + \cos\theta \cdot \eta^2\right)^2 + \sin^2\theta \cdot \eta^4 \right]^{1/2} - \left(|\xi'|^2 + \cos\theta \cdot \eta^2\right) \right]^{1/2}.
\end{aligned}$$

Hence we have the formula

$$\widetilde{p}_1(x', \xi', y, \eta; \theta)^2 + \widetilde{q}_1(x', \xi', y, \eta; \theta)^2 = \left[\left(|\xi'|^2 + \cos\theta \cdot \eta^2\right)^2 + \sin^2\theta \cdot \eta^4 \right]^{1/2}.$$

Moreover, it is easy to see that

$$\widetilde{p}_1(x', \xi', y, \eta; \theta) \geq \begin{cases} \sqrt{\dfrac{1 + \cos\theta}{2}} \left(|\xi'|^2 + \eta^2\right)^{1/2} & \text{if } |\theta| \leq \pi/2, \\ \dfrac{1}{\sqrt{2}} \sqrt[4]{\dfrac{1 + \cos\theta}{1 - \cos\theta}} \left(|\xi'|^2 + \eta^2\right)^{1/2} & \text{if } \pi/2 < |\theta| < \pi, \end{cases} \tag{10.1}$$

and that

$$\widetilde{p}_1(x',\xi',y,\eta;\theta)^2+\widetilde{q}_1(x',\xi',y,\eta;\theta)^2\geq\frac{1+\cos\theta}{2}\left(|\xi'|^2+\eta^2\right). \tag{10.2}$$

Therefore, we obtain that the operator

$$\widetilde{T}(\theta)=a(x')\widetilde{\Pi}(\theta)+\alpha(x')$$

is a classical, pseudo-differential operator of first order on the boundary $\Gamma\times S$ and further that its complete symbol is given by the following formula:

$$\begin{aligned}&a(x')\,\widetilde{p}_1(x',\xi',y,\eta;\theta)\\&\quad+\sqrt{-1}\,\left(a(x')\,\widetilde{q}_1(x',\xi',y,\eta;\theta)+\left[\sum_{k=1}^{n-1}\alpha^k(x')\xi_k\right]\right)\\&\quad+\text{ terms of order}\leq 0,\quad(x',\xi',y,\eta)=(x_1,\ldots,x_{n-1},\xi_1,\ldots,\xi_{n-1},y,\eta).\end{aligned}$$

By straightening out the vector field $\alpha(x')$ as in Proposition 6.8, we may assume that the principal symbol $\widetilde{t}_1(x'',t,\xi'',\tau,y,\eta;\theta)$ of $\widetilde{T}(\theta)$ is equal to the following:

$$\begin{aligned}&\widetilde{t}_1(x'',t,\xi'',\tau,y,\eta;\theta)\\&\quad=a(x'',t)\,\widetilde{p}_1(x'',t,\xi'',\tau,y,\eta;\theta)+\sqrt{-1}\,\left(a(x'',t)\,\widetilde{q}_1(x'',t,\xi'',\tau,y,\eta;\theta)+\tau\right),\\&\quad\ (x',\xi',y,\eta)=((x'',t),(\xi'',\tau),y,\eta).\end{aligned} \tag{10.3}$$

Step 2: Secondly, we decompose the principal symbol $\widetilde{t}_1(x'',t,\xi'',\tau,y,\eta;\theta)$ of $\widetilde{T}(\theta)$ as follows:

$$\begin{aligned}&\widetilde{t}_1(x'',t,\xi'',\tau,y,\eta;\theta)\\&\quad=\sqrt{-1}\,\left(\tau+a(x'',t)\,\widetilde{q}_1(x'',t,\xi'',\tau,y,\eta;\theta)\right)+a(x'',t)\,\widetilde{p}_1(x'',t,\xi'',\tau,y,\eta;\theta)\\&\quad=\left(\frac{\sqrt{-1}\,\left(\tau+a(x'',t)\,\widetilde{q}_1(x'',t,\xi'',\tau,y,\eta;\theta)\right)+a(x'',t)\,\widetilde{p}_1(x'',t,\xi'',\tau,y,\eta;\theta)}{\sqrt{-1}\,\tau+a(x'',t)\,\sqrt{|\xi''|^2+\tau^2+\eta^2}}\right)\\&\qquad\times\left(\sqrt{-1}\,\tau+a(x'',t)\sqrt{|\xi''|^2+\tau^2+\eta^2}\right).\end{aligned}$$

However, the middle term

$$\left(\frac{\sqrt{-1}\,\left(\tau+a(x'',t)\,\widetilde{q}_1(x'',t,\xi'',\tau,y,\eta;\theta)\right)+a(x'',t)\,\widetilde{p}_1(x'',t,\xi'',\tau,y,\eta;\theta)}{\sqrt{-1}\,\tau+a(x'',t)\,\sqrt{|\xi''|^2+\tau^2+\eta^2}}\right)$$

is an *elliptic symbol* on $\Gamma\times S$. Indeed, we can prove the following:

LEMMA 10.1. *There exists a constant $\gamma(\theta)>0$ depending on $\theta\in(-\pi,\pi)$ such that*

$$\left|\frac{\sqrt{-1}\,\left(\tau+a(x'',t)\widetilde{q}_1(x'',t,\xi'',\tau,y,\eta;\theta)\right)+a(x'',t)\widetilde{p}_1(x'',t,\xi'',\tau,y,\eta;\theta)}{\sqrt{-1}\,\tau+a(x'',t)\sqrt{|\xi''|^2+\tau^2+\eta^2}}\right|^2$$

$$= \frac{\left(\tau + a(x'',t)\widetilde{q}_1(x'',t,\xi'',\tau,y,\eta;\theta)\right)^2 + a(x'',t)^2\widetilde{p}_1(x'',t,\xi'',\tau,y,\eta;\theta)^2}{\tau^2 + a(x'',t)^2\left(|\xi''|^2+\tau^2+\eta^2\right)}$$
$$\geq \gamma(\theta). \tag{10.4}$$

Summing up, we have proved the following proposition:

PROPOSITION 10.2. *We may assume that the principal symbol $\widetilde{t}_1(x',\xi',y,\eta;\theta)$ of the pseudo-differential operator*

$$\widetilde{T}(\theta) = a(x')\widetilde{\Pi}(\theta) + \alpha(x'), \quad -\pi < \theta < \pi,$$

is equal to the following formula ($\theta := 0$ *in formula* (10.3)):

$$\begin{aligned}\widetilde{t}_1(x',\xi',y,\eta;\theta) &= \widetilde{t}_1(x'',t,\xi'',\tau,y,\eta;\theta)\\ &= a(x'',t)\sqrt{|\xi''|^2+\tau^2+\eta^2} + \sqrt{-1}\,\tau.\end{aligned} \tag{10.5}$$

In this way, we are completely reduced to the study of the principal symbol $\widetilde{t}_1(x'',t,\xi'',\tau,y,\eta;\theta)$ when $A = \Delta$ and $\theta = 0$.

10.2. Subellipticity of $\widetilde{T}(\theta)$ and $\widetilde{T}(\theta)'$.

In light of formula (10.5), by applying Theorem 4.1 (Egorov–Hörmander) to the pseudo-differential operators $\widetilde{T}(\theta)$ and $\widetilde{T}(\theta)'$ just as in Proposition 6.8 we can prove the following proposition:

PROPOSITION 10.3. *Assume that the hypothesis* (H) *is satisfied. Then the pseudo-differential operators $\widetilde{T}(\theta)$ and $\widetilde{T}(\theta)'$ are both* subelliptic *with loss of some δ on $\Gamma \times S$ where $2k/(2k+1) \leq \delta < 1$.*

Now we associate with the oblique derivative problem (7.1) a densely defined, closed linear operator

$$\widetilde{\mathcal{T}}_p(\theta) : L^p(\Gamma \times S) \longrightarrow L^p(\Gamma \times S)$$

in the Banach space $L^p(\Gamma \times S)$ as follows:

(1) The domain $\mathcal{D}(\widetilde{\mathcal{T}}_p(\theta))$ of definition of $\widetilde{\mathcal{T}}_p(\theta)$ is the space

$$\mathcal{D}(\widetilde{\mathcal{T}}_p(\theta)) = \left\{\widetilde{\varphi} \in L^p(\Gamma \times S) : \widetilde{T}(\theta)\widetilde{\varphi} \in L^p(\Gamma \times S)\right\}.$$

(2) $\widetilde{\mathcal{T}}_p(\theta)\widetilde{\varphi} = \widetilde{T}(\theta)\widetilde{u}$ for every $\widetilde{\varphi} \in \mathcal{D}(\widetilde{\mathcal{T}}_p(\theta))$.

By Smith [**31**] and Guan–Sawyer [**18**], we find that if the condition (H) is satisfied, then the pseudo-differential operator $\widetilde{\mathcal{T}}_p(\theta)$ has a *parametrix* (see Remark 2.1).

Summing up, we have proved the following fundamental proposition:

PROPOSITION 10.4. *If the condition* (H) *is satisfied, then the operator $\widetilde{\mathcal{T}}_p(\theta)$ is a* Fredholm operator *for every $1 < p < \infty$.*

11. Proof of Theorem 7.1.

This section is devoted to the proof of Theorem 7.1. More precisely, we show how Theorem 7.1 follows from Propositions 9.1 and 10.4. The proof of Theorem 7.1 is divided into three steps.

Step 1: By virtue of Propositions 9.1 and 10.4, we can obtain the following theorem essentially due to Smith [**31**, Main Theorem] and Guan–Sawyer [**18**, Theorem 2, part (i)]:

THEOREM 11.1. *Let $1 < p < \infty$ and $\theta \in (-\pi, \pi)$. Assume that the condition* (H) *is satisfied. If $\widetilde{f} \in L^p(\Omega \times S)$, then every solution $\widetilde{u}$ of the homogeneous oblique derivative problem*

$$\begin{cases} \widetilde{\Lambda}(\theta)\widetilde{u} = \widetilde{f} & \text{in } \Omega \times S, \\ B\boldsymbol{\gamma}\widetilde{u} = 0 & \text{on } \Gamma \times S \end{cases} \tag{7.1}$$

belongs to the Sobolev space $W^{2,p}(\Omega \times S)$ and can be expressed, unique modulo the null space $\mathcal{N}(\widetilde{\mathfrak{A}}_p(\theta))$, in the form

$$\begin{aligned} \widetilde{u} = {} & \widetilde{Q}(\theta)\, \widetilde{f_0}\Big|_{\Omega\times S} - \widetilde{P}(\theta) \left(\widetilde{S}(\theta) \left(\alpha(x') - a(x')\widetilde{\Pi}^{\mathrm{ext}}(\theta) \right) \left(\widetilde{Q}(\theta)\widetilde{f_0}\Big|_{\Gamma\times S} \right) \right) \\ & - \widetilde{P}(\theta) \left(\widetilde{\Pi}_{\mathrm{c}}(\theta) \left(\widetilde{Q}(\theta)\widetilde{f_0}\Big|_{\Gamma\times S} \right) \right) \quad \text{in } \Omega \times S. \end{aligned} \tag{11.1}$$

Here:

1. $\widetilde{Q}(\theta)$ *is the fundamental solution of* $\widetilde{\Lambda}(\theta)$.
2. $\widetilde{P}(\theta)$ *is the Poisson kernel for* $\widetilde{\Lambda}(\theta)$.
3. $\widetilde{T}(\theta) = B\boldsymbol{\gamma}\widetilde{P}(\theta)$ *is the first-order pseudo-differential operator on* $\Gamma \times S$.
4. $\widetilde{S}(\theta)$ *is a unique* right inverse *of $\widetilde{T}(\theta)$ that annihilates the cokernel of the range $\mathcal{R}(\widetilde{T}(\theta))$ and has range perpendicular to the null space $\mathcal{N}(\widetilde{T}(\theta))$.*
5. $\widetilde{R}(\theta) := \widetilde{S}(\theta) \left(\alpha(x') - a(x')\widetilde{\Pi}^{\mathrm{ext}}(\theta) \right)$ *is* bounded *on $B^{\sigma-1/p,p}(\Gamma \times S)$ for every* $\sigma \in \mathbf{R}$ (see [**31**, Theorem 3.13], [**18**, Section 6]).
6. $\widetilde{\Pi}_{\mathrm{c}}(\theta)$ *is the* Calderón projector *onto the null space $\mathcal{N}(\widetilde{T}(\theta))$ of $\widetilde{T}(\theta)$, and is* bounded *on $B^{\sigma-1/p,p}(\Gamma \times S)$ for every $\sigma \in \mathbf{R}$.*

Indeed, the proof of Theorem 11.1 can be carried out as in Diagram 7 below.

Step 2: By using the representation formula (11.1) of the solution $\widetilde{u}$, we can obtain the following regularity result for the homogeneous oblique derivative problem (7.1) (cf. Guan–Sawyer [**18**, Theorem 2, part (i)]):

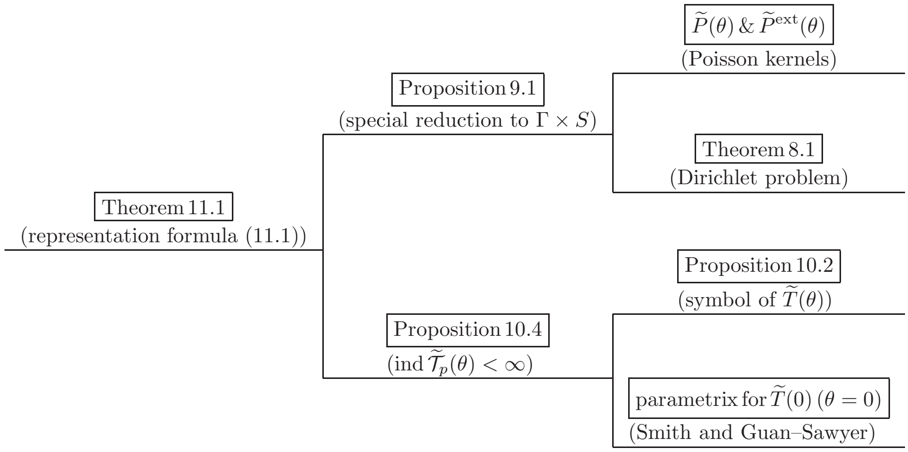

Diagram 7.

COROLLARY 11.2. *Let $1 < p < \infty$ and $\theta \in (-\pi, \pi)$. Assume that the condition* (H) *is satisfied. If $\widetilde{f} \in W^{s,p}(\Omega \times S)$ with $s > -1 + 1/p$, then every solution $\widetilde{u}$ of the homogeneous oblique derivative problem* (7.1) *belongs to the Sobolev space $W^{s+2,p}(\Omega \times S)$.*

Rephrased, Corollary 11.2 asserts that every solution $\widetilde{u}$ of the homogeneous oblique derivative problem (7.1) has the *elliptic gain* of 2 derivatives from $\widetilde{f}$ in L^p Sobolev spaces. In particular, it follows that the null space $\mathcal{N}(\widetilde{\mathfrak{A}}_p(\theta))$ is a subspace of $C^\infty(\overline{\Omega} \times S)$.

Step 3: Finally, it remains to prove the *a priori* estimate (7.2).

By combining Proposition 10.4 and Theorem 6.3, we find that the null space $\mathcal{N}(\widetilde{\mathfrak{A}}_p(\theta))$ is a finite dimensional subspace of $C^\infty(\overline{\Omega} \times S)$. Let $\{\widetilde{u}_j\}_{j=1}^{\ell}$ be a basis of $\mathcal{N}(\widetilde{\mathfrak{A}}_p(\theta))$ and $\{\widetilde{v}_j\}_{j=1}^{\ell}$ its dual basis in the space $L^{p'}(\Omega \times S)$. Then it follows from formula (11.1) that every solution $\widetilde{u}$ of the homogeneous oblique derivative problem (7.1) can be uniquely written in the form

$$\begin{aligned}\widetilde{u} = {} & \widetilde{Q}(\theta)\, \widetilde{f}_0\Big|_{\Omega\times S} - \widetilde{P}(\theta)\left(\widetilde{R}(\theta)\left(\widetilde{Q}(\theta)\widetilde{f}_0\Big|_{\Gamma\times S}\right)\right) - \widetilde{P}(\theta)\left(\widetilde{\Pi}_{\rm c}(\theta)\left(\widetilde{Q}(\theta)\widetilde{f}_0\Big|_{\Gamma\times S}\right)\right) \\ & + \sum_{j=1}^{\ell}\left(\int_{\Omega\times S} \widetilde{u}(x',y')\, \widetilde{v}_j(x',y')\, dx'\, dy'\right)\widetilde{u}_j \quad \text{in } \Omega\times S. \end{aligned} \tag{11.2}$$

However, by using Hölder's inequality we can estimate the last term as follows:

$$\begin{aligned}&\left\| \sum_{j=1}^{\ell}\left(\int_{\Omega\times S} \widetilde{u}(x',y')\, \widetilde{v}_j(x',y')\, dx'\, dy'\right)\widetilde{u}_j \right\|_{2,p} \\ &\quad \leq \sum_{j=1}^{\ell}\left|\int_{\Omega\times S} \widetilde{u}(x',y')\, \widetilde{v}_j(x',y')\, dx'\, dy'\right| \|\widetilde{u}_j\|_{2,p} \leq \sum_{j=1}^{\ell} \|\widetilde{u}\|_p \; \|\widetilde{v}_j\|_{p'} \; \|\widetilde{u}_j\|_{2,p}\end{aligned}$$

$$\leq \left(\sum_{j=1}^{\ell} \|\widetilde{v}_j\|_{p'} \, \|\widetilde{u}_j\|_{2,p} \right) \|\widetilde{u}\|_p \,.$$

Therefore, we obtain from formula (11.2) that the desired *a priori* estimate (7.2) holds true for all functions $\widetilde{u} \in W^{2,p}(\Omega \times S)$ satisfying $B\boldsymbol{\gamma}\widetilde{u} = 0$ on $\Gamma \times S$.

Now the proof of Theorem 7.1 is complete. $\square$

12. Proof of Theorem 2.4.

The proof of Theorem 2.4 is divided into four steps.

Step 1: Let $p = 2$ and $2k/(2k+1) \leq \delta < 1$. We associate with the oblique derivative problem (1.1) a densely defined, closed linear operator

$$\mathcal{A}_2 : L^2(\Omega) \longrightarrow L^2(\Omega)$$

in the Hilbert space $L^2(\Omega)$ as follows:

(a) The domain $\mathcal{D}(\mathcal{A}_2)$ of definition of $\mathcal{A}_2$ is the space

$$\mathcal{D}(\mathcal{A}_2) = \left\{ u \in W^{2-\delta,2}(\Omega) : Au \in L^2(\Omega), \; B\boldsymbol{\gamma}u = 0 \text{ on } \Gamma \right\}.$$

(b) $\mathcal{A}_2 u = Au$ for every $u \in \mathcal{D}(\mathcal{A}_2)$.

Here Au and $B\boldsymbol{\gamma}u$ are taken in the sense of distributions.

By Theorem 2.1 with $\theta := 0$ and $\varphi := 0$, we know that if the condition (H) is satisfied, then, for every real number $\mu \geq R_2(0)$ the homogeneous oblique derivative problem

$$\begin{cases} (A - \mu)u = f & \text{in } \Omega, \\ B\boldsymbol{\gamma}u = a(x') \dfrac{\partial u}{\partial \boldsymbol{n}} + \alpha(x') \cdot u = 0 & \text{on } \Gamma \end{cases}$$

has a unique solution $u \in W^{2-\delta,2}(\Omega)$ for any $f \in L^2(\Omega)$. Hence we find that the operator

$$\mathcal{A}_2 - \mu I : L^2(\Omega) \longrightarrow L^2(\Omega)$$

is *bijective* for all real number $\mu \geq R_2(0)$. However, by the Rellich–Kondrachov theorem (Theorem 3.3) it follows that the operator

$$(\mu - \lambda)I : W^{2-\delta,2}(\Omega) \longrightarrow L^2(\Omega)$$

is *compact* for all complex number $\lambda \in \mathbf{C}$.

Therefore, we have proved that the index of the operator

$$\mathcal{A}_2 - \lambda I = (\mathcal{A}_2 - \mu I) + (\mu - \lambda)I : L^2(\Omega) \longrightarrow L^2(\Omega)$$

is equal to *zero* for all complex number $\lambda \in \mathbf{C}$.

Step 2: Moreover, by applying [**18**, Theorem 1, part (i)] to the differential operator A (see Remark 2.1) we obtain that (the elliptic gain of 2 derivatives from f in the framework of L^2 Sobolev spaces)

$$\begin{cases} Au = f \in L^2(\Omega), \\ B\boldsymbol{\gamma} u = 0 \quad \text{on } \Gamma \end{cases} \implies u \in W^{2,2}(\Omega).$$

Hence, if $\mathfrak{A}_2$ is a closed linear operator defined by formula (2.2), we obtain that

$$\begin{aligned} \mathcal{D}(\mathcal{A}_2) &= \left\{ u \in W^{2-\delta,2}(\Omega) : Au \in L^2(\Omega),\ B\boldsymbol{\gamma} u = 0 \text{ on } \Gamma \right\} \\ &= \left\{ u \in W^{2,2}(\Omega) : B\boldsymbol{\gamma} u = 0 \text{ on } \Gamma \right\} \\ &= \mathcal{D}(\mathfrak{A}_2). \end{aligned}$$

Therefore, we have proved that the index of the operator

$$\mathfrak{A}_2 - \lambda I = \mathcal{A}_2 - \lambda I : L^2(\Omega) \longrightarrow L^2(\Omega)$$

is equal to *zero* for all complex number $\lambda \in \mathbf{C}$. By combining this fact with Theorem 6.11, we have proved the following fundamental theorem:

THEOREM 12.1. *Let $1 < p < \infty$. If condition* (H) *is satisfied, then the index of the operator $\mathfrak{A}_p - \lambda I$ is equal to* zero *for all complex number $\lambda \in \mathbf{C}$.*

Step 3: The next theorem plays an essential role in the proof of the *a priori* estimate (2.4) (and estimate (2.1)):

THEOREM 12.2. *Let $1 < p < \infty$ and $\theta \in (-\pi, \pi)$. Assume that the* a priori *estimate (7.2) holds true for all functions $\widetilde{u} \in W^{2,p}(\Omega \times S)$ satisfying $B\boldsymbol{\gamma}\widetilde{u} = 0$ on $\Gamma \times S$. Then, for every $-\pi < \theta < \pi$ there exists a constant $R(\theta) > 0$ depending on θ such that if $\lambda = r^2\, e^{i\theta}$ and $|\lambda| = r^2 \geq R(\theta)$, we have, for all functions $u \in W^{2,p}(\Omega)$ satisfying $B\boldsymbol{\gamma} u = 0$ on Γ (that is, $u \in D(\mathfrak{A}_p)$),*

$$|u|_{2,p} + |\lambda|^{1/2} \cdot |u|_{1,p} + |\lambda| \cdot \|u\|_p \leq C(\theta)\, \|(A - \lambda)u\|_p, \tag{12.1}$$

with a constant $C(\theta) > 0$ depending on θ. Here $|\cdot|_{j,p}$ ($j = 1,\, 2$) is the seminorm on the Sobolev space $W^{2,p}(\Omega)$ defined by the formula

$$|u|_{j,p} = \left(\int_\Omega \sum_{|\beta|=j} |D^\beta u(x)|^p \, dx \right)^{1/p}.$$

PROOF. Now let $u(x)$ be an arbitrary function in the domain $\mathcal{D}(\mathfrak{A}_p)$:

$$u \in W^{2,p}(\Omega) \text{ and } B\boldsymbol{\gamma} u = 0 \text{ on } \Gamma.$$

We choose a function $\zeta(y)$ in $C^\infty(S)$ such that

$$\begin{cases} 0 \leq \zeta(y) \leq 1 \quad \text{on } S, \\ \operatorname{supp} \zeta \subset \left[\dfrac{\pi}{3}, \dfrac{5\pi}{3} \right], \\ \zeta(y) = 1 \quad \text{for} \quad \dfrac{\pi}{2} \leq y \leq \dfrac{3\pi}{2}, \end{cases}$$

and let

$$\widetilde{v}_\eta(x,y) = u(x) \otimes \zeta(y) e^{i\eta y} \quad \text{for all } x \in \Omega,\ y \in S \text{ and } \eta \geq 0.$$

Then we have the assertions

$$\begin{aligned} \widetilde{v}_\eta &\in W^{2,p}(\Omega \times S), \\ \widetilde{\Lambda}(\theta)\widetilde{v}_\eta &= \left(A + e^{i\theta} \frac{\partial^2}{\partial y^2} \right) \widetilde{v}_\eta \\ &= (A - \eta^2 e^{i\theta}) u \otimes \zeta(y) e^{i\eta y} + 2(i\eta) e^{i\theta} u \otimes \zeta'(y) e^{i\eta y} + e^{i\theta} u \otimes \zeta''(y) e^{i\eta y}, \end{aligned}$$

and also

$$B\boldsymbol{\gamma} \left(\widetilde{v}_\eta(x', y) \right) = (B\boldsymbol{\gamma} u(x')) \otimes \zeta(y) e^{i\eta y} = 0 \quad \text{on } \Gamma \times S.$$

Thus, by applying the *a priori* estimate (7.2) to the functions

$$\widetilde{v}_\eta(x,y) = u(x) \otimes \zeta(y) e^{i\eta y} \in \mathcal{D}(\widetilde{\mathfrak{A}}_p(\theta)) \quad \text{for all } \eta \geq 0,$$

we obtain that

$$\left\| u \otimes \zeta e^{i\eta y} \right\|_{2,p} \leq \widetilde{C}(\theta) \left(\left\| \widetilde{\Lambda}(\theta)(u \otimes \zeta e^{i\eta y}) \right\|_p + \left\| u \otimes \zeta e^{i\eta y} \right\|_p \right). \tag{12.2}$$

We can estimate each term of inequality (12.2) as follows:

$$\left\| u \otimes \zeta e^{i\eta y} \right\|_p = \left(\int_{\Omega \times S} |u(x)|^p |\zeta(y)|^p \, dxdy \right)^{1/p} = \|\zeta\|_p \cdot \|u\|_p. \tag{12.3}$$

$$\begin{aligned} \left\| \widetilde{\Lambda}(\theta)(u \otimes \zeta e^{i\eta y}) \right\|_p &\leq \left\| (A - \eta^2 e^{i\theta}) u \otimes \zeta e^{i\eta y} \right\|_p \\ &\quad + 2\eta \left\| u \otimes \zeta' e^{i\eta y} \right\|_p + \left\| u \otimes \zeta'' e^{i\eta y} \right\|_p \\ &\leq \|\zeta\|_p \cdot \left\| (A - \eta^2 e^{i\theta}) u \right\|_p \\ &\quad + (2\eta \|\zeta'\|_p + \|\zeta''\|_p) \|u\|_p. \end{aligned} \tag{12.4}$$

$$\begin{aligned} \left\| u \otimes \zeta e^{i\eta y} \right\|_{2,p}^p &= \sum_{|\alpha| \leq 2} \int_{\Omega \times S} \left| D^\alpha_{x,y} (u(x) \otimes \zeta(y) e^{i\eta y}) \right|^p dxdy \\ &\geq \sum_{|\alpha| \leq 2} \int_\Omega \int_{\pi/2}^{3\pi/2} \left| D^\alpha_{x,y} (u(x) \otimes e^{i\eta y}) \right|^p dxdy \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k+|\beta|\leq 2} \int_\Omega \int_{\pi/2}^{3\pi/2} \left|\eta^k D^\beta u(x)\right|^p dxdy \\
&\geq \pi \biggl(\sum_{|\beta|=2} \int_\Omega \left|D^\beta u(x)\right|^p dx + \eta^p \sum_{|\beta|=1} \int_\Omega \left|D^\beta u(x)\right|^p dx \\
&\qquad + \eta^{2p} \int_\Omega |u(x)|^p dx \biggr) \\
&= \pi \left(|u|_{2,p}^p + \eta^p |u|_{1,p}^p + \eta^{2p} \|u\|_p^p \right). \qquad (12.5)
\end{aligned}
$$

Therefore, by carrying these three inequalities (12.3), (12.4) and (12.5) into inequality (12.2) we obtain that

$$
|u|_{2,p} + \eta\, |u|_{1,p} + \eta^2 \|u\|_p \leq \widetilde{C}'(\theta) \left(\left\|(A - \eta^2 e^{i\theta})u\right\|_p + \eta\, \|u\|_p \right),
$$

with a constant $\widetilde{C}'(\theta) > 0$ independent of η. If η is so large that

$$
\eta \geq 2\widetilde{C}'(\theta),
$$

then we can eliminate the last term on the right-hand side to obtain that

$$
|u|_{2,p} + \eta\, |u|_{1,p} + \eta^2 \|u\|_p \leq 2\widetilde{C}'(\theta) \left\|(A - \eta^2 e^{i\theta})u\right\|_p .
$$

This proves the desired *a priori* estimate (12.1) if we take

$$
\lambda := \eta^2 e^{i\theta}, \quad R(\theta) := 4\,\widetilde{C}'(\theta)^2, \quad C(\theta) := 2\,\widetilde{C}'(\theta).
$$

The proof of Theorem 12.2 is now complete. □

By combining Theorems 7.1 and 12.2, we have the desired *a priori* estimate (2.4) (and estimate (2.1)) for the operator $\mathfrak{A}_p - \lambda I$:

Corollary 12.3. *Let $1 < p < \infty$. Assume that the condition* (H) *is satisfied. Then, for every $0 < \varepsilon < \pi/2$ there exist constants $r_p(\varepsilon) > 0$ and $c_p(\varepsilon) > 0$ such that we have, for all $\lambda = r^2\, e^{i\theta}$ satisfying $r \geq r_p(\varepsilon)$ and $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$,*

$$
|u|_{2,p} + |\lambda|^{1/2} \cdot |u|_{1,p} + |\lambda| \cdot \|u\|_p \leq c_p(\varepsilon)\, \|(\mathfrak{A}_p - \lambda I)u\|_p, \quad u \in \mathcal{D}(\mathfrak{A}_p). \qquad (12.6)
$$

Proof. By the *a priori* estimate (12.1), it follows that if $\lambda = r^2\, e^{i\theta}$, $-\pi < \theta < \pi$ and if $|\lambda| = r^2 \geq R(\theta)$, then we have, for all functions $u \in \mathcal{D}(\mathfrak{A}_p)$,

$$
|u|_{2,p} + |\lambda|^{1/2} \cdot |u|_{1,p} + |\lambda| \cdot \|u\|_p \leq C(\theta) \|(\mathfrak{A}_p - \lambda I)u\|_p.
$$

However, we find from the proof of Theorem 12.2 that the constants $R(\theta)$ and $C(\theta)$ depend *continuously* on $\theta \in (-\pi, \pi)$, so that they may be chosen uniformly in $\theta \in [-\pi + \varepsilon, \pi - \varepsilon]$, for every $\varepsilon > 0$. This proves the existence of the constants $r_p(\varepsilon)$ and $c_p(\varepsilon)$. Namely, the *a priori* estimate (12.6) holds true for all $\lambda = r^2\, e^{i\theta}$ satisfying $r \geq r_p(\varepsilon)$ and $\theta \in [-\pi + \varepsilon, \pi - \varepsilon]$.

The proof of Corollary 12.3 is complete. □

Step 4: The *a priori* estimate (12.6) asserts that the operator $\mathfrak{A}_p - \lambda I$ is injective if λ belongs to the set

$$\Sigma_p(\varepsilon) = \left\{ \lambda = r^2 e^{i\theta} : r \geq r_p(\varepsilon), -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon \right\}.$$

Hence it is *bijective* for all $\lambda \in \Sigma_p(\varepsilon)$, since the index of the operator $\mathfrak{A}_p - \lambda I$ is equal to zero for all complex number $\lambda \in \mathbf{C}$ (Theorem 12.1).

Summing up, we have proved that the resolvent set of $\mathfrak{A}_p$ contains the set $\Sigma_p(\varepsilon)$ and that the resolvent $(\mathfrak{A}_p - \lambda I)^{-1}$ satisfies the estimate

$$\left\| (\mathfrak{A}_p - \lambda I)^{-1} \right\| \leq \frac{c_p(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma_p(\varepsilon). \tag{2.4}$$

Now the proof of Theorem 2.4 (and also Theorem 2.2) is complete. □

13. Proof of Theorem 2.5.

This last section is devoted to the proof of Theorem 2.5. The proof is divided into three steps.

Step 1: We start with the following proposition:

PROPOSITION 13.1. *Let $n < p < \infty$. Assume that the condition* (H) *is satisfied. Then, for every $\varepsilon > 0$ there exists a constant $r'_p(\varepsilon) > 0$ such that if $\lambda = r^2 e^{i\theta}$ with $r \geq r'_p(\varepsilon)$ and $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$, we have, for all functions $u \in \mathcal{D}(\mathfrak{A}_p)$,*

$$|\lambda|^{1/2} |u|_{C^1(\overline{\Omega})} + |\lambda| \cdot |u|_{C(\overline{\Omega})} \leq c'_p(\varepsilon) |\lambda|^{n/2p} \|(A - \lambda)u\|_p, \tag{13.1}$$

with a constant $c'_p(\varepsilon) > 0$ depending on p and ε, but independent of u and λ.

PROOF. First, by applying Theorem 3.2 with $p := r > n$, $\theta := n/p$ and $\nu := 0$ we obtain from the Gagliardo–Nirenberg inequality (3.1) that

$$|u|_{C(\overline{\Omega})} \leq C |u|_{1,p}^{n/p} \|u\|_p^{1-n/p}. \tag{13.2}$$

Here and in the following the letter C denotes a generic positive constant depending on p and ε, but independent of u and λ.

By using inequality (12.6), we obtain from inequality (13.2) that

$$\begin{aligned} |u|_{C(\overline{\Omega})} &\leq C \left(|\lambda|^{-1/2} \|(A - \lambda)u\|_p \right)^{n/p} \left(|\lambda|^{-1} \|(A - \lambda)u\|_p \right)^{1-n/p} \\ &= C |\lambda|^{-1+n/2p} \|(A - \lambda)u\|_p. \end{aligned}$$

This proves that

$$|\lambda| \cdot |u|_{C(\overline{\Omega})} \leq C |\lambda|^{n/2p} \|(A - \lambda)u\|_p \quad \text{for all functions } u \in \mathcal{D}(\mathfrak{A}_p). \tag{13.3}$$

Similarly, by applying inequality (13.2) to the functions $D_i u \in W^{1,p}(\Omega)$, $1 \leq i \leq n$, we obtain that

$$\begin{aligned} |D_i u|_{C(\overline{\Omega})} &\leq C |D_i u|_{1,p}^{n/p} \|D_i u\|_p^{1-n/p} \leq C |u|_{2,p}^{n/p} |u|_{1,p}^{1-n/p} \\ &\leq C \left(\|(A-\lambda)u\|_p\right)^{n/p} \left(|\lambda|^{-1/2} \|(A-\lambda)u\|_p\right)^{1-n/p} \\ &= C |\lambda|^{-1/2+n/2p} \|(A-\lambda)u\|_p. \end{aligned}$$

This proves that

$$|\lambda|^{1/2} |u|_{C^1(\overline{\Omega})} \leq C |\lambda|^{n/2p} \|(A-\lambda)u\|_p \quad \text{for all functions } u \in \mathcal{D}(\mathfrak{A}_p). \tag{13.4}$$

Therefore, the desired *a priori* estimate (13.1) follows by combining inequalities (13.3) and (13.4).

The proof of Proposition 13.1 is complete. □

Step 2: The next proposition plays an essential role in the proof of the *a priori* estimate (2.6) for the operator $\mathfrak{A}$:

PROPOSITION 13.2. *Assume that the conditions* (G) *and* (H) *are satisfied. Then, for every $\varepsilon > 0$ there exists a constant $r(\varepsilon) > 0$ such that if $\lambda = r^2 e^{i\theta}$ with $r \geq r(\varepsilon)$ and $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$, we have, for all functions $u \in \mathcal{D}(\mathfrak{A})$,*

$$|\lambda|^{1/2} |u|_{C^1(\overline{\Omega})} + |\lambda| \cdot |u|_{C(\overline{\Omega})} \leq c(\varepsilon) |(A-\lambda)u|_{C(\overline{\Omega})}, \tag{13.5}$$

with a constant $c(\varepsilon) > 0$.

PROOF. We shall make use of a λ-*dependent localization* argument in order to adjust the term $\|(A-\lambda)u\|_p$ in inequality (13.1) to obtain inequality (13.5), just as in [**37**] (see Masuda [**26**] for the Dirichlet case). The proof of Proposition 13.2 is divided into four substeps.

Substep 2-1: We remark that

$$\mathfrak{A} \subset \mathfrak{A}_p \quad \text{for all } 1 < p < \infty.$$

Indeed, since we have, for any $u \in \mathcal{D}(\mathfrak{A})$,

$$u \in C(\overline{\Omega}) \subset L^p(\Omega), \ Au \in C(\overline{\Omega}) \subset L^p(\Omega) \ \text{and } B\gamma u = 0 \text{ on } \Gamma,$$

it follows from an application of [**18**, Theorem 1, part (i)] (see Remark 2.1) that

$$u \in W^{2,p}(\Omega).$$

Now let x_0 be an arbitrary point of the closure $\overline{\Omega} = \Omega \cup \Gamma$.

(1) First, let x_0' be a boundary point of the *submanifold* $\Gamma_0 = \{x' \in \Gamma : a(x') = 0\}$. Just as in Egorov–Kondratev [**13**, Section 2, A special partition of unity], we make use of a smooth coordinate transformation χ defined in a neighborhood of x_0' such that:

(a) the transformation χ flattens a part of the boundary Γ into the plane $x_n = 0$.

(b) the transformation χ maps a part of the submanifold Γ_0 into the hyperplane $x_1 = 0$ of the plane $x_n = 0$.

(c) the transformation χ straightens out the oblique vector field $\boldsymbol{\nu}$ as

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \frac{\partial}{\partial x_1}.$$

If $Q(x,r)$ is an open cube with side length $2r$ and with center x, we may assume that

$$\chi : Q(x_0, \eta_0) \cap \Omega \longrightarrow Q(0, \varepsilon) \cap \mathbf{R}^n_+ \quad \text{for some constant } \eta_0 > 0.$$

Then we let

$$\begin{aligned} G_0 &= Q(x_0', \eta_0) \cap \Omega, \\ G' &= Q(x_0', \eta) \cap \Omega, \quad 0 < \eta < \eta_0, \\ G'' &= Q(x_0', \eta/2) \cap \Omega, \quad 0 < \eta < \eta_0. \end{aligned}$$

Now we choose a function $\varphi(x)$ in $C_0^\infty(\mathbf{R}^n)$ such that

$$\begin{cases} \varphi(x) = 0 & \text{outside } Q(0,1), \\ \dfrac{\partial \varphi}{\partial x_1}(x) = 0 & \text{on } Q\left(0, \dfrac{1}{2}\right), \\ \varphi(x) > 0 & \text{on } Q(0,1). \end{cases}$$

If we introduce a localizing function

$$\varphi_0(x,\eta) := \varphi\left(\frac{x - x_0'}{\eta}\right), \quad x_0' \in \Gamma_0, \quad 0 < \eta < \eta_0,$$

then we find that

$$\begin{cases} \varphi_0(x,\eta) = 0 & \text{outside } Q(x_0', \eta), \\ \dfrac{\partial \varphi_0}{\partial x_1}(x,\eta) = 0 & \text{on } Q\left(x_0', \dfrac{\eta}{2}\right), \\ \varphi_0(x,\eta) > 0 & \text{on } Q(x_0', \eta), \end{cases}$$

and further that the function $\varphi_0(x,\eta)$ satisfies the oblique derivative condition

$$\frac{\partial}{\partial \boldsymbol{\nu}}(\varphi_0(x,\eta)) = \frac{\partial}{\partial x_1}(\varphi_0(x,\eta)) = 0 \quad \text{on } Q\left(x_0', \frac{\eta}{2}\right).$$

Moreover, from an open covering of Γ_0 by cubes $Q(x', \eta)$ we can choose a finite subcovering $\{Q(x_\ell', \eta_\ell)\}$ together with smooth functions $\{\varphi_0(x, \eta_\ell)\}$. Then we have the assertion

$$\frac{\partial}{\partial \boldsymbol{\nu}}(\varphi_0(x, \eta_\ell)) = 0 \quad \text{in a tubular open neighborhood } Q \text{ of } \Gamma_0.$$

(2) Secondly, let x_0' be a boundary point outside the tubular neighborhood Q of Γ_0 in Γ. In this case, since the oblique vector field $\boldsymbol{\nu}$ is not tangent to the boundary Γ on $\Gamma \setminus Q$, we make use of a smooth coordinate transformation χ defined in a neighborhood of x_0' such that:

(d) the transformation χ flattens a part of the boundary Γ into the plane $x_n = 0$.

(e) the transformation χ straightens out the oblique vector field $\boldsymbol{\nu}$ as

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \frac{\partial}{\partial x_n}.$$

Similarly, we choose a function $\varphi(x)$ in $C_0^\infty(\mathbf{R}^n)$ such that

$$\begin{cases} \varphi(x) = 0 & \text{outside } Q(0,1), \\ \dfrac{\partial \varphi}{\partial x_n}(x) = 0 & \text{on } Q\left(0, \dfrac{1}{2}\right), \\ \varphi(x) > 0 & \text{on } Q(0,1). \end{cases}$$

Then we find that a localizing function

$$\varphi_0(x, \eta) := \varphi\left(\frac{x - x_0'}{\eta}\right), \quad x_0' \in \Gamma \setminus Q, \quad 0 < \eta < \eta_0,$$

satisfies the conditions

$$\begin{cases} \varphi_0(x, \eta) = 0 & \text{outside } Q(x_0', \eta), \\ \dfrac{\partial \varphi_0}{\partial x_n}(x, \eta) = 0 & \text{on } Q\left(x_0', \dfrac{\eta}{2}\right), \\ \varphi_0(x, \eta) > 0 & \text{on } Q(x_0', \eta), \end{cases}$$

and also the oblique derivative condition

$$\frac{\partial}{\partial \boldsymbol{\nu}}(\varphi_0(x, \eta)) = \frac{\partial}{\partial x_n}(\varphi_0(x, \eta)) = 0 \quad \text{on } \Gamma \setminus Q.$$

(3) Thirdly, let x_0 be an interior point of an open set $W \Subset \Omega$, bounded away from Γ. In this case, we make use of a smooth coordinate transformation such that

$$\chi : Q(x_0, \eta_0) \longrightarrow Q(0, \varepsilon).$$

Then we let

$$\begin{aligned} G_0 &= Q(x_0, \eta_0), \\ G' &= Q(x_0, \eta), \quad 0 < \eta < \eta_0, \\ G'' &= Q(x_0, \eta/2), \quad 0 < \eta < \eta_0. \end{aligned}$$

Now we choose a function $\varphi(x)$ in $C_0^\infty(\mathbf{R}^n)$ such that

$$\begin{cases} \varphi(x) = 0 & \text{outside } Q(0,1), \\ \varphi(x) > 0 & \text{on } Q(0,1). \end{cases}$$

We find that a localizing function

$$\varphi_0(x,\eta) := \varphi\left(\frac{x - x_0}{\eta}\right), \quad x_0 \in W \Subset \Omega, \quad 0 < \eta < \eta_0,$$

satisfies the conditions

$$\begin{cases} \varphi_0(x,\eta) = 0 & \text{outside } Q(x_0,\eta), \\ \varphi_0(x,\eta) > 0 & \text{on } Q(x_0,\eta). \end{cases}$$

Substep 2-2: For the localizing function $\varphi_0(x,\eta)$, we can prove the following lemma:

LEMMA 13.3. *If $u \in \mathcal{D}(\mathfrak{A})$, then $\varphi_0(x,\eta)u \in \mathcal{D}(\mathfrak{A}_p)$ for all $0 < \eta < \eta_0$ and $1 < p < \infty$.*

PROOF. (i) First, we recall that

$$u \in W^{2,p}(\Omega) \quad \text{for all } 1 < p < \infty.$$

Hence we have the assertion

$$\varphi_0(x,\eta)u \in W^{2,p}(\Omega).$$

(ii) Secondly, it is easy to verify that the function $\varphi_0(x,\eta)u$, $x \in \overline{\Omega}$, satisfies the boundary condition

$$B\boldsymbol{\gamma}\left(\varphi_0(x,\eta)u\right) = \frac{\partial}{\partial \boldsymbol{\nu}}\left(\varphi_0(x,\eta)u\right) = 0 \quad \text{on } \Gamma.$$

Indeed, this is obvious if we have the condition

$$\operatorname{supp}\left(\varphi_0(x,\eta)u\right) \subset Q(x_0,\eta), \quad x_0 \in W \Subset \Omega.$$

Moreover, if we have the condition

$$\operatorname{supp}\left(\varphi_0(x,\eta)u\right) \subset Q(x_0,\eta) \cap \overline{\Omega}, \quad x_0 \in \Gamma,$$

then it follows that

$$\begin{aligned} B\boldsymbol{\gamma}\left(\varphi_0(x,\eta)\right) &= \frac{\partial}{\partial \boldsymbol{\nu}}\left(\varphi_0(x,\eta)\right) \\ &= \begin{cases} \dfrac{\partial}{\partial x_1}\left(\varphi_0(x,\eta)\right) = 0 & \text{in the tubular neighborhood } Q \text{ of } \Gamma_0 \text{ in } \Gamma, \\ \dfrac{\partial}{\partial x_n}\left(\varphi_0(x,\eta)\right) = 0 & \text{on } \Gamma \setminus Q. \end{cases} \end{aligned}$$

Therefore, we have the assertion

$$B\boldsymbol{\gamma}\left(\varphi_0(x,\eta)u\right) = \varphi_0(x,\eta)(B\boldsymbol{\gamma}u) + u\left(B\boldsymbol{\gamma}\varphi_0(x,\eta)\right) = 0 \quad \text{on } \Gamma,$$

since $B\boldsymbol{\gamma}u = 0$ on Γ.

Summing up, we have proved that

$$\varphi_0(x,\eta)u \in \mathcal{D}(\mathfrak{A}_p) \quad \text{for all } 0 < \eta < \eta_0 \text{ and } 1 < p < \infty.$$

The proof of Lemma 13.3 is complete. □

Substep 2-3: Now we take a positive number p such that

$$n < p < \infty.$$

Then, by Lemma 13.3 we can apply inequality (13.1) to the function $\varphi_0(x,\eta)u$ with $u \in \mathcal{D}(\mathfrak{A})$ to obtain that

$$\begin{aligned}
&|\lambda|^{1/2}|u|_{C^1(\overline{G''})} + |\lambda|\cdot|u|_{C(\overline{G''})} \\
&\quad \leq |\lambda|^{1/2}\left|\varphi_0(x,\eta)u\right|_{C^1(\overline{G'})} + |\lambda|\cdot\left|\varphi_0(x,\eta)u\right|_{C(\overline{G'})} \\
&\quad = |\lambda|^{1/2}\left|\varphi_0(x,\eta)u\right|_{C^1(\overline{\Omega})} + |\lambda|\cdot\left|\varphi_0(x,\eta)u\right|_{C(\overline{\Omega})} \\
&\quad \leq C|\lambda|^{n/2p}\left\|(A-\lambda)(\varphi_0(x,\eta)u)\right\|_{L^p(\Omega)} \\
&\quad = C|\lambda|^{n/2p}\left\|(A-\lambda)(\varphi_0(x,\eta)u)\right\|_{L^p(G')} \quad \text{for all } 0 < \eta < \eta_0,
\end{aligned} \tag{13.6}$$

since we have the assertions

$$\begin{cases} \varphi_0(x,\eta) = 1 & \text{on } G'', \\ \operatorname{supp}\left(\varphi_0(x,\eta)u\right) \subset \overline{G'}. \end{cases}$$

However, we have the formula

$$(A-\lambda)(\varphi_0(x,\eta)u) = \varphi_0(x,\eta)\left((A-\lambda)u\right) + \left[A,\varphi_0(x,\eta)\right]u, \tag{13.7}$$

where $[A,\varphi_0(x,\eta)]$ is the commutator of A and $\varphi_0(x,\eta)$ defined by the formula

$$\begin{aligned}
\left[A,\varphi_0(x,\eta)\right]u &= A(\varphi_0(x,\eta)u) - \varphi_0(x,\eta)Au \\
&= 2\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial\varphi_0}{\partial x_i}\frac{\partial u}{\partial x_j} \\
&\quad + \left(\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2\varphi_0}{\partial x_i\partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial\varphi_0}{\partial x_i}\right)u.
\end{aligned} \tag{13.8}$$

Here we need the following elementary inequality:

Lemma 13.4. *We have, for all functions $v \in C^j(\overline{G'})$, $j = 0, 1, 2,$*

$$\|v\|_{W^{j,p}(G')} \leq |G'|^{1/p}\, \|v\|_{C^j(\overline{G'})},$$

where $|G'|$ denotes the measure of G'.

PROOF. It suffices to note that we have, for all functions $w \in C(\overline{G'})$,

$$\int_{G'} |w(x)|^p dx \leq |G'|\, |w|^p_{C(\overline{G'})}.$$

This proves Lemma 13.4. □

Since we have, for some constant $c > 0$,

$$|G'| \leq |Q(x_0, \eta)| \leq c\eta^n, \quad 0 < \eta < \eta_0,$$

it follows from an application of Lemma 13.4 that

$$\|\varphi_0(x,\eta)((A-\lambda)u)\|_{L^p(G')} \leq c^{1/p}\eta^{n/p}|(A-\lambda)u|_{C(\overline{G'})}, \quad 0 < \eta < \eta_0. \tag{13.9}$$

Furthermore, we remark that

$$|D^\beta \varphi_0(x,\eta)| = O\left(\eta^{-|\beta|}\right) \quad \text{as } \eta \downarrow 0.$$

Hence it follows from an application of Lemma 13.4 that

$$\left\| \frac{\partial \varphi_0}{\partial x_j} \frac{\partial u}{\partial x_j} \right\|_{L^p(G')} \leq \frac{C}{\eta} |u|_{1,p,G'} \leq C\eta^{-1+n/p}|u|_{C^1(\overline{G'})}. \tag{13.10}$$

$$\|(A\varphi_0(x,\eta))\, u\|_{L^p(G')} \leq \frac{C}{\eta^2}|u|_{L^p(G')} \leq C\eta^{-2+n/p}|u|_{C(\overline{G'})}. \tag{13.11}$$

By using inequalities (13.10) and (13.11), we obtain from formula (13.8) that

$$\begin{aligned}
&\|[A, \varphi_0(x,\eta)]\, u\|_{L^p(G')} \\
&\leq C\left(\eta^{-1+n/p}|u|_{C^1(\overline{G'})} + \eta^{-2+n/p}|u|_{C(\overline{G'})} + \eta^{-1+n/p}|u|_{C(\overline{G'})}\right) \\
&\leq C\left(\eta^{-1+n/p}|u|_{C^1(\overline{\Omega})} + \eta^{-2+n/p}|u|_{C(\overline{\Omega})}\right).
\end{aligned} \tag{13.12}$$

In view of formula (13.7), it follows from inequalities (13.9) and (13.12) that

$$\begin{aligned}
&\|(A-\lambda)(\varphi_0(x,\eta)u)\|_{L^p(G')} \\
&\quad \leq \|\varphi_0(x,\eta)((A-\lambda)u)\|_{L^p(G')} + \|[A, \varphi_0(x,\eta)]\, u\|_{L^p(G')} \\
&\quad \leq C\eta^{n/p}\left(|(A-\lambda)u|_{C(\overline{G'})} + \eta^{-1}|u|_{C^1(\overline{\Omega})} + \eta^{-2}|u|_{C(\overline{\Omega})}\right) \\
&\quad\quad \text{for all } 0 < \eta < \eta_0.
\end{aligned} \tag{13.13}$$

Therefore, by combining inequalities (13.6) and (13.13) we obtain that

$$|\lambda|^{1/2}|u|_{C^1(\overline{G''})} + |\lambda| \cdot |u|_{C(\overline{G''})}$$

$$
\begin{aligned}
&\leq C|\lambda|^{n/2p} \, \|(A-\lambda)(\varphi_0(x,\eta)u)\|_{L^p(G')} \\
&\leq C|\lambda|^{n/2p} \, \eta^{n/p} \left(|(A-\lambda)u|_{C(\overline{G'})} + \eta^{-1}|u|_{C^1(\overline{G'})} + \eta^{-2}|u|_{C(\overline{G'})} \right) \\
&\leq C|\lambda|^{n/2p} \, \eta^{n/p} \left(|(A-\lambda)u|_{C(\overline{\Omega})} + \eta^{-1}|u|_{C^1(\overline{\Omega})} + \eta^{-2}|u|_{C(\overline{\Omega})} \right) \\
&\quad \text{for all } 0 < \eta < \eta_0. \qquad (13.14)
\end{aligned}
$$

We remark that the closure $\overline{\Omega} = \Omega \cup \Gamma$ can be covered by a finite number of sets of the forms

$$
Q(x_0', \eta/2) \cap \overline{\Omega}, \quad x_0' \in \Gamma,
$$

and

$$
Q(x_0, \eta/2), \quad x_0 \in W \Subset \Omega.
$$

Hence, by taking the supremum of inequality (13.14) over $x \in \overline{\Omega}$ we find that

$$
\begin{aligned}
&|\lambda|^{1/2}|u|_{C^1(\overline{\Omega})} + |\lambda| \cdot |u|_{C(\overline{\Omega})} \\
&\quad \leq C|\lambda|^{n/2p}\eta^{n/p} \left(|(A-\lambda)u|_{C(\overline{\Omega})} + \eta^{-1} \, |u|_{C^1(\overline{\Omega})} + \eta^{-2} \, |u|_{C(\overline{\Omega})} \right), \\
&\quad \text{for all functions } u \in \mathcal{D}(\mathfrak{A}). \qquad (13.15)
\end{aligned}
$$

Here we recall that

$$
0 < \eta < \eta_0. \tag{13.16}
$$

Substep 2-4: We are in a position to choose the localization parameter η. To do so, we let

$$
\widetilde{r}_p(\varepsilon) := \max \left\{ r_p'(\varepsilon), r_p''(\varepsilon) \right\}, \tag{13.17}
$$

$$
0 < K < \widetilde{r}_p(\varepsilon), \tag{13.18}
$$

where the constant K will be chosen later on.

For a complex number $\lambda = r^2 \, e^{i\theta}$ with $r \geq \widetilde{r}_p(\varepsilon)$, we let

$$
\eta := \frac{\eta_0}{|\lambda|^{1/2}} K. \tag{13.19}
$$

Then the parameter η satisfies condition (13.16), since we have, by formulas (13.17) and (13.18),

$$
\eta = \frac{\eta_0}{|\lambda|^{1/2}} K = \frac{\eta_0}{r} K \leq \frac{\eta_0}{\widetilde{r}_p(\varepsilon)} K < \eta_0.
$$

Hence it follows from inequality (13.15) that

$$
|\lambda|^{1/2}|u|_{C^1(\overline{\Omega})} + |\lambda| \cdot |u|_{C(\overline{\Omega})}
$$

$$\le C\,\eta_0^{n/p} K^{n/p} |(A-\lambda)u|_{C(\overline{\Omega})} + \left(C\,\eta_0^{n/p-1} K^{-1+n/p}\right) |\lambda|^{1/2} \cdot |u|_{C^1(\overline{\Omega})}$$
$$+ \left(C\,\eta_0^{n/p-2} K^{-2+n/p}\right) |\lambda| \cdot |u|_{C(\overline{\Omega})} \quad \text{for all functions } u \in \mathcal{D}(\mathfrak{A}). \tag{13.20}$$

However, since the exponents $-1+n/p$ and $-2+n/p$ are both negative for $n<p<\infty$, we can choose the constant K so large that

$$C\,\eta_0^{n/p-1} K^{-1+n/p} < 1, \qquad C\,\eta_0^{n/p-2} K^{-2+n/p} < 1.$$

For example, we may take

$$K > \widetilde{C} := \frac{1}{\eta_0} \max\left\{C^{1/\sigma}, C^{1/(\sigma+1)}\right\}, \quad \sigma = 1 - \frac{n}{p} > 0. \tag{13.21}$$

Then the desired inequality (13.5) follows from inequality (13.20). Indeed, if we let

$$r(\varepsilon) := \max\left\{\widetilde{r}_p(\varepsilon), \widetilde{C}+1\right\},$$

and choose the constant K such that

$$\widetilde{C} < K < r(\varepsilon),$$

then, for all complex numbers $\lambda = r^2\,e^{i\theta}$ with $r \ge r(\varepsilon)$ we have, by conditions (13.17), (13.18), (13.19) and (13.21),

$$\begin{aligned}
&0 < \eta < \eta_0,\\
&0 < K < |\lambda|^{1/2},\\
&C\,\eta_0^{n/p-1} K^{-1+n/p} < 1,\\
&C\,\eta_0^{n/p-1} K^{-1+n/p} < 1.
\end{aligned}$$

Now the proof of Proposition 13.2 is complete. □

Step 3: Finally, the next proposition (together with Proposition 13.2) proves that the resolvent set of $\mathfrak{A}$ contains the set

$$\Sigma(\varepsilon) = \left\{\lambda = r^2\,e^{i\theta} : r \ge r(\varepsilon),\ -\pi+\varepsilon \le \theta \le \pi - \varepsilon\right\},$$

that is, the resolvent $(\mathfrak{A} - \lambda I)^{-1}$ exists for all $\lambda \in \Sigma(\varepsilon)$.

PROPOSITION 13.5. *If $\lambda \in \Sigma(\varepsilon)$, then, for any function $f \in C(\overline{\Omega})$ there exists a unique function $u \in \mathcal{D}(\mathfrak{A})$ such that $(\mathfrak{A} - \lambda I)u = f$.*

PROOF. Since we have the assertion

$$f \in C(\overline{\Omega}) \subset L^p(\Omega) \quad \text{for all } 1<p<\infty,$$

it follows from an application of Theorem 2.4 that if $\lambda \in \Sigma(\varepsilon)$ there exists a unique function $u(x) \in W^{2,p}(\Omega)$ such that

$$(A - \lambda)u = f \quad \text{in } \Omega \tag{13.22}$$

and that

$$B\boldsymbol{\gamma} u = a(x')\frac{\partial u}{\partial \boldsymbol{n}} + \alpha(x') \cdot u = 0 \quad \text{on } \Gamma.$$

However, the part (ii) of Theorem 3.1 asserts that

$$u \in W^{2,p}(\Omega) \subset C^{2-n/p}(\overline{\Omega}) \subset C^1(\overline{\Omega}) \quad \text{if} \quad n < p < \infty.$$

Furthermore, in view of formula (13.22) it follows that

$$Au = f + \lambda u \in C(\overline{\Omega}).$$

Summing up, we have proved that

$$\begin{cases} u \in \mathcal{D}(\mathfrak{A}), \\ (\mathfrak{A} - \lambda I)u = f. \end{cases}$$

The proof of Proposition 13.5 is complete. □

Now Theorem 2.5 follows by combining Propositions 13.5 and 13.2. □

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