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On positive solutions generated by semi-strong saturation effect for the Gierer-Meinhardt system

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Abstract. In this paper, we study the existence and the asymptotic behavior of a positive solution to the one-dimensional stationary shadow system of the Gierer-Meinhardt system with saturation. We equip a reaction term of activator with saturation effect $\kappa_0\varepsilon^{2\alpha}$ for $\alpha \in (0,1)$ (semi-strong saturation effect). Here, $\varepsilon > 0$ stands for the diffusion constant of activator. For sufficiently small ε, we show the existence of a new type of solutions which has the following properties:
(a) the solution has an internal transition-layer of $O(\varepsilon)$ in width,
(b) the transition-layer is located in the position of $O(\varepsilon^\alpha)$ from the boundary $x = 0$,
(c) the solution concentrates at $x = 0$ with the amplitude of the order of $O(\varepsilon^{-\alpha})$ when $\varepsilon \ll 1$.

1. Introduction.

In this paper, we are concerned with the following stationary problem of the shadow system for the Gierer-Meinhardt system with saturation:

$$\begin{cases} 0 = \varepsilon^2 A'' - A + \dfrac{A^2}{\xi(1+\kappa_0\varepsilon^{2\alpha}A^2)}, & x \in (0,1), \\ \xi = \displaystyle\int_0^1 A^2 dx, \\ A'(0) = A'(1) = 0, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, $\kappa_0 > 0$ and $\alpha \geq 0$ are constants. The unknowns are $A = A(x)$ and ξ. $A(x)$ and ξ represent the concentrations of an activator and an inhibitor at $x \in (0,1)$. In the shadow system case, the concentration of an inhibitor is considered to be uniform, and hence ξ does not depend on x. The value $\kappa_0\varepsilon^{2\alpha}$ stands for the saturation effect of activator.

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The Gierer-Meinhardt system [**5**] is a reaction-diffusion system of an activator and an inhibitor and is a model of biological pattern formations. The general Gierer-Meinhardt system is written by

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A - A + \dfrac{A^2}{H(1+\kappa A^2)}, & x \in \Omega,\ t > 0, \\ \tau \dfrac{\partial H}{\partial t} = D\Delta H - H + A^2, & x \in \Omega,\ t > 0, \\ \dfrac{\partial A}{\partial \nu} = \dfrac{\partial H}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ A(x,0) = A_0(x),\ H(x,0) = H_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where ε, D, τ are positive constants, and $\kappa \geq 0$. Ω is a domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. ν is the unit outer normal to $\partial\Omega$. $A(x,t)$ and $H(x,t)$ are concentrations of an activator and an inhibitor at x and time t. A_0 and H_0 are their initial data. Dividing the second equation in (1.2) by D and taking the limit $D \to \infty$ formally, we have $\Delta H = 0$ in Ω and $\partial H/\partial\nu = 0$ on $\partial\Omega$. This means that $H(x,t)$ does not depends on x, and hence we can regard $H(x,t) = \xi(t)$ under the limit $D \to \infty$. By this consideration, we have the following shadow system:

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A - A + \dfrac{A^2}{\xi(1+\kappa A^2)}, \quad x \in \Omega,\ t > 0, \\ \tau \dfrac{\partial \xi}{\partial t} = \xi - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega A^2 dx, \quad t > 0, \\ \dfrac{\partial A}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ A(x,0) = A_0(x), \quad \xi(0) = \xi_0,\ x \in \Omega. \end{cases} \tag{1.3}$$

The system (1.1) is the one-dimensional stationary problem of (1.3) with $\kappa = \kappa_0\varepsilon^{2\alpha}$ and $\Omega = (0,1)$.

It is known that (1.2) and (1.3) have various kinds of striking solutions when ε is small and D is large. In this paper, we are concerned with stationary solutions only. It is known that different types of solutions appear according to the value κ. Dividing three cases, let us present several known facts.

(a) No saturation case. When $\kappa = 0$, the saturation effect is neglected. In this case, there are many results on the stationary solutions to (1.2) or (1.3). In particular, spike-layer solutions (or point-condensation solutions) appear. The first result on the existence of spike-layer solutions was established by I. Takagi [**25**]. He treated the one-dimensional problem of (1.3) and (1.2), and constructed stationary boundary spike-layer solutions to (1.3) and (1.2) for sufficiently small ε and large D. In the one-dimensional case, it is known that, for the shadow system (1.3), only monotone decreasing or increasing stationary solutions could be stable (see [**17**], [**22**]). Hence, if a spike-layer solution to the one-dimensional shadow system is stable, then the number of the spikes must be one, and the location of the spike must be on the boundary. However, in the case where $D > 0$ is not so large, the situation changes dramatically. For sufficiently small ε, if D becomes smaller and smaller, then multi-spike layer solutions get back their stability (see [**7**], [**32**]). In two dimensional case, interior multi-spike layer solutions were constructed, and their stability were studied in [**28**], [**29**], [**30**], [**33**]. For other related result, see [**4**], [**10**], [**12**], [**13**], [**18**], [**19**], [**20**], [**26**], [**27**], [**16**] and the references therein.

(b) Weak saturation case. When $\kappa > 0$ is small enough according to ε, it is known that spike-layer solutions may appear similarly to the no saturation case. By J. Wei and M. Winter [**31**], the following condition was subjected:

(A) $\kappa > 0$ depends on ε, and there exists a limit $\kappa\varepsilon^{-2N} \to \kappa_0$ as $\varepsilon \to 0$ for some $\kappa_0 \in [0, \infty)$.

Under this assumption, for sufficiently small ε, they showed the existence of a boundary spike-layer solution to the shadow system (1.3) and studied its stability. After their work, K. Kurata and the author [**9**] showed the existence of boundary multi-spike layer solutions on axially symmetric domains Ω to the shadow system (1.3) and the original system (1.2) for sufficiently small ε and large D under the same assumption (A). The case where D is not so large (so-called the strong coupling case) was treated in [**15**] in the one-dimensional case. For other related results, see [**8**], [**14**], [**23**].

(c) Strong saturation case. When $\kappa > 0$ is fixed, it is known that spike-layer solutions could not appear by numerical simulation. Moreover, if $\kappa > 0$ is very large, then stationary problem of (1.2) possesses the constant solution only under some suitable conditions (see [**2**]). Therefore, $\kappa > 0$ must be small suitably to obtain nonconstant solutions. For small fixed constant $\kappa > 0$, internal transition-layer solutions may appear due to the bistable structure. In fact, M. Mimura, M. Tabata and Y. Hosono [**11**] showed the existence of internal transition-layer solutions to (1.2) in one-dimensional case when ε is sufficiently small and D is large. See also [**21**]. In higher dimension case, the existence of an internal transition-layer solution was studied in [**3**], [**24**]. However, in higher dimension case, the problem becomes more difficult, and hence the existence of such a solution has been proven rigorously only in the case where Ω is a ball.

In the case where $\kappa = \kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$ but $\varepsilon^{-2N}\kappa \to \infty$ as $\varepsilon \to 0$, there is no result as far as the author knows. We call such a saturation effect a *semi-strong* *saturation effect.* Thus, as a preliminary step, we intend to study the existence and the asymptotic behavior of a solution to the one-dimensional shadow system (1.1) for $\alpha \in (0,1)$.

1.1. Main results.

Before stating our main results, we reformulate our problem. For (1.1), we put $A(x) = \varepsilon^{-\alpha}\tilde{a}(x)$ and $\xi = \varepsilon^{-\alpha}\tilde{\xi}$. Then we have

$$\begin{cases} 0 = \varepsilon^2 \tilde{a}'' - \tilde{a} + \dfrac{\tilde{a}^2}{\tilde{\xi}(1+\kappa_0 \tilde{a}^2)}, \quad x \in (0,1), \\ \tilde{\xi} = \dfrac{1}{\varepsilon^\alpha} \displaystyle\int_0^1 \tilde{a}^2 dx, \\ \tilde{a}'(0) = \tilde{a}'(1) = 0. \end{cases} \tag{1.4}$$

Moreover, we put $a(y) = \tilde{a}(\varepsilon^\alpha y)$, $x = \varepsilon^\alpha y$, and we rewrite $\tilde{\xi} \mapsto \xi$ simply. Then we have the following problem:

$$\begin{cases} 0 = \varepsilon^{2(1-\alpha)} a'' - a + \dfrac{a^2}{\xi(1+\kappa_0 a^2)}, \quad y \in \Omega_\varepsilon := \left(0, \dfrac{1}{\varepsilon^\alpha}\right), \\ \xi = \displaystyle\int_0^{1/\varepsilon^\alpha} a^2 dy, \\ a'(0) = a'\left(\dfrac{1}{\varepsilon^\alpha}\right) = 0. \end{cases} \tag{1.5}$$

As the first step, we start with a formal consideration. With respect to the nonlinear term above, let us consider the graph of the function $g_\xi(t)$ with parameter $\xi > 0$ defined by

$$g_\xi(t) := -t + \frac{1}{\xi}\frac{t^2}{1+\kappa_0 t^2}, \quad t \in \mathbb{R}. \tag{1.6}$$

Then it is easy to see that there exist numbers $0 < \underline{\xi} < \overline{\xi} < \infty$ such that $g_\xi(t) = 0$, $t \in \mathbb{R}$, possesses three roots $t = 0$, $t_1(\xi)$, $t_2(\xi)$ $(0 < t_1(\xi) < t_2(\xi))$ for each $\xi \in (\underline{\xi}, \overline{\xi})$. $g'_\xi(0) < 0$, $g'_\xi(t_1(\xi)) > 0$ and $g'_\xi(t_2(\xi)) < 0$ hold for each $\xi \in (\underline{\xi}, \overline{\xi})$. Moreover, there exists a unique $\xi_* \in (\underline{\xi}, \overline{\xi})$ such that

$$J(\xi_*) = 0, \quad J'(\xi_*) < 0, \quad J(\xi) := \int_0^{t_2(\xi)} g_\xi(s) ds. \tag{1.7}$$

In the case of $\xi = \xi_*$, $g_{\xi_*}(t)$ has an equi-stable nonlinearity. For simplicity of notation, we write $\beta := t_2(\xi_*)$ henceforth. Under those situation, it is known that the problem

$$\begin{cases} w''(z) - w(z) + \dfrac{1}{\xi_*} f(w(z)) = 0, \quad z \in \mathbb{R}, \quad f(w) := \dfrac{w^2}{1+\kappa_0 w^2}, \\ w(-\infty) = \beta, \quad w(+\infty) = 0, \end{cases} \tag{1.8}$$

possesses a unique solution provided translations are neglected (see, e.g., [**6**]). Using this solution $w(z)$, if we consider $\xi \sim \xi_*$, then we could expect a solution to the first equation in (1.5) such that $a(y)$ has a transition-layer with the width of the order of $O(\varepsilon^{1-\alpha})$ around some point. Namely, if we take $y_c \in \Omega_\varepsilon$ suitably, then we can expect a one-parameter family of solutions $a_\varepsilon(y)$ to (1.5) such that

$$a_\varepsilon(y) \sim \beta, \quad y \in (0, y_c - \delta), \tag{1.9}$$

$$a_\varepsilon(y) \sim 0, \quad y \in (y_c + \delta, 1/\varepsilon^\alpha), \tag{1.10}$$

as $\varepsilon \to 0$ for any small $\delta > 0$. Indeed, we introduce a new z-coordinate in the transition-layer by $y = y_c + \varepsilon^{1-\alpha} z$. Let us assume that $a(y)$ is a solution to the first equation in (1.5), which has a transition-layer at certain point y_c. Putting $u(z) = a(y_c + \varepsilon^{1-\alpha} z)$ and taking a limit $\varepsilon \to 0$ formally, we see that $u(z)$ satisfies the same equation as that in (1.8). Therefore, we see that the solution $w(z)$ to (1.8) gives a good approximation of an inner solution $u(x)$ which connects outer solutions β and 0. On the other hand, if $a(y)$ has an asymptotic behavior such as (1.9) and (1.10), and if $a(y)$ decays exponentially at each point in (1.10), then we can approximate roughly as follows:

$$\int_0^{1/\varepsilon^\alpha} a^2(y) dy \sim \beta^2 y_c \tag{1.11}$$

for sufficiently small ε. Hence, considering the second equation in (1.5), in order to ensure that $\xi \sim \xi_*$ as $\varepsilon \to 0$ we see that y_c should be taken so that

$$y_c := \frac{\xi_*}{\beta^2}. \tag{1.12}$$

Let us define an approximate solution to (1.5). We define y_c by (1.12). Let $\chi_0 \in C_0^\infty(\Omega_\varepsilon)$ be a function such that, $0 \le \chi_0 \le 1$, and satisfies

$$\chi_0(y) = \begin{cases} 1, & y \in \Big[\frac{3}{4}y_c, \frac{5}{4}y_c\Big], \\ 0, & y \in \Big(0, \frac{1}{2}y_c\Big] \cup \Big[\frac{3}{2}y_c, \frac{1}{\varepsilon^\alpha}\Big). \end{cases} \tag{1.13}$$

Define χ_1 by

$$\chi_1(y) = \begin{cases} 1, & y \in \Big(0, \frac{1}{2}y_c\Big), \\ 1 - \chi_0(y), & y \in \Big[\frac{1}{2}y_c, \frac{3}{4}y_c\Big), \\ 0, & y \in \Big[\frac{3}{4}y_c, \frac{1}{\varepsilon^\alpha}\Big). \end{cases} \tag{1.14}$$

Let w_γ be a unique solution to (1.8) with initial value $w(0) = \gamma$ for each $\gamma \in (0, \beta)$. We define an approximate solution $U_{\varepsilon,\gamma}(y)$ by

$$U_{\varepsilon,\gamma}(y) := w_\gamma\Big(\frac{y - y_c}{\varepsilon^{1-\alpha}}\Big)\chi_0(y) + \beta\chi_1(y), \quad y \in \Big(0, \frac{1}{\varepsilon^\alpha}\Big). \tag{1.15}$$

Now, we are ready to state our main results.

Theorem 1.1. *Let $\alpha \in (0,1)$. For sufficiently small $\varepsilon > 0$, there exists a one-parameter family of positive solutions $(a_\varepsilon, \xi_\varepsilon)$ to (1.5), and $(a_\varepsilon, \xi_\varepsilon)$ has the following asymptotic behavior*:

$$a_\varepsilon(y) = U_{\varepsilon,\gamma_\varepsilon}(y) + \exp\Big\{-\frac{c_1}{\varepsilon^{1-\alpha}}\Big\}\tilde{\phi}_\varepsilon(y), \quad y \in \Omega_\varepsilon, \tag{1.16}$$

$$\xi_\varepsilon = \xi_* + O\Big(\varepsilon^{(1-\alpha)/2}\exp\Big\{-\frac{c_1}{\varepsilon^{1-\alpha}}\Big\}\Big), \tag{1.17}$$

as $\varepsilon \to 0$, for some $c_1 > 0$ independent of ε and some $\gamma_\varepsilon \in (0,\beta)$. The function $\tilde{\phi}_\varepsilon$ is bounded independently of ε. Moreover, γ_ε accumulate at a certain unique $\gamma_* \in (0,\beta)$ as $\varepsilon \to 0$. In particular, for any small $\delta > 0$, it holds that

$$\sup_{y \in (0, y_c - \delta)} |a_\varepsilon(y) - \beta| \to 0, \qquad \sup_{y \in (y_c + \delta, 1/\varepsilon^\alpha)} |a_\varepsilon(y)| \to 0, \tag{1.18}$$

as $\varepsilon \to 0$.

By this theorem, we obtain an internal transition-layer solution to (1.5), and hence we obtain a solution to (1.1).

COROLLARY 1.1. *Let $\alpha \in (0,1)$. For sufficiently small $\varepsilon > 0$, there exists a one-parameter family of positive solutions $(A_\varepsilon, \xi_\varepsilon)$ to (1.1) such that*

$$A_\varepsilon(x) = \frac{1}{\varepsilon^\alpha}\left\{U_{\varepsilon,\gamma_\varepsilon}\left(\frac{x}{\varepsilon^\alpha}\right) + \exp\left\{-\frac{c_1}{\varepsilon^{1-\alpha}}\right\}\tilde{\phi}_\varepsilon\left(\frac{x}{\varepsilon^\alpha}\right)\right\}, \quad x \in (0,1), \tag{1.19}$$

$$\xi_\varepsilon = \frac{1}{\varepsilon^\alpha}\left\{\xi_* + O\left(\varepsilon^{(1-\alpha)/2}\exp\left\{-\frac{c_1}{\varepsilon^{1-\alpha}}\right\}\right)\right\}, \tag{1.20}$$

as $\varepsilon \to 0$. In particular, for any small $\delta > 0$, it holds that

$$\sup_{x\in(0,\varepsilon^\alpha(y_c-\delta))}\left|A_\varepsilon(x) - \frac{1}{\varepsilon^\alpha}\beta\right| \to 0, \qquad \sup_{x\in(\varepsilon^\alpha(y_c+\delta),1)}|A_\varepsilon(x)| \to 0, \tag{1.21}$$

as $\varepsilon \to 0$.

We note that

$$U_{\varepsilon,\gamma_\varepsilon}\left(\frac{x}{\varepsilon^\alpha}\right) = w_{\gamma_\varepsilon}\left(\frac{x-\varepsilon^\alpha y_c}{\varepsilon}\right)\chi_0\left(\frac{x}{\varepsilon^\alpha}\right) + \beta\chi_1\left(\frac{x}{\varepsilon^\alpha}\right). \tag{1.22}$$

Hence, we notice that $A_\varepsilon(x)$ has a transition-layer of $O(\varepsilon)$ in width at $x = \varepsilon^\alpha y_c$.

REMARK 1. In this paper, we do not treat the stability of our solution because the main purpose is to show the existence and the asymptotic behavior of solutions to the shadow system (1.1) with semi-strong saturation effect. However, the solution obtaind in Corollary 1.1 seems to be stable by a simple numerical simulation if τ is small enough. The author thinks that it is not easy to study the stability of our solution rigorously because it is harder to study the properties of the linearized operator in the case $\kappa > 0$ than in the case of $\kappa = 0$. However, dynamics of the problem is also important. For study of the dynamics of the Gierer-Meinhardt system, see [**7**], [**8**], [**12**], [**13**], [**17**], [**21**], [**26**], [**27**], [**31**], [**32**], [**33**].

1.2. Preliminaries and outline of our construction.

We first state some properties of solutions to (1.8). We recall that (1.8) admits a unique solution w_γ provided the initial value $w(0) = \gamma$, $\gamma \in (0,\beta)$, is given. $w_\gamma(z)$ is monotone decreasing:

$$w_\gamma'(z) < 0, \quad z \in \mathbb{R}. \tag{1.23}$$

Moreover, for each $\gamma \in (0,\beta)$, the following estimates hold:

$$\max\left\{\beta - w_\gamma(z), |w_\gamma'(z)|\right\} \le Ce^{cz}, \qquad z<0, \tag{1.24}$$

$$\max\left\{w_\gamma(z), |w_\gamma'(z)|\right\} \le Ce^{-cz}, \quad z\ge 0, \tag{1.25}$$

for some constants $C, c>0$. Furthermore, from the equation (1.8) and the estimates above, we have

$$\begin{aligned}\left|w_\gamma''(z)\right| &\le w_\gamma(z) + \frac{1}{\xi_*} f(w_\gamma(z)) \le w_\gamma(z) + \frac{1}{\xi_*} w_\gamma^2(z)\\ &\le C' w^{c'z}, \quad z<0,\end{aligned}$$

for some $C', c'>0$, and

$$\begin{aligned}\left|w_\gamma''(z)\right| &= \left|w_\gamma(z) - \frac{1}{\xi_*} f(w_\gamma)\right| = \left|w_\gamma(z) - \beta - \frac{1}{\xi_*}(f(w_\gamma(z)) - f(\beta))\right|\\ &\le |w_\gamma(z) - \beta| + \frac{1}{\xi_*}|f(w_\gamma(z)) - f(\beta)|\\ &\le C'' e^{-c''z}, \quad z\ge 0,\end{aligned}$$

for some $C'', c''>0$. Hence, $|w_\gamma''(z)|$ decays exponentially at infinity. Moreover, from the equation (1.8), the following equation holds:

$$w_\gamma'''(z) - w_\gamma'(z) + \frac{1}{\xi_*} f'(w_\gamma(z)) w_\gamma'(z) = 0, \quad z\in\mathbb{R}. \tag{1.26}$$

Therefore, we see that $|w_\gamma'''(z)|$ also decays exponentially at infinity. Here, we note that, for γ, $\gamma'\in(0,\beta)$, there is only difference of translations, namely,

$$w_{\gamma'}(z) = w_\gamma(z - z(\gamma')), \quad z\in\mathbb{R},$$

for some $z(\gamma')$. Therefore, if we restrict $\gamma\in[\gamma_1,\gamma_2]$ for fixed $\gamma_1,\gamma_2\in(0,\beta)$, then we have uniform estimates with respect to γ and we obtain the following lemma.

Lemma 1.1. *For fixed γ_1, $\gamma_2\in(0,\beta)$, $\gamma_1<\gamma_2$, there exist constants C, $c>0$ such that*

$$\max\left\{\beta - w_\gamma(z), |w_\gamma'(z)|, |w_\gamma''(z)|, |w_\gamma'''(z)|\right\} \le Ce^{cz}, \qquad z<0, \tag{1.27}$$

$$\max\left\{w_\gamma(z), |w_\gamma'(z)|, |w_\gamma''(z)|, |w_\gamma'''(z)|\right\} \le Ce^{-cz}, \quad z\ge 0, \tag{1.28}$$

hold for any $\gamma \in [\gamma_1, \gamma_2]$.

Now, let us state our construction of a solution to (1.5). We put $u(z) = a(y_c + \varepsilon^{1-\alpha} z)$, $y = y_c + \varepsilon^{1-\alpha} z$. Then we have the following equations equivalent to (1.5):

$$0 = u'' - u + \frac{1}{\xi} f(u), \quad z \in \tilde{\Omega}_\varepsilon := \left(-\frac{1}{\varepsilon^{1-\alpha}} y_c, \frac{1}{\varepsilon^{1-\alpha}} \left(\frac{1}{\varepsilon^\alpha} - y_c \right) \right), \tag{1.29}$$

$$\xi = \varepsilon^{1-\alpha} \int_{\tilde{\Omega}_\varepsilon} u^2(z) dz. \tag{1.30}$$

Throughout this paper, we set $f(u) = u^2/(1 + \kappa_0 u^2)$. We put

$$\overline{U}_{\varepsilon,\gamma}(z) := U_{\varepsilon,\gamma}(y_c + \varepsilon^{1-\alpha} z) \tag{1.31}$$

$$= w_\gamma(z) \overline{\chi}_0(z) + \beta \overline{\chi}_1(z), \tag{1.32}$$

where $U_{\varepsilon,\gamma}$ is defined by (1.15), and $\overline{\chi}_0(z)$ and $\overline{\chi}_1(z)$ mean

$$\overline{\chi}_0(z) = \chi_0(y_c + \varepsilon^{1-\alpha} z), \quad \overline{\chi}_1(z) = \chi_1(y_c + \varepsilon^{1-\alpha} z).$$

REMARK 2. Fundamentally, we should write like $\overline{\chi}_0^\varepsilon(z)$ and $\overline{\chi}_1^\varepsilon(z)$ instead of $\overline{\chi}_0(z)$ and $\overline{\chi}_1(z)$ because they depend on ε. However, only the bounds of $\overline{\chi}_0$ and $\overline{\chi}_1$ are important for our argument. We note that $\overline{\chi}_0(z), \overline{\chi}_1(z)$ and their derivatives are bounded independently of ε small. Therefore, we omit the index of ε.

Let us first consider the single equation (1.29). We expect that there exists a solution to (1.29), near the approximate function $\overline{U}_{\varepsilon,\gamma}$, provided $|\xi - \xi_*| \ll 1$, for sufficiently small ε. Let us consider ξ to be a parameter of the equation (1.29) and restrict the range so that

$$\xi \in I_\varepsilon := \left\{ \xi \in \mathbb{R} : |\xi - \xi_*| \le \varepsilon^{(1-\alpha)/2} e^{-c_1/\varepsilon^{1-\alpha}} \right\}, \tag{1.33}$$

where $c_1 > 0$ is some fixed constant decided later (see (2.6)). For our purpose, we need to analyze the following linearized operator:

$$\tilde{L}_{\varepsilon,\gamma,\xi} \phi := \phi'' - \phi + \frac{1}{\xi} f'(\overline{U}_{\varepsilon,\gamma}) \phi. \tag{1.34}$$

From (1.26), $\tilde{L}_{\varepsilon,\gamma,\xi}$ is likely to have an eigenvalue near 0 for ε sufficiently small.

Hence, we use the Liapunov-Schmidt reduction method. We follow the method used in [**32**]. Let us define an approximate kernel of $\tilde{L}_{\varepsilon,\gamma,\xi}$ as follows:

$$\operatorname{span}\left\{\frac{d\overline{U}_{\varepsilon,\gamma}}{dz}\right\} \subset H^2(\tilde{\Omega}_\varepsilon). \tag{1.35}$$

We write $\overline{U}'_{\varepsilon,\gamma}(z) = d\overline{U}_{\varepsilon,\gamma}/dz$ simply, henceforth. Define

$$E_{\varepsilon,\gamma}u := \frac{(\overline{U}'_{\varepsilon,\gamma}, u)_{L^2(\tilde{\Omega}_\varepsilon)}}{\|\overline{U}'_{\varepsilon,\gamma}\|^2_{L^2(\tilde{\Omega}_\varepsilon)}}\overline{U}'_{\varepsilon,\gamma}, \quad u \in L^2(\tilde{\Omega}_\varepsilon), \tag{1.36}$$

and

$$\pi^\perp_{\varepsilon,\gamma} := I - E_{\varepsilon,\gamma}, \tag{1.37}$$

where I is an identity map on $L^2(\tilde{\Omega}_\varepsilon)$. Then $\pi^\perp_{\varepsilon,\gamma}$ is a projection from $L^2(\tilde{\Omega}_\varepsilon)$ into $C^\perp_{\varepsilon,\gamma}$, where

$$C^\perp_{\varepsilon,\gamma} := \left\{u \in L^2(\tilde{\Omega}_\varepsilon) : \int_{\tilde{\Omega}_\varepsilon} u(z)\overline{U}'_{\varepsilon,\gamma}(z)dz = 0\right\}. \tag{1.38}$$

We put

$$K^\perp_{\varepsilon,\gamma} := \left\{u \in H^2(\tilde{\Omega}_\varepsilon) : \int_{\tilde{\Omega}_\varepsilon} u(z)\overline{U}'_{\varepsilon,\gamma}(z)dz = 0\right\}, \tag{1.39}$$

$$H^2_\nu(\tilde{\Omega}_\varepsilon) := \left\{u \in H^2(\tilde{\Omega}_\varepsilon) : u'\left(-\frac{1}{\varepsilon^{1-\alpha}}y_c\right) = u'\left(\frac{1}{\varepsilon^{1-\alpha}}\left(\frac{1}{\varepsilon^\alpha} - y_c\right)\right) = 0\right\}. \tag{1.40}$$

We define an operator $L_{\varepsilon,\gamma,\xi}$ on $C^\perp_{\varepsilon,\gamma}$ by

$$Dom(L_{\varepsilon,\gamma,\xi}) = K^\perp_{\varepsilon,\gamma} \cap H^2_\nu(\tilde{\Omega}_\varepsilon), \quad L_{\varepsilon,\gamma,\xi} := \pi^\perp_{\varepsilon,\gamma} \circ \tilde{L}_{\varepsilon,\gamma,\xi}. \tag{1.41}$$

It is easy to see that $L_{\varepsilon,\gamma,\xi}$ is a self-adjoint operator. Then we will see that $L_{\varepsilon,\gamma,\xi}$ is invertible as an operator from $K^\perp_{\varepsilon,\gamma} \cap H^2_\nu(\tilde{\Omega}_\varepsilon)$ into $C^\perp_{\varepsilon,\gamma}$ for ε sufficiently small (Lemma 3.1). We divide the problem (1.29) into

$$\pi^\perp_{\varepsilon,\gamma}S[u;\xi] = 0, \tag{1.42}$$

$$E_{\varepsilon,\gamma}S[u;\xi]=0, \tag{1.43}$$

where

$$S[u;\xi]:=u''-u+\frac{1}{\xi}f(u). \tag{1.44}$$

Our construction consists of three steps:

Step 1. For sufficiently small ε, we will construct a solution $u_\varepsilon(\cdot;\gamma,\xi)\in H^2_\nu(\tilde{\Omega}_\varepsilon)$ of the equation (1.42), near $\overline{U}_{\varepsilon,\gamma}$, for each $\gamma\in[\gamma_1,\gamma_2]$ and $\xi\in I_\varepsilon$.

Step 2. For each ε sufficiently small and each $\gamma\in[\gamma_1,\gamma_2]$, we will find $\xi=\xi_{\varepsilon,\gamma}\in I_\varepsilon$ such that (1.43) holds, namely,

$$\int_{\tilde{\Omega}_\varepsilon} S[u_\varepsilon(z;\gamma,\xi_{\varepsilon,\gamma});\xi_{\varepsilon,\gamma}]\overline{U}'_{\varepsilon,\gamma}(z)dz=0.$$

Step 3. For each ε sufficiently small, we will find $\gamma=\gamma_\varepsilon\in[\gamma_1,\gamma_2]$ such that (1.30) holds, namely,

$$\xi_{\varepsilon,\gamma_\varepsilon}=\varepsilon^{1-\alpha}\int_{\tilde{\Omega}_\varepsilon}u_\varepsilon^2(z;\gamma_\varepsilon,\xi_{\varepsilon,\gamma_\varepsilon})dz.$$

In Section 2, we lead some basic estimates. In Section 3, we complete Step 1. In Section 4, we show the continuity and the differentiability of $u_\varepsilon(\cdot;\gamma,\xi)$ with respect to γ and ξ. In Sections 5 and 6, we complete Step 2 and 3, respectively.

2. Basic estimates.

In this section, we show some estimates. The following lemma is the one on the estimate of error term.

LEMMA 2.1. *Let $M>0$ and $\gamma_1,\gamma_2\in(0,\beta)$ be fixed. Then there exist constants $\overline{C}_1$, $\bar{c}_1>0$ such that*

$$\left\|S[\overline{U}_{\varepsilon,\gamma};\xi]\right\|_{L^2(\tilde{\Omega}_\varepsilon)}\le\left|\frac{1}{\xi_*}-\frac{1}{\xi}\right|\|f(w_\gamma)\|_{L^2(\tilde{\Omega}_\varepsilon)}+\overline{C}_1e^{-\bar{c}_1/\varepsilon^{1-\alpha}} \tag{2.1}$$

holds for all $\gamma\in[\gamma_1,\gamma_2]$ and all $\xi>0$ such that $1/\xi<M$.

PROOF. We calculate the term $\overline{U}''_{\varepsilon,\gamma}-\overline{U}_{\varepsilon,\gamma}$:

$$\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} = w''_\gamma\overline{\chi}_0 + 2w'_\gamma\overline{\chi}'_0 + w_\gamma\overline{\chi}''_0 + \beta\overline{\chi}''_1 - w_\gamma\overline{\chi}_0 - \beta\overline{\chi}_1.$$

By using Lemma 1.1, we can estimate as follows.

For $|z| \le y_c/(4\varepsilon^{1-\alpha})$, because $\overline{\chi}_0 \equiv 1$ and $\overline{\chi}_1 \equiv 0$, we have

$$\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} = w''_\gamma - w_\gamma.$$

For $z \in (-y_c/\varepsilon^{1-\alpha}, -y_c/(2\varepsilon^{1-\alpha}))$, because $\overline{\chi}_0 \equiv 0$ and $\overline{\chi}_1 \equiv 1$, we have

$$\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} = -\beta = w''_\gamma - w_\gamma + \big(w_\gamma - \beta - w''_\gamma\big) = w''_\gamma - w_\gamma + O(e^{-c|z|}).$$

For $z \in (-y_c/(2\varepsilon^{1-\alpha}), -y_c/(4\varepsilon^{1-\alpha}))$, because $\overline{\chi}_1 = 1 - \overline{\chi}_0$, we have

$$\begin{aligned}\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} &= w''_\gamma\overline{\chi}_0 + 2w'_\gamma\overline{\chi}'_0 + w_\gamma\overline{\chi}''_0 - \beta\overline{\chi}''_0 - w_\gamma\overline{\chi}_0 - \beta(1-\overline{\chi}_0)\\ &= w''_\gamma - w_\gamma - (1-\overline{\chi}_0)w''_\gamma + (w_\gamma - \beta) + \overline{\chi}_0(\beta - w_\gamma) + 2w'_\gamma\overline{\chi}'_0\\ &\quad + \overline{\chi}''_0(w_\gamma - \beta)\\ &= w''_\gamma - w_\gamma + O(e^{-c|z|}).\end{aligned}$$

For $z \in \tilde{\Omega}_\varepsilon \backslash (-y_c/\varepsilon^{1-\alpha}, y_c/(4\varepsilon^{1-\alpha}))$, because w_γ and $|w''_\gamma|$ are estimated by $Ce^{-c|z|}$, we have

$$\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} = w''_\gamma - w_\gamma + O(\varepsilon^{-c|z|}).$$

Then we can see that

$$\begin{aligned}\big\|S[\overline{U}_{\varepsilon,\gamma};\xi]\big\|_{L^2(\tilde{\Omega}_\varepsilon)} &= \left\|\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} + \frac{1}{\xi}f(\overline{U}_{\varepsilon,\gamma})\right\|_{L^2(\tilde{\Omega}_\varepsilon)}\\ &\le \left\|w''_\gamma - w_\gamma + \frac{1}{\xi}f(\overline{U}_{\varepsilon,\gamma})\right\|_{L^2(\tilde{\Omega}_\varepsilon)} + C'e^{-c'/\varepsilon^{1-\alpha}}\\ &= \left\| -\frac{1}{\xi_*}f(w_\gamma) + \frac{1}{\xi}f(\overline{U}_{\varepsilon,\gamma})\right\|_{L^2(\tilde{\Omega}_\varepsilon)} + C'e^{-c'/\varepsilon^{1-\alpha}}\\ &\le \left|\frac{1}{\xi_*} - \frac{1}{\xi}\right| \|f(w_\gamma)\|_{L^2(\tilde{\Omega}_\varepsilon)} + \frac{1}{\xi}\big\|f(w_\gamma) - f(\overline{U}_{\varepsilon,\gamma})\big\|_{L^2(\tilde{\Omega}_\varepsilon)}\\ &\quad + C'e^{-c'/\varepsilon^{1-\alpha}},\end{aligned}$$

for some $C', c' > 0$. Here, we see that

$$\left|f(w_\gamma) - f(\overline{U}_{\varepsilon,\gamma})\right| \le \left|w_\gamma^2 - \overline{U}_{\varepsilon,\gamma}^2\right| \le 2\beta\left|w_\gamma - \overline{U}_{\varepsilon,\gamma}\right|$$

by direct calculation. Noting $w_\gamma \equiv \overline{U}_{\varepsilon,\gamma}$ for $z \in (-y_c/(4\varepsilon^{1-\alpha}), y_c/(4\varepsilon^{1-\alpha}))$, we see by the same consideration as above that

$$\left\|f(w_\gamma) - f(\overline{U}_{\varepsilon,\gamma})\right\|_{L^2(\tilde{\Omega}_\varepsilon)} \le C'' e^{-c''/\varepsilon^{1-\alpha}}$$

holds for some C'', $c'' > 0$. Thus we have a conclusion. □

Next, we see that the same type of estimate holds for $\tilde{L}_{\varepsilon,\gamma,\xi}\overline{U}'_{\varepsilon,\gamma}$.

Lemma 2.2. *Let $M > 0$ and γ_1, $\gamma_2 \in (0,\beta)$ be fixed. Then there exist constants $\overline{C}_2$, $\overline{c}_2 > 0$ such that*

$$\left\|\tilde{L}_{\varepsilon,\gamma,\xi}\overline{U}'_{\varepsilon,\gamma}\right\|_{L^2(\tilde{\Omega}_\varepsilon)} \le \left|\frac{1}{\xi_*} - \frac{1}{\xi}\right| \left\|f'(w_\gamma)w'_\gamma\right\|_{L^2(\tilde{\Omega}_\varepsilon)} + \overline{C}_2 e^{-\overline{c}_2/\varepsilon^{1-\alpha}} \tag{2.2}$$

holds for all $\gamma \in [\gamma_1, \gamma_2]$ and all $\xi > 0$ such that $1/\xi < M$.

Proof. The proof can be carried out by the same argument as that in the proof of Lemma 2.1. Thus we omit the proof. □

Lemma 2.3. *Let $M > 0$ and γ_1, $\gamma_2 \in (0,\beta)$ be fixed. Then the following identity holds*

$$\int_{\tilde{\Omega}_\varepsilon} S[\overline{U}_{\varepsilon,\gamma};\xi]\overline{U}'_{\varepsilon,\gamma}dz = \left(\frac{1}{\xi} - \frac{1}{\xi_*}\right)\int_{-D_\varepsilon}^{D_\varepsilon} f(w_\gamma)w'_\gamma dz + k(\varepsilon), \tag{2.3}$$

$$D_\varepsilon := \frac{1}{4\varepsilon^{1-\alpha}}y_c,$$

for all $\gamma \in [\gamma_1, \gamma_2]$ and all $\xi > 0$ such that $1/\xi < M$. The term $k(\varepsilon)$ satisfies

$$|k(\varepsilon)| \le \overline{C}_3 e^{-\overline{c}_3/\varepsilon^{1-\alpha}} \tag{2.4}$$

for some constants $\overline{C}_3$, $\overline{c}_3 > 0$ independent of ε, γ and ξ.

Proof. By the definition of $\overline{U}_{\varepsilon,\gamma}$, we have $\overline{U}'_{\varepsilon,\gamma} = w'_\gamma\overline{\chi}_0 + w_\gamma\overline{\chi}'_0 + \beta\overline{\chi}'_1$. Therefore,

$$\overline{U}'_{\varepsilon,\gamma} = \begin{cases} w'_\gamma, & |z| < D_\varepsilon, \\ w'_\gamma \overline{\chi}_0 + (w_\gamma - \beta)\overline{\chi}'_0, & z \in (-2D_\varepsilon, -D_\varepsilon), \\ w'_\gamma \overline{\chi}_0 + w_\gamma \overline{\chi}'_0, & z \in (D_\varepsilon, 2D_\varepsilon), \\ 0, & \text{elsewhere}. \end{cases} \tag{2.5}$$

Noting $w''_\gamma - w_\gamma + f(w_\gamma)/\xi_* = 0$, we have

$$\begin{aligned} \int_{\tilde{\Omega}_\varepsilon} S\big[\overline{U}_{\varepsilon,\gamma};\xi\big]\overline{U}'_{\varepsilon,\gamma} dz &= \int_{\tilde{\Omega}_\varepsilon} \left(\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} + \frac{1}{\xi} f(\overline{U}_{\varepsilon,\gamma})\right)\overline{U}'_{\varepsilon,\gamma} dz \\ &= \int_{-D_\varepsilon}^{D_\varepsilon} \left(w''_\gamma - w_\gamma + \frac{1}{\xi} f(w_\gamma)\right) w'_\gamma dz \\ &\quad + \int_{\tilde{\Omega}_\varepsilon \setminus [-D_\varepsilon, D_\varepsilon]} \left(\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} + \frac{1}{\xi} f(\overline{U}_{\varepsilon,\gamma})\right)\overline{U}'_{\varepsilon,\gamma} dz \\ &= \left(\frac{1}{\xi} - \frac{1}{\xi_*}\right) \int_{-D_\varepsilon}^{D_\varepsilon} f(w_\gamma) w'_\gamma dz \\ &\quad + \int_{\tilde{\Omega}_\varepsilon \setminus [-D_\varepsilon, D_\varepsilon]} \left(\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} + \frac{1}{\xi} f(\overline{U}_{\varepsilon,\gamma})\right)\overline{U}'_{\varepsilon,\gamma} dz. \end{aligned}$$

Using the estimates in Lemma 1.1 and (2.5), and noting $\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} + f(\overline{U}_{\varepsilon,\gamma})/\xi$ is bounded, we can estimate as follows:

$$\left| \int_{\tilde{\Omega}_\varepsilon \setminus [-D_\varepsilon, D_\varepsilon]} \left(\overline{U}''_{\varepsilon,\gamma} - \overline{U}_{\varepsilon,\gamma} + \frac{1}{\xi} f(\overline{U}_{\varepsilon,\gamma})\right)\overline{U}'_{\varepsilon,\gamma} dz \right| \le \overline{C}_3 e^{-\overline{c}_3/\varepsilon^{1-\alpha}}$$

for some $\overline{C}_3$, $\overline{c}_3 > 0$. Thus we complete the proof. □

Here, we define the number $c_1 > 0$ in (1.33) so that

$$c_1 < \min\{\overline{c}_1, \overline{c}_2, \overline{c}_3\}. \tag{2.6}$$

We always consider $\xi \in I_\varepsilon = \{\xi \in \mathbb{R} : |\xi - \xi_*| \le \varepsilon^{(1-\alpha)/2} e^{-c_1/\varepsilon^{1-\alpha}}\}$, henceforth.

3. Liapunov-Schmidt reduction method.

In this section, we will show the invertibility of $L_{\varepsilon,\gamma,\xi}$ and solve the equation (1.42).

Lemma 3.1. *There exist $\varepsilon_1 > 0$ and $\lambda > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$, $\gamma \in [\gamma_1, \gamma_2]$ and $\xi \in I_\varepsilon$, the following hold:*

$$\left\|L_{\varepsilon,\gamma,\xi}\phi\right\|_{L^2(\tilde{\Omega}_\varepsilon)} \geq \lambda\|\phi\|_{H^2(\tilde{\Omega}_\varepsilon)}, \quad \phi \in K_{\varepsilon,\gamma}^{\perp} \cap H_\nu^2(\tilde{\Omega}_\varepsilon), \tag{3.1}$$

$$\mathrm{Ran}(L_{\varepsilon,\gamma,\xi}) = C_{\varepsilon,\gamma}^{\perp}. \tag{3.2}$$

Therefore, $L_{\varepsilon,\gamma,\xi} : K_{\varepsilon,\gamma}^{\perp} \cap H_\nu^2(\tilde{\Omega}_\varepsilon) \to C_{\varepsilon,\gamma}^{\perp}$ has a bounded inverse $L_{\varepsilon,\gamma,\xi}^{-1}$.

Proof. We first show (3.1). Let the contrary be true. Then there exist sequences ε_n, $\gamma_n \in [\gamma_1, \gamma_2]$, $\xi_n \in I_{\varepsilon_n}$ and $\phi_n \in K_{\varepsilon_n,\gamma_n}^{\perp} \cap H_\nu^2(\tilde{\Omega}_{\varepsilon_n})$, $n = 1, 2, \ldots$, such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\left\|L_{\varepsilon_n,\gamma_n,\xi_n}\phi_n\right\|_{L^2(\tilde{\Omega}_{\varepsilon_n})} \leq \frac{1}{n}, \quad \|\phi_n\|_{H^2(\tilde{\Omega}_{\varepsilon_n})} = 1, \tag{3.3}$$

for $n = 1, 2, \ldots$. We may assume $\gamma_n \to \gamma' \in [\gamma_1, \gamma_2]$ as $n \to \infty$ by the compactness of $[\gamma_1, \gamma_2]$. Moreover, we note that $\xi_n \to \xi_*$ as $n \to \infty$. We extend ϕ_n into $H^2(\mathbb{R})$-function for each n. We keep the same notation for the extended function for simplicity of notation. Then we can see that

$$\|\phi_n\|_{H^2(\mathbb{R})} \leq M \tag{3.4}$$

holds for some $M > 0$ independent of n. Hence we can pick up a subsequence (we denote the subsequence by $\{\phi_n\}$ simply) such that,

$$\phi_n \rightharpoonup \phi \text{ in } H^2(\mathbb{R}), \tag{3.5}$$

$$\phi_n \to \phi \text{ in } L_{loc}^2(\mathbb{R}) \text{ and } L_{loc}^\infty(\mathbb{R}), \tag{3.6}$$

as $n \to \infty$, for some $\phi \in H^2(\mathbb{R})$, where "$\rightharpoonup$" means the weak-limit. Now, we take a test function $\varphi \in C_0^\infty(\mathbb{R})$. Set $K = \mathrm{supp}(\varphi)$. We may assume $K \subset \tilde{\Omega}_{\varepsilon_n}$ considering ε_n to be small enough. Let

$$\begin{aligned}&\left(L_{\varepsilon_n,\gamma_n,\xi_n}\phi_n, \varphi\right)_{L^2(\tilde{\Omega}_{\varepsilon_n})}\\ &\quad = \left(\tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\phi_n, \varphi\right)_{L^2(K)} - \left(E_{\varepsilon_n,\gamma_n}\tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\phi_n, \varphi\right)_{L^2(K)}.\end{aligned} \tag{3.7}$$

Then it is easy to see that

$$\left(\tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\phi_n, \varphi\right)_{L^2(K)} \to \int_K \left\{\phi'' - \phi + \frac{1}{\xi_*} f'(w_{\gamma'})\phi\right\}\varphi dz \tag{3.8}$$

as $n \to \infty$. Recall that

$$E_{\varepsilon_n,\gamma_n}\tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\phi_n = \frac{(\overline{U}'_{\varepsilon_n,\gamma_n}, \tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\phi_n)_{L^2(\tilde{\Omega}_{\varepsilon_n})}}{\|\overline{U}'_{\varepsilon_n,\gamma_n}\|^2_{L^2(\tilde{\Omega}_{\varepsilon_n})}}\overline{U}'_{\varepsilon_n,\gamma_n}. \tag{3.9}$$

In this term, we can calculate as follows

$$\begin{aligned}\left|(\overline{U}'_{\varepsilon_n,\gamma_n}, \tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\phi_n)_{L^2(\tilde{\Omega}_{\varepsilon_n})}\right| &= \left|(\tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\overline{U}'_{\varepsilon_n,\gamma_n}, \phi_n)_{L^2(\tilde{\Omega}_{\varepsilon_n})}\right| \\ &\le \left\|\tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\overline{U}'_{\varepsilon_n,\xi_n}\right\|_{L^2(\tilde{\Omega}_{\varepsilon_n})}.\end{aligned}$$

Hence, we see by Lemma 2.2 that

$$\left|(E_{\varepsilon_n,\gamma_n}\tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\phi_n, \varphi)_{L^2(K)}\right| \to 0 \tag{3.10}$$

as $n \to \infty$. On the other hand, we notice that

$$\left|(L_{\varepsilon_n,\gamma_n,\xi_n}\phi_n, \varphi)_{L^2(\tilde{\Omega}_{\varepsilon_n})}\right| \le \left\|L_{\varepsilon_n,\gamma_n,\xi_n}\phi_n\right\|_{L^2(\tilde{\Omega}_{\varepsilon_n})}\|\varphi\|_{L^2(\tilde{\Omega}_{\varepsilon_n})} \to 0 \tag{3.11}$$

as $n \to \infty$ from (3.3). Combining (3.7), (3.8), (3.10) and (3.11), we have

$$\int_{\mathbb{R}}\left\{\phi'' - \phi + \frac{1}{\xi_*}f'(w_{\gamma'})\phi\right\}\varphi dz = 0 \tag{3.12}$$

for any $\varphi \in C_0^\infty(\mathbb{R})$. Therefore, we see that

$$\phi'' - \phi + \frac{1}{\xi_*}f'(w_{\gamma'})\phi = 0 \quad \text{in } \mathbb{R}. \tag{3.13}$$

It is known that such a bounded function ϕ satisfying (3.13) is only a multiple of $w'_{\gamma'}$ (see, e.g, [**1**], [**6**]). However, we can see that $\phi \perp w'_{\gamma'}$ in $L^2(\mathbb{R})$ from $\phi_n \in K^\perp_{\varepsilon_n,\gamma_n}$. Therefore, we have $\phi = 0$.

Now, we claim that $\|\phi_n\|_{H^2(\tilde{\Omega}_{\varepsilon_n})} \to 0$ as $n \to \infty$, and we lead a contradiction. For the purpose, we divide $\tilde{\Omega}_{\varepsilon_n}$ into two intervals as follows

$$\tilde{\Omega}^1_{\varepsilon_n} := \left(-\frac{1}{\varepsilon_n^{1-\alpha}}y_c, -a\right), \tag{3.14}$$

$$\tilde{\Omega}^2_{\varepsilon_n} := \left(-a, \frac{1}{\varepsilon^{1-\alpha}}\left(\frac{1}{\varepsilon^\alpha} - y_c\right)\right), \tag{3.15}$$

where $a > 0$ is a fixed large constant. Here, we note that we may regard ϕ_n as a C^1-function by Sobolev's embedding theorem. Moreover, because the C^1-norm of ϕ_n is bounded from (3.4) and $\phi_n \to 0$ in $L^\infty_{loc}(\mathbb{R})$, we see that $|\phi_n'(-a)\phi_n(-a)| \to 0$ as $n \to \infty$.

We first claim that $\|\phi_n\|_{H^2(\tilde\Omega^2_{\varepsilon_n})} \to 0$ as $n \to \infty$. Let

$$\phi_n'' - \phi_n = L_{\varepsilon_n,\gamma_n,\xi_n}\phi_n + E_{\varepsilon_n,\gamma_n}\tilde L_{\varepsilon_n,\gamma_n,\xi_n}\phi_n - \frac{1}{\xi_n}f'(\overline{U}_{\varepsilon_n,\gamma_n})\phi_n =: \tilde g_n. \tag{3.16}$$

Let us estimate the $L^2(\tilde\Omega^2_{\varepsilon_n})$-norm of $\tilde g_n$. Let

$$\begin{aligned}\|\tilde g_n\|_{L^2(\tilde\Omega^2_{\varepsilon_n})} \le \left\|L_{\varepsilon_n,\gamma_n,\xi_n}\phi_n\right\|_{L^2(\tilde\Omega^2_{\varepsilon_n})} + \left\|E_{\varepsilon_n,\gamma_n}\tilde L_{\varepsilon_n,\gamma_n,\xi_n}\phi_n\right\|_{L^2(\tilde\Omega^2_{\varepsilon_n})} \\ + \frac{1}{\xi_n}\left\|f'(\overline{U}_{\varepsilon_n,\gamma_n})\phi_n\right\|_{L^2(\tilde\Omega^2_{\varepsilon_n})}.\end{aligned} \tag{3.17}$$

We have already known that the first term and the second term in (3.17) tend to 0 as $n \to \infty$. We see that the third term also tends to 0 as $n \to \infty$ by using $\phi_n \to \phi = 0$ in $L^\infty_{loc}(\mathbb{R})$ and the estimates: $|\overline{U}_{\varepsilon_n,\gamma_n}(z)| \le Ce^{-cz}$, $z > 0$, and $\|\phi_n\|_{L^\infty(\mathbb{R})} \le C'$ for some $C, c, C' > 0$ independent of n. Hence, we have $\|\tilde g_n\|_{L^2(\tilde\Omega^2_{\varepsilon_n})} \to 0$ as $n \to \infty$. Now, integrating (3.16) after multiplying both sides by ϕ_n, we see that

$$\int_{\tilde\Omega^2_{\varepsilon_n}} (\phi_n')^2 + \int_{\tilde\Omega^2_{\varepsilon_n}} \phi_n^2 = -\int_{\tilde\Omega^2_{\varepsilon_n}} \tilde g_n\phi_n - \phi_n'(-a)\phi_n(-a) \tag{3.18}$$

holds by integration by parts and the Neumann boundary condition. Then we easily see that the right hand side of (3.18) tends to 0 as $n \to \infty$ because the L^2-norm of ϕ_n is bounded independently of n and $\|\tilde g_n\|_{L^2(\tilde\Omega^2_{\varepsilon_n})} \to 0$ as $n \to \infty$. Hence $\|\phi_n\|_{H^1(\tilde\Omega^2_{\varepsilon_n})} \to 0$ holds as $n \to 0$. Moreover, from (3.16), we have

$$\begin{aligned}\int_{\tilde\Omega^2_{\varepsilon_n}} (\phi_n'')^2 &= \int_{\tilde\Omega^2_{\varepsilon_n}} (\phi_n + \tilde g_n)^2 \\ &= \int_{\tilde\Omega^2_{\varepsilon_n}} \phi_n^2 + 2\int_{\tilde\Omega^2_{\varepsilon_n}} \phi_n\tilde g_n + \int_{\tilde\Omega^2_{\varepsilon_n}} \tilde g_n^2 \\ &\le \|\phi_n\|^2_{L^2(\tilde\Omega^2_{\varepsilon_n})} + 2\|\phi_n\|_{L^2(\tilde\Omega^2_{\varepsilon_n})}\|\tilde g_n\|_{L^2(\tilde\Omega^2_{\varepsilon_n})} + \|\tilde g_n\|^2_{L^2(\tilde\Omega^2_{\varepsilon_n})} \to 0\end{aligned}$$

as $n \to \infty$. Therefore, we have $\|\phi_n\|_{H^2(\tilde\Omega^2_{\varepsilon_n})} \to 0$ as $n \to \infty$.

Next, let us show that $\|\phi_n\|_{H^2(\tilde{\Omega}^1_{\varepsilon_n})} \to 0$ as $n \to \infty$. Put

$$V_n(z) := 1 - \frac{1}{\xi_n} f'(\overline{U}_{\varepsilon_n,\gamma_n}(z)). \tag{3.19}$$

Recall that $1 - f'(\beta)/\xi_* > 0$. Hence, if we take $a > 0$ in (3.15) large enough, then we can take a constant $\mu > 0$ such that

$$\mu \leq V_n(z) \leq \frac{1}{\mu}, \quad z \in \tilde{\Omega}^1_{\varepsilon_n},$$

holds for all n sufficiently large. Then ϕ_n satisfies the following equation:

$$\phi_n'' - V_n\phi_n = L_{\varepsilon_n,\gamma_n,\xi_n}\phi_n + E_{\varepsilon_n,\gamma_n}\tilde{L}_{\varepsilon_n,\gamma_n,\xi_n}\phi_n =: g_n \ \text{ in } \ \tilde{\Omega}^1_{\varepsilon_n}. \tag{3.20}$$

We note that $\|g_n\|_{L^2(\tilde{\Omega}^1_{\varepsilon_n})} \to 0$ as $n \to \infty$. Integrating (3.20) after multiplying both sides by ϕ_n, we have

$$\int_{\tilde{\Omega}^1_{\varepsilon_n}} \phi_n''\phi_n - \int_{\tilde{\Omega}^1_{\varepsilon_n}} V_n\phi_n^2 = \int_{\tilde{\Omega}^1_{\varepsilon_n}} g_n\phi_n.$$

Noting $\phi_n'(z) = 0$ at $z = -y_c/\varepsilon^{1-\alpha}$, we have

$$\begin{aligned}\int_{\tilde{\Omega}^1_{\varepsilon_n}} (\phi_n')^2 + \int_{\tilde{\Omega}^1_{\varepsilon_n}} V_n\phi_n^2 &= -\int_{\tilde{\Omega}^1_{\varepsilon_n}} g_n\phi_n - \phi_n'(-a)\phi_n(-a)\\ &\leq \|g_n\|_{L^2(\tilde{\Omega}^1_{\varepsilon_n})}\|\phi_n\|_{L^2(\tilde{\Omega}^1_{\varepsilon_n})} + \big|\phi_n'(-a)\phi_n(-a)\big|.\end{aligned}$$

Hence, we see that $\|\phi_n\|_{H^1(\tilde{\Omega}'_{\varepsilon_n})} \to 0$ as $n \to \infty$. Moreover, from (3.20), we have

$$\begin{aligned}\int_{\tilde{\Omega}^1_{\varepsilon_n}} (\phi_n'')^2 &= \int_{\tilde{\Omega}^1_{\varepsilon_n}} (V_n\phi_n + g_n)^2\\ &= \int_{\tilde{\Omega}^1_{\varepsilon_n}} V_n^2\phi_n^2 + 2\int_{\tilde{\Omega}^1_{\varepsilon_n}} V_n\phi_n g_n + \int_{\tilde{\Omega}^1_{\varepsilon_n}} g_n^2\\ &\leq \frac{1}{\mu^2}\|\phi_n\|^2_{L^2(\tilde{\Omega}^1_{\varepsilon_n})} + \frac{2}{\mu}\|\phi_n\|_{L^2(\tilde{\Omega}^1_{\varepsilon_n})}\|g_n\|_{L^2(\tilde{\Omega}^1_{\varepsilon_n})} + \|g_n\|^2_{L^2(\tilde{\Omega}^1_{\varepsilon_n})} \to 0,\end{aligned}$$

as $n \to \infty$. Thus we have $\|\phi_n\|_{H^2(\tilde{\Omega}^1_{\varepsilon_n})} \to 0$ as $n \to \infty$. Hence we have

$\|\phi_n\|_{H^2(\tilde{\Omega}_{\varepsilon_n})} \to 0$ as $n \to \infty$. This contradicts (3.3). Thus (3.1) is verified.

Next, we show (3.2). We recall that $L_{\varepsilon,\gamma,\xi}$ is a self-adjoint operator, and we note that (3.1) means that $\mathrm{Ran}(L_{\varepsilon,\gamma,\xi})$ is closed. Then we notice that (3.1) means also that $L_{\varepsilon,\gamma,\xi}$ is surjective by the theory of adjoint operators. Thus we complete the proof. □

Now, we solve (1.42) in the form $u = \overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi$ for some $\phi \in H^2_\nu(\tilde{\Omega}_\varepsilon)$. We substitute $u = \overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi$ into (1.42). Then we have

$$\pi^{\perp}_{\varepsilon,\gamma}\left\{S[\overline{U}_{\varepsilon,\gamma};\xi] + e^{-c_1/\varepsilon^{1-\alpha}}\tilde{L}_{\varepsilon,\gamma,\xi}\phi + \frac{1}{\xi}M_{\varepsilon,\gamma}[\phi]\right\} = 0, \tag{3.21}$$

where

$$M_{\varepsilon,\gamma}[\phi] := f\big(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi\big) - f(\overline{U}_{\varepsilon,\gamma}) - e^{-c_1/\varepsilon^{1-\alpha}}f'(\overline{U}_{\varepsilon,\gamma})\phi. \tag{3.22}$$

Then, by Lemma 3.1, for $\varepsilon \in (0, \varepsilon_1)$, (3.21) is equivalent to the following

$$\phi = e^{c_1/\varepsilon^{1-\alpha}}\left\{-L^{-1}_{\varepsilon,\gamma,\xi}\big(\pi^{\perp}_{\varepsilon,\gamma}S[\overline{U}_{\varepsilon,\gamma};\xi]\big) - \frac{1}{\xi}L^{-1}_{\varepsilon,\gamma,\xi}\big(\pi^{\perp}_{\varepsilon,\gamma}M_{\varepsilon,\gamma}[\phi]\big)\right\} =: T_{\varepsilon,\gamma,\xi}[\phi]. \tag{3.23}$$

Hence, we only need to find a fixed point of $T_{\varepsilon,\gamma,\xi}$ in a suitable function space. For the purpose, we prepare the following lemma.

LEMMA 3.2. *For $M_{\varepsilon,\gamma}$ defined by* (3.22)*, there exists a constant $C_1 > 0$ such that, for any $\phi, \phi_1, \phi_2 \in H^2(\tilde{\Omega}_\varepsilon)$,*

$$\big\|M_{\varepsilon,\gamma}[\phi]\big\|_{L^2(\tilde{\Omega}_\varepsilon)} \le C_1 e^{-2c_1/\varepsilon^{1-\alpha}}\|\phi\|^2_{H^2(\tilde{\Omega}_\varepsilon)}, \tag{3.24}$$

$$\begin{aligned}&\big\|M_{\varepsilon,\gamma}[\phi_1] - M_{\varepsilon,\gamma}[\phi_2]\big\|_{L^2(\tilde{\Omega}_\varepsilon)}\\ &\quad \le C_1 e^{-2c_1/\varepsilon^{1-\alpha}}\big\{\|\phi_1\|_{H^2(\tilde{\Omega}_\varepsilon)} + \|\phi_2\|_{H^2(\tilde{\Omega}_\varepsilon)}\big\}\|\phi_1 - \phi_2\|_{H^2(\tilde{\Omega}_\varepsilon)},\end{aligned} \tag{3.25}$$

hold for all $\varepsilon > 0$ sufficiently small and all $\gamma \in (0, \beta)$.

PROOF. We first note that

$$|f''(t)| = \left|\frac{2 - 6\kappa_0 t^2}{(1 + \kappa t^2)^3}\right| \le C$$

holds for some $C > 0$. Then, by making use of the mean value theorem, we can estimate as follows

$$
\begin{aligned}
\left|M_{\varepsilon,\gamma}[\phi]\right| &= \left|\int_0^1 \{f'(\overline{U}_{\varepsilon,\gamma}+e^{-c_1/\varepsilon^{1-\alpha}}t\phi)-f'(\overline{U}_{\varepsilon,\gamma})\}dt\right|\left|e^{-c_1/\varepsilon^{1-\alpha}}\phi\right| \\
&\le Ce^{-2c_1/\varepsilon^{1-\alpha}}|\phi|^2.
\end{aligned}
$$

Hence, we have

$$
\|M_{\varepsilon,\gamma}[\phi]\|_{L^2(\tilde{\Omega}_\varepsilon)} \le Ce^{-2c_1/\varepsilon^{1-\alpha}}\|\phi\|_{L^\infty(\tilde{\Omega}_\varepsilon)}\|\phi\|_{L^2(\tilde{\Omega}_\varepsilon)}.
$$

Using the fact:

$$
\|\phi\|_{L^\infty(\tilde{\Omega}_\varepsilon)} \le C'\|\phi\|_{H^2(\tilde{\Omega}_\varepsilon)}, \quad \phi\in H^2(\tilde{\Omega}_\varepsilon), \tag{3.26}
$$

holds for some $C'>0$ independent of ϕ, and $C'>0$ can be taken uniformly on ε sufficiently small, we have (3.24).

Similarly, we can estimate as follows

$$
\begin{aligned}
&\left|M_{\varepsilon,\gamma}[\phi_1]-M_{\varepsilon,\gamma}[\phi_2]\right| \\
&= \left|f\big(\overline{U}_{\varepsilon,\gamma}+e^{-c_1/\varepsilon^{1-\alpha}}\phi_1\big)-f\big(\overline{U}_{\varepsilon,\gamma}+e^{-c_1/\varepsilon^{1-\alpha}}\phi_2\big)-e^{-c_1/\varepsilon^{1-\alpha}}f'(\overline{U}_{\varepsilon,\gamma})(\phi_1-\phi_2)\right| \\
&= e^{-c_1/\varepsilon^{1-\alpha}}\left|\int_0^1\{f'\big(\overline{U}_{\varepsilon,\gamma}+e^{-c_1/\varepsilon^{1-\alpha}}\phi_2+e^{-c_1/\varepsilon^{1-\alpha}}t(\phi_1-\phi_2)\big)-f'(\overline{U}_{\varepsilon,\gamma})\}dt\right| \\
&\quad\times|\phi_1-\phi_2| \\
&\le Ce^{-2c_1/\varepsilon^{1-\alpha}}\int_0^1|\phi_2+t(\phi_1-\phi_2)|dt\cdot|\phi_1-\phi_2| \\
&\le Ce^{-2c_1/\varepsilon^{1-\alpha}}\{|\phi_1|+|\phi_2|\}|\phi_1-\phi_2|.
\end{aligned}
$$

Taking L^2-norm and using (3.26), we have (3.25). □

Here, we remember the inequality (2.1). We note that the term $\|f(w_\gamma)\|_{L^2(\tilde{\Omega}_\varepsilon)}$ tends to infinity as $\varepsilon\to 0$. However, the following estimate holds

$$
\begin{aligned}
\left|\frac{1}{\xi_*}-\frac{1}{\xi}\right|\|f(w_\gamma)\|_{L^2(\tilde{\Omega}_\varepsilon)} &\le \frac{2}{\xi_*^2}|\xi-\xi_*|\|f(w_\gamma)\|_{L^2(\tilde{\Omega}_\varepsilon)} \\
&\le \frac{2}{\xi_*^2}e^{-c_1/\varepsilon^{1-\alpha}}\varepsilon^{(1-\alpha)/2}\|f(w_\gamma)\|_{L^2(\tilde{\Omega}_\varepsilon)}
\end{aligned}
$$

$$= \frac{2}{\xi_*^2} e^{-c_1/\varepsilon^{1-\alpha}} \left(\varepsilon^{1-\alpha} \int_{\tilde{\Omega}_\varepsilon} f^2(w_\delta) dz \right)^{1/2},$$

for $\xi \in I_\varepsilon$ and ε sufficiently small. Because of (1.28), we see that

$$\left(\varepsilon^{1-\alpha} \int_{\tilde{\Omega}_\varepsilon} f^2(w_\delta) dz \right)^{1/2} \le C_0 \tag{3.27}$$

holds for some constant $C_0 > 0$ independent of $\gamma \in [\gamma_1, \gamma_2]$ and ε sufficiently small.

To show the existence of a fixed point of $T_{\varepsilon,\gamma,\xi}$, we set

$$B := \left\{ \phi \in H^2_\nu(\tilde{\Omega}_\varepsilon) : \|\phi\|_{H^2(\tilde{\Omega}_\varepsilon)} \le \frac{4}{\lambda \xi_*^2} C_0 \right\}, \tag{3.28}$$

where λ is a constant given in Lemma 3.1 and C_0 is a constant given in (3.27).

PROPOSITION 3.1. *There exists $\varepsilon_2 > 0$ such that, for $\varepsilon \in (0, \varepsilon_2)$, $\gamma \in [\gamma_1, \gamma_2]$ and $\xi \in I_\varepsilon$, $T_{\varepsilon,\gamma,\xi}$ is a contraction mapping on B, and hence $T_{\varepsilon,\gamma,\xi}$ admits a unique fixed point $\phi_{\varepsilon,\gamma,\xi} \in B$. Moreover, $\phi_{\varepsilon,\gamma,\xi} \in K^\perp_{\varepsilon,\gamma} \cap H^2_\nu(\tilde{\Omega}_\varepsilon)$.*

PROOF. Let $\phi \in B$. Then, by Lemmas 2.1, 3.1, 3.2 and the estimate (3.27), we can estimate as follows:

$$\begin{aligned}
&\left\| T_{\varepsilon,\gamma,\xi}[\phi] \right\|_{H^2(\tilde{\Omega}_\varepsilon)} \\
&\quad \le e^{c_1/\varepsilon^{1-\alpha}} \frac{1}{\lambda} \left\{ \left\| S[\overline{U}_{\varepsilon,\gamma}; \xi] \right\|_{L^2(\tilde{\Omega}_\varepsilon)} + \frac{1}{\xi} \| M_{\varepsilon,\gamma}[\phi] \|_{L^2(\tilde{\Omega}_\varepsilon)} \right\} \\
&\quad \le e^{c_1/\varepsilon^{1-\alpha}} \frac{1}{\lambda} \left\{ \frac{2}{\xi_*^2} C_0 e^{-c_1/\varepsilon^{1-\alpha}} + \overline{C}_1 e^{-\overline{c}_1/\varepsilon^{1-\alpha}} + \frac{2C_1}{\xi_*^2} \|\phi\|^2_{H^2(\tilde{\Omega}_\varepsilon)} e^{-2c_1/\varepsilon^{1-\alpha}} \right\} \\
&\quad = \frac{2}{\lambda \xi_*^2} C_0 + \frac{1}{\lambda} \overline{C}_1 e^{-(\overline{c}_1 - c_1)/\varepsilon^{1-\alpha}} + \frac{2}{\lambda \xi_*^2} C_1 \cdot \left(\frac{4}{\lambda \xi_*^2} C_0 \right)^2 e^{-c_1/\varepsilon^{1-\alpha}}.
\end{aligned}$$

We recall that $\overline{c}_1 - c_1 > 0$. Hence, if ε is small enough, then

$$\|T_{\varepsilon,\gamma,\xi}[\phi]\|_{H^2(\tilde{\Omega}_\varepsilon)} \le \frac{4}{\lambda \xi_*^2} C_0 \tag{3.29}$$

holds for any $\gamma \in [\gamma_1, \gamma_2]$, $\xi \in I_\varepsilon$. Thus $T_{\varepsilon,\gamma,\xi}$ becomes a mapping from B into itself.

Next, let $\phi_1, \phi_2 \in B$. Then, by Lemmas 3.1 and 3.2, we can estimate as follows:

$$
\begin{aligned}
&\big\|T_{\varepsilon,\gamma,\xi}[\phi_1]-T_{\varepsilon,\gamma,\xi}[\phi_2]\big\|_{H^2(\tilde{\Omega}_\varepsilon)}\\
&\quad\le \frac{2}{\lambda\xi_*}e^{c_1/\varepsilon^{1-\alpha}}\big\|M_{\varepsilon,\gamma}[\phi_1]-M_{\varepsilon,\gamma}[\phi_2]\big\|_{L^2(\tilde{\Omega}_\varepsilon)}\\
&\quad\le \frac{2}{\lambda\xi_*}C_1\big\{\|\phi_1\|_{H^2(\tilde{\Omega}_\varepsilon)}+\|\phi_2\|_{H^2(\tilde{\Omega}_\varepsilon)}\big\}\|\phi_1-\phi_2\|_{H^2(\tilde{\Omega}_\varepsilon)}e^{-c_1/\varepsilon^{1-\alpha}}\\
&\quad\le \frac{2}{\lambda\xi_*}C_1\frac{8}{\lambda\xi_*^2}C_0\|\phi_1-\phi_2\|_{H^2(\tilde{\Omega}_\varepsilon)}e^{-c_1/\varepsilon^{1-\alpha}}.
\end{aligned}
$$

Therefore, if ε is small enough, then it holds that

$$
\big\|T_{\varepsilon,\gamma,\xi}[\phi_1]-T_{\varepsilon,\gamma,\xi}[\phi_2]\big\|_{H^2(\tilde{\Omega}_\varepsilon)}\le\frac{1}{2}\|\phi_1-\phi_2\|_{H^2(\tilde{\Omega}_\varepsilon)}
$$

for all $\gamma\in[\gamma_1,\gamma_2]$, $\xi\in I_\varepsilon$. Thus, $T_{\varepsilon,\gamma,\xi}$ is a contraction mapping on B, and there exists a unique fixed point $\phi_{\varepsilon,\gamma,\xi}\in B$. Moreover, from (3.23), we see that $\phi_{\varepsilon,\gamma,\xi}\in K^{\perp}_{\varepsilon,\gamma}\cap H^2_\nu(\tilde{\Omega}_\varepsilon)$. Thus, we complete the proof. $\square$

Thus, we complete Step 1 with

$$
u_\varepsilon(z;\gamma,\xi):=\overline{U}_{\varepsilon,\gamma}(z)+e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi}(z). \tag{3.30}
$$

4. Continuity on parameters.

Before going to the next step, we must show the continuity and differentiability of $\phi_{\varepsilon,\gamma,\xi}$ on γ and ξ. It is known that, if $T_{\varepsilon,\gamma,\xi}[\phi]$ is continuous with respect to γ and ξ for each $\phi\in B$, then $\phi_{\varepsilon,\gamma,\xi}$ is also continuous with respect to γ and ξ (see Proposition 1.2 of [**34**]). That is, we only need to show the following lemma to ensure the continuity of $\phi_{\varepsilon,\gamma,\xi}$ on γ and ξ.

Proposition 4.1. *Let $\varepsilon\in(0,\varepsilon_2)$. For any γ', $\gamma\in[\gamma_1,\gamma_2]$ and ξ', $\xi\in I_\varepsilon$, it holds that*

$$
\big\|T_{\varepsilon,\gamma',\xi'}[\phi]-T_{\varepsilon,\gamma,\xi}[\phi]\big\|_{H^2(\tilde{\Omega}_\varepsilon)}\to 0 \tag{4.1}
$$

as $\gamma\to\gamma'$ and $\xi\to\xi'$ for every $\phi\in B$.

To show this, we prepare some lemmas.

LEMMA 4.1. *Let $\gamma, \gamma' \in [\gamma_1, \gamma_2]$ and $\xi \in I_\varepsilon$. Then, for each $\varepsilon > 0$ sufficiently small, the following estimate holds*

$$\big\| L^{-1}_{\varepsilon,\gamma',\xi} \pi^{\perp}_{\varepsilon,\gamma'} \phi - L^{-1}_{\varepsilon,\gamma,\xi} \pi^{\perp}_{\varepsilon,\gamma} \phi \big\|_{H^2(\tilde{\Omega}_\varepsilon)} \leq \omega_1(\gamma, \gamma') \|\phi\|_{L^2(\tilde{\Omega}_\varepsilon)}, \quad \phi \in L^2(\tilde{\Omega}_\varepsilon), \tag{4.2}$$

where $\omega_1(\gamma,\gamma') > 0$ is a certain quantity independent of ϕ and ξ and satisfies $\omega_1(\gamma,\gamma') \to 0$ as $\gamma \to \gamma'$.

PROOF. In this proof, we omit the indexes of ε and ξ. Let

$$v_\gamma := L^{-1}_\gamma \pi^{\perp}_\gamma \phi, \quad v_{\gamma'} := L^{-1}_{\gamma'} \pi^{\perp}_{\gamma'} \phi.$$

Then

$$\pi^{\perp}_\gamma \tilde{L}_\gamma v_\gamma = \pi^{\perp}_\gamma \phi = \phi - b_\gamma \overline{U}'_\gamma, \quad b_\gamma := \frac{(\overline{U}_\gamma, \phi)_{L^2(\tilde{\Omega}_\varepsilon)}}{\|\overline{U}_\gamma\|^2_{L^2(\tilde{\Omega}_\varepsilon)}}. \tag{4.3}$$

Similarly, we have $\pi^{\perp}_{\gamma'} \tilde{L}_{\gamma'} v_{\gamma'} = \phi - b_{\gamma'} \overline{U}'_{\gamma'}$. From (4.3), we have $\phi = \pi^{\perp}_\gamma \tilde{L}_\gamma v_\gamma + b_\gamma \overline{U}'_\gamma$. Hence,

$$\begin{aligned}
\pi^{\perp}_{\gamma'} \tilde{L}_{\gamma'} v_{\gamma'} &= \phi - b_{\gamma'} \overline{U}'_{\gamma'} \\
&= \pi^{\perp}_\gamma \tilde{L}_\gamma v_\gamma + b_\gamma \overline{U}'_\gamma - b_{\gamma'} \overline{U}'_{\gamma'} \\
&= \tilde{L}_\gamma v_\gamma - E_\gamma \tilde{L}_\gamma v_\gamma + b_\gamma \overline{U}'_\gamma - b_{\gamma'} \overline{U}'_{\gamma'} \\
&= \tilde{L}_{\gamma'} v_{\gamma'} + \big(\tilde{L}_\gamma - \tilde{L}_{\gamma'}\big) v_\gamma - E_\gamma \tilde{L}_\gamma v_\gamma + b_\gamma \overline{U}'_\gamma - b_{\gamma'} \overline{U}'_{\gamma'} \\
&= \tilde{L}_{\gamma'} v_{\gamma'} + \frac{1}{\xi}\big(f'(\overline{U}_{\gamma'}) - f'(\overline{U}_\gamma)\big) v_\gamma - E_\gamma \tilde{L}_\gamma v_\gamma + b_\gamma \overline{U}'_\gamma - b_{\gamma'} \overline{U}'_{\gamma'}.
\end{aligned}$$

Multiplying both sides by $\pi^{\perp}_{\gamma'}$, we have

$$\pi^{\perp}_{\gamma'} \tilde{L}_{\gamma'} v_{\gamma'} = \pi^{\perp}_{\gamma'} \tilde{L}_{\gamma'} v_\gamma + \frac{1}{\xi} \pi^{\perp}_{\gamma'} \big\{ (f'(\overline{U}_{\gamma'}) - f'(\overline{U}_\gamma)) v_\gamma \big\} - \pi^{\perp}_{\gamma'} E_\gamma \tilde{L}_\gamma v_\gamma + b_\gamma \pi^{\perp}_{\gamma'} \overline{U}'_\gamma.$$

Multiplying both sides by $L^{-1}_{\gamma'}$, we have

$$v_{\gamma'} - v_\gamma = \frac{1}{\xi} L^{-1}_{\gamma'} \big[\pi^{\perp}_{\gamma'} \{ (f'(\overline{U}_{\gamma'}) - f'(\overline{U}_\gamma)) v_\gamma \} \big] - L^{-1}_{\gamma'} \pi^{\perp}_{\gamma'} E_\gamma \tilde{L}_\gamma v_\gamma + b_\gamma L^{-1}_{\gamma'} \pi^{\perp}_{\gamma'} \overline{U}'_\gamma.$$

Thus we have

$$\|v_\gamma - v_{\gamma'}\|_{H^2(\tilde\Omega_\varepsilon)} \le \frac{1}{\lambda}\biggl\{\frac{1}{\xi}\bigl\|f'(\overline{U}_{\gamma'}) - f'(\overline{U}_\gamma)\bigr\|_{L^\infty(\tilde\Omega_\varepsilon)}\|v_\gamma\|_{L^2(\tilde\Omega_\varepsilon)} + \bigl\|\pi_{\gamma'}^\perp E_\gamma \tilde L_\gamma v_\gamma\bigr\|_{L^2(\tilde\Omega_\varepsilon)} + b_\gamma\bigl\|\pi_{\gamma'}^\perp \overline{U}'_\gamma\bigr\|_{L^2(\tilde\Omega_\varepsilon)}\biggr\}. \tag{4.4}$$

We may consider $1/\xi < 2/\xi_*$ for $\xi \in I_\varepsilon$. It is easy to see that $\|f'(\overline{U}_{\gamma'}) - f'(\overline{U}_\gamma)\|_{L^\infty(\tilde\Omega_\varepsilon)} = o(1)$ holds as $\gamma \to \gamma'$ (see the proof of Lemma 4.3), and $\|v_\gamma\|_{L^2(\tilde\Omega_\varepsilon)} \le \|\phi\|_{L^2(\tilde\Omega_\varepsilon)}/\lambda$ holds. Hence, the first term of the right hand side of (4.4) is estimated as follows:

$$\frac{1}{\xi}\bigl\|f'(\overline{U}_{\gamma'}) - f'(\overline{U}_\gamma)\bigr\|_{L^\infty(\tilde\Omega_\varepsilon)}\|v_\gamma\|_{L^2(\tilde\Omega_\varepsilon)} \le \omega_1(\gamma,\gamma')\|\phi\|_{L^2(\tilde\Omega_\varepsilon)}, \tag{4.5}$$

the quantity $\omega_1(\gamma,\gamma')$ satisfies the assertion of this lemma. For the second term of the right hand side of (4.4), we can calculate as follows:

$$\begin{aligned}
\pi_{\gamma'}^\perp E_\gamma \tilde L_\gamma v_\gamma &= (I - E_{\gamma'})E_\gamma \tilde L_\gamma v_\gamma \\
&= E_\gamma \tilde L_\gamma v_\gamma - E_{\gamma'}E_\gamma \tilde L_\gamma v_\gamma \\
&= \frac{(\overline{U}'_\gamma, \tilde L_\gamma v_\gamma)_{L^2(\tilde\Omega_\varepsilon)}}{\|\overline{U}'_\gamma\|^2_{L^2(\tilde\Omega_\varepsilon)}}\biggl\{\overline{U}'_\gamma - \frac{(\overline{U}'_{\gamma'}, \overline{U}'_\gamma)_{L^2(\tilde\Omega_\varepsilon)}}{\|\overline{U}'_{\gamma'}\|^2_{L^2(\tilde\Omega_\varepsilon)}}\overline{U}'_{\gamma'}\biggr\}.
\end{aligned}$$

Thus we can estimate as follows:

$$\begin{aligned}
&\bigl\|\pi_{\gamma'}^\perp E_\gamma \tilde L_\gamma v_\gamma\bigr\|_{L^2(\tilde\Omega_\varepsilon)} \\
&\quad\le \frac{|(\overline{U}'_\gamma, \tilde L_\gamma v_\gamma)_{L^2(\tilde\Omega_\varepsilon)}|}{\|\overline{U}'_\gamma\|^2_{L^2(\tilde\Omega_\varepsilon)}}\biggl\|\overline{U}'_\gamma - \frac{(\overline{U}'_{\gamma'}, \overline{U}'_\gamma)_{L^2(\tilde\Omega_\varepsilon)}}{\|\overline{U}'_{\gamma'}\|^2_{L^2(\tilde\Omega_\varepsilon)}}\overline{U}'_{\gamma'}\biggr\|_{L^2(\tilde\Omega_\varepsilon)} \\
&\quad= \frac{|(\tilde L_\gamma \overline{U}'_\gamma, v_\gamma)_{L^2(\tilde\Omega_\varepsilon)}|}{\|\overline{U}'_\gamma\|^2_{L^2(\tilde\Omega_\varepsilon)}}\biggl\|\overline{U}'_\gamma - \frac{(\overline{U}'_{\gamma'}, \overline{U}'_\gamma)_{L^2(\tilde\Omega_\varepsilon)}}{\|\overline{U}'_{\gamma'}\|^2_{L^2(\tilde\Omega_\varepsilon)}}\overline{U}'_{\gamma'}\biggr\|_{L^2(\tilde\Omega_\varepsilon)} \\
&\quad\le \frac{\|\tilde L_\gamma \overline{U}'_\gamma\|_{L^2(\tilde\Omega_\varepsilon)}\|L_\gamma^{-1}\pi_\gamma^\perp\phi\|_{L^2(\tilde\Omega_\varepsilon)}}{\|\overline{U}'_\gamma\|^2_{L^2(\tilde\Omega_\varepsilon)}}\biggl\|\overline{U}'_\gamma - \frac{(\overline{U}'_{\gamma'}, \overline{U}'_\gamma)_{L^2(\tilde\Omega_\varepsilon)}}{\|\overline{U}'_{\gamma'}\|^2_{L^2(\tilde\Omega_\varepsilon)}}\overline{U}'_{\gamma'}\biggr\|_{L^2(\tilde\Omega_\varepsilon)} \\
&\quad\le C\biggl\|\overline{U}'_\gamma - \frac{(\overline{U}'_{\gamma'}, \overline{U}'_\gamma)_{L^2(\tilde\Omega_\varepsilon)}}{\|\overline{U}'_{\gamma'}\|^2_{L^2(\tilde\Omega_\varepsilon)}}\overline{U}'_{\gamma'}\biggr\|_{L^2(\tilde\Omega_\varepsilon)}\|\phi\|_{L^2(\tilde\Omega_\varepsilon)},
\end{aligned}$$

for some constant $C > 0$. It is easy to see that

$$\left\| \overline{U}'_{\gamma} - \frac{(\overline{U}'_{\gamma'}, \overline{U}'_{\gamma})_{L^2(\tilde{\Omega}_\varepsilon)}}{\|\overline{U}'_{\gamma'}\|^2_{L^2(\tilde{\Omega}_\varepsilon)}} \overline{U}'_{\gamma'} \right\|_{L^2(\tilde{\Omega}_\varepsilon)} = o(1), \quad \text{as } \gamma \to \gamma'.$$

Therefore,

$$\left\| \pi^{\perp}_{\gamma'} E_\gamma \tilde{L}_\gamma v_\gamma \right\|_{L^2(\tilde{\Omega}_\varepsilon)} \le \omega_1(\gamma, \gamma') \|\phi\|_{L^2(\tilde{\Omega}_\varepsilon)}$$

holds for some quantity $\omega_1(\gamma, \gamma') > 0$ satisfying the assertion. For the third term in (4.3), we easily see that

$$\left\| \pi^{\perp}_{\gamma'} \overline{U}'_{\gamma} \right\|_{L^2(\tilde{\Omega}_\varepsilon)} = \left\| \overline{U}'_{\gamma} - \frac{(\overline{U}'_{\gamma'}, \overline{U}'_{\gamma})_{L^2(\tilde{\Omega}_\varepsilon)}}{\|\overline{U}'_{\gamma'}\|^2_{L^2(\tilde{\Omega}_\varepsilon)}} \overline{U}'_{\gamma'} \right\|_{L^2(\tilde{\Omega}_\varepsilon)} = o(1) \text{ as } \gamma \to \gamma',$$

and

$$|b_\gamma| = \frac{|(\overline{U}'_{\gamma}, \phi)_{L^2(\tilde{\Omega}_\varepsilon)}|}{\|\overline{U}'_{\gamma}\|^2_{L^2(\tilde{\Omega}_\varepsilon)}} \le \frac{\|\phi\|_{L^2(\tilde{\Omega}_\varepsilon)}}{\|\overline{U}'_{\gamma}\|_{L^2(\tilde{\Omega}_\varepsilon)}} \le C \|\phi\|_{L^2(\tilde{\Omega}_\varepsilon)},$$

for some $C > 0$. By taking $\omega_1(\gamma, \gamma')$ suitably, we have a conclusion. □

LEMMA 4.2. *Let ξ, $\xi' \in I_\varepsilon$. For each $\varepsilon > 0$ sufficiently small, the following estimate holds*

$$\left\| (L^{-1}_{\varepsilon,\gamma,\xi} - L^{-1}_{\varepsilon,\gamma,\xi'}) \phi \right\|_{H^1(\tilde{\Omega}_\varepsilon)} \le \omega_2(\xi, \xi') \|\phi\|_{L^2(\tilde{\Omega}_\varepsilon)}, \quad \phi \in C^{\perp}_{\varepsilon,\gamma}, \tag{4.6}$$

where $\omega_2(\xi, \xi') > 0$ is a certain quantity independent of ϕ and $\gamma \in [\gamma_1, \gamma_2]$ and it satisfies $\omega_2(\xi, \xi') \to 0$ as $\xi \to \xi'$.

PROOF. In this proof, we omit the indexes of ε and γ of $L_{\varepsilon,\gamma,\xi}$. For $\phi \in C^{\perp}_{\varepsilon,\gamma}$, let

$$\begin{aligned} \phi &= L_{\xi'} L^{-1}_{\xi'} \phi \\ &= \{L_\xi + (L_{\xi'} - L_\xi)\} L^{-1}_{\xi'} \phi \\ &= \left\{ L_\xi + \left(\frac{1}{\xi'} - \frac{1}{\xi} \right) f'(\overline{U}_{\varepsilon,\gamma}) \right\} L^{-1}_{\xi'} \phi \end{aligned}$$

$$L_\xi^{-1}\phi = L_{\xi'}^{-1}\phi + \left(\frac{1}{\xi'} - \frac{1}{\xi}\right) L_\xi^{-1}\left[f'(\overline{U}_{\varepsilon,\gamma}) L_{\xi'}^{-1}\phi\right].$$

Then we have

$$\begin{aligned}\left\|L_\xi^{-1}\phi - L_{\xi'}^{-1}\phi\right\|_{H^2(\tilde\Omega_\varepsilon)} &\le \frac{1}{\lambda}\left|\frac{1}{\xi'} - \frac{1}{\xi}\right| \left\|f'(\overline{U}_{\varepsilon,\gamma}) L_{\xi'}^{-1}\phi\right\|_{L^2(\tilde\Omega_\varepsilon)} \\ &\le \frac{1}{\lambda^2}\left|\frac{1}{\xi'} - \frac{1}{\xi}\right| \left\|f'(\overline{U}_{\varepsilon,\gamma})\right\|_{L^\infty(\tilde\Omega_\varepsilon)} \|\phi\|_{L^2(\tilde\Omega_\varepsilon)}.\end{aligned}$$

We note that $\|\overline{U}_{\varepsilon,\gamma}\|_{L^\infty(\tilde\Omega_\varepsilon)}$ is bounded independently of ε and γ, and hence $\|f'(\overline{U}_{\varepsilon,\gamma})\|_{L^\infty(\tilde\Omega_\varepsilon)}$ is also bounded independently of ε and γ. From the estimate above, we have a conclusion. □

Lemma 4.3. *Let $\gamma, \gamma' \in [\gamma_1, \gamma_2]$ and $\xi, \xi' \in I_\varepsilon$. For each ε sufficiently small, it holds that*

$$\left\|S[\overline{U}_{\varepsilon,\gamma'}; \xi'] - S[\overline{U}_{\varepsilon,\gamma}; \xi]\right\|_{L^2(\tilde\Omega_\varepsilon)} \to 0$$

as $\gamma \to \gamma'$ and $\xi \to \xi'$.

Proof. By the definition, we can estimate as follows:

$$\begin{aligned}&\left\|S[\overline{U}_{\varepsilon,\gamma'}; \xi'] - S[\overline{U}_{\varepsilon,\gamma}; \xi]\right\|_{L^2(\tilde\Omega_\varepsilon)} \\ &\quad = \left\|\overline{U}''_{\varepsilon,\gamma'} - \overline{U}_{\varepsilon,\gamma'} + \frac{1}{\xi'} f(\overline{U}_{\gamma'}) - \overline{U}''_{\varepsilon,\gamma} + \overline{U}_{\varepsilon,\gamma} - \frac{1}{\xi} f(\overline{U}_{\gamma'})\right\|_{L^2(\tilde\Omega_\varepsilon)} \\ &\quad \le \left\|\overline{U}''_{\varepsilon,\gamma'} - \overline{U}''_{\varepsilon,\gamma}\right\|_{L^2(\tilde\Omega_\varepsilon)} + \left\|\overline{U}_{\varepsilon,\gamma'} - \overline{U}_{\varepsilon,\gamma}\right\|_{L^2(\tilde\Omega_\varepsilon)} + \left|\frac{1}{\xi'} - \frac{1}{\xi}\right| \left\|f(\overline{U}_{\varepsilon,\gamma'})\right\|_{L^2(\tilde\Omega_\varepsilon)} \\ &\qquad + \frac{1}{\xi}\left\|f(\overline{U}_{\varepsilon,\gamma'}) - f(\overline{U}_{\varepsilon,\gamma})\right\|_{L^2(\tilde\Omega_\varepsilon)}.\end{aligned}$$

The third term obviously tends to 0 as $\xi \to \xi'$. By the definition of $\overline{U}_{\varepsilon,\gamma}$, we have

$$\overline{U}_{\varepsilon,\gamma'} - \overline{U}_{\varepsilon,\gamma} = (w_{\gamma'} - w_\gamma)\overline{\chi}_0.$$

Because $w_\gamma(z) = w_{\gamma'}(z - z(\gamma))$ for some $z(\gamma)$ which is continuous in γ and satisfies $|z(\gamma)| \to 0$ as $\gamma \to \gamma'$, using the mean value theorem, we can estimate as follows:

$$\begin{aligned}\int_{\tilde{\Omega}_\varepsilon}\big(\overline{U}_{\varepsilon,\gamma'}(z)-\overline{U}_{\varepsilon,\gamma}(z)\big)^2dz &= \int_{\tilde{\Omega}_\varepsilon}\big(w_{\gamma'}(z)-w_{\gamma'}(z-z(\gamma))\big)^2\overline{\chi}_0^2(z)dz \\ &\le \sup_{z\in\mathbb{R}}\big|w'_{\gamma'}(z)\big|^2\int_{\tilde{\Omega}_\varepsilon}|z(\gamma)|^2dz \\ &= \sup_{z\in\mathbb{R}}\big|w'_{\gamma'}(z)\big|^2|\tilde{\Omega}_\varepsilon||z(\gamma)|^2.\end{aligned}$$

Note that $\sup_{z\in\mathbb{R}}|w'_{\gamma'}(z)|$ is finite. Therefore, $\|\overline{U}_{\varepsilon,\gamma'}-\overline{U}_{\varepsilon,\gamma}\|_{L^2(\tilde{\Omega}_\varepsilon)}\to 0$ as $\gamma\to\gamma'$. Next, we see that

$$\overline{U}''_{\varepsilon,\gamma'}-\overline{U}''_{\varepsilon,\gamma}=\big(w''_{\gamma'}-w''_{\gamma}\big)\overline{\chi}_0+2\big(w'_{\gamma'}-w'_{\gamma}\big)\overline{\chi}'_0+(w_{\gamma'}-w_{\gamma})\overline{\chi}''_0.$$

By the same argument above, we have $\|\overline{U}''_{\varepsilon,\gamma'}-\overline{U}''_{\varepsilon,\gamma}\|_{L^2(\tilde{\Omega}_\varepsilon)}\to 0$ as $\gamma\to\gamma'$. Finally, noting $0\le\overline{U}_{\varepsilon,\gamma}\le\beta$, we can see that

$$\big|f(\overline{U}_{\varepsilon,\gamma'})-f(\overline{U}_{\varepsilon,\gamma})\big|\le C\big|\overline{U}_{\varepsilon,\gamma'}-\overline{U}_{\varepsilon,\gamma}\big|$$

holds for some $C>0$ by using the mean value theorem or a direct calculation. Hence, we see that

$$\big\|f(\overline{U}_{\varepsilon,\gamma'})-f(\overline{U}_{\varepsilon,\gamma})\big\|_{L^2(\tilde{\Omega}_\varepsilon)}\to 0$$

as $\gamma\to\gamma'$. □

Lemma 4.4. *Let $\gamma,\gamma'\in[\gamma_1,\gamma_2]$. For each ε sufficiently small and $\phi\in B$, it holds that*

$$\|M_{\varepsilon,\gamma'}[\phi]-M_{\varepsilon,\gamma}[\phi]\|_{L^2(\tilde{\Omega}_\varepsilon)}\to 0$$

as $\gamma\to\gamma'$.

Proof. The proof can be done by an argument similar to that in the proof of Lemma 4.3. Thus we omit the proof. □

Proof of Proposition 4.1. From Lemmas 3.1, 4.1, 4.2, 4.3 and 4.4, we can easily verify (4.1) by a simple calculation. Thus we omit the details. □

Thus, we see that $\phi_{\varepsilon,\gamma,\xi}$, which is a unique fixed point of $T_{\varepsilon,\gamma,\xi}$ in B, is continuous with respect to γ and ξ in the space $H^2(\tilde{\Omega}_\varepsilon)$. However, it is insufficient for going to the next step. Let us show that $\phi_{\varepsilon,\gamma,\xi}$ is of class C^1 with respect to ξ. For the purpose, we first show the following lemma.

Lemma 4.5. *Let $\phi_{\varepsilon,\gamma,\xi}$ be a fixed point of $T_{\varepsilon,\gamma,\xi}$ given in Proposition* 3.1. *Define*

$$\tilde{\mathcal{L}}_{\varepsilon,\gamma,\xi}\psi := \psi'' - \psi + \frac{1}{\xi} f'\big(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi}\big)\psi$$

and $\mathcal{L}_{\varepsilon,\gamma,\xi} := \pi^{\perp}_{\varepsilon,\gamma} \circ \tilde{\mathcal{L}}_{\varepsilon,\gamma,\xi}$. Then $\mathcal{L}_{\varepsilon,\gamma,\xi}$ is invertible as an operator from $K^{\perp}_{\varepsilon,\gamma} \cap H^2_\nu(\tilde{\Omega}_\varepsilon)$ into $C^{\perp}_{\varepsilon,\gamma}$ for sufficiently small ε. The inverse $\mathcal{L}^{-1}_{\varepsilon,\gamma,\xi}$ satisfies

$$\big\|\mathcal{L}^{-1}_{\varepsilon,\gamma,\xi} g\big\|_{H^2(\tilde{\Omega}_\varepsilon)} \le \frac{2}{\lambda}\|g\|_{L^2(\tilde{\Omega}_\varepsilon)}, \quad g \in C^{\perp}_{\varepsilon,\gamma}, \tag{4.7}$$

where $\lambda > 0$ is a constant given in Proposition 3.1.

Proof. For $g \in C^{\perp}_{\varepsilon,\gamma}$ and $u \in K^{\perp}_{\varepsilon,\gamma} \cap H^2_\nu(\tilde{\Omega}_\varepsilon)$, the following equations are equivalent:

$$\mathcal{L}_{\varepsilon,\gamma,\xi} u = g,$$

$$L_{\varepsilon,\gamma,\xi} u + (\mathcal{L}_{\varepsilon,\gamma,\xi} - L_{\varepsilon,\gamma,\xi})u = g$$

$$L_{\varepsilon,\gamma,\xi} u + \frac{1}{\xi}\pi^{\perp}_{\varepsilon,\gamma}\big[\big(f'(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi}) - f'(\overline{U}_{\varepsilon,\gamma})\big)u\big] = g \tag{4.8}$$

$$u + \frac{1}{\xi} L^{-1}_{\varepsilon,\gamma,\xi}\big[\pi^{\perp}_{\varepsilon,\gamma}\big[(f'(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi}) - f'(\overline{U}_{\varepsilon,\gamma}))u\big]\big] = L^{-1}_{\varepsilon,\gamma,\xi} g. \tag{4.9}$$

By the mean value theorem and (3.26), we have

$$\begin{aligned}
\big\|f'\big(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi}\big) - f'\big(\overline{U}_{\varepsilon,\gamma}\big)\big\|_{L^\infty(\tilde{\Omega}_\varepsilon)} &\le C e^{-c_1/\varepsilon^{1-\alpha}}\|\phi_{\varepsilon,\gamma,\xi}\|_{L^\infty(\tilde{\Omega}_\varepsilon)} \\
&\le CC' e^{-c_1/\varepsilon^{1-\alpha}}\|\phi_{\varepsilon,\gamma,\xi}\|_{H^2(\tilde{\Omega}_\varepsilon)} \\
&\le C'' e^{-c_1/\varepsilon^{1-\alpha}}
\end{aligned}$$

for some $C'' > 0$ independent of ε, $\gamma \in [\gamma_1, \gamma_2]$ and $\xi \in I_\varepsilon$ because of $\phi_{\varepsilon,\gamma,\xi} \in B$. Hence it holds that

$$
\begin{aligned}
&\left\| \frac{1}{\xi} L_{\varepsilon,\gamma,\xi}^{-1} \big[\pi_{\varepsilon,\gamma}^{\perp} \big[(f'(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}} \phi_{\varepsilon,\gamma,\xi}) - f'(\overline{U}_{\varepsilon,\gamma}))u\big]\big] \right\|_{L^2(\tilde{\Omega}_\varepsilon)} \\
&\le \frac{2}{\xi_* \lambda} \big\| \big[(f'(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}} \phi_{\varepsilon,\gamma,\xi}) - f'(\overline{U}_{\varepsilon,\gamma})\big) u \big\|_{L^2(\tilde{\Omega}_\varepsilon)} \\
&\le C''' e^{-c_1/\varepsilon^{1-\alpha}} \|u\|_{L^2(\tilde{\Omega}_\varepsilon)} \\
&\le \frac{1}{2} \|u\|_{L^2(\tilde{\Omega}_\varepsilon)}
\end{aligned}
$$

for ε sufficiently small. Hence, by the Neumann series theory, we know that the equation (4.9) is solvable, namely, there exists a unique $u \in C_{\varepsilon,\gamma}^{\perp}$ satisfying (4.9) for each $g \in C_{\varepsilon,\gamma}^{\perp}$. Moreover, we see that $u \in K_{\varepsilon,\gamma}^{\perp} \cap H_\nu^2(\tilde{\Omega}_\varepsilon)$ from (4.9). Then, we can estimate as follows:

$$
\begin{aligned}
\|u\|_{H^2(\tilde{\Omega}_\varepsilon)} &\le \frac{1}{\xi} \big\| L_{\varepsilon,\gamma,\xi}^{-1} \big[\pi_{\varepsilon,\gamma}^{\perp} [(f'(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}} \phi_{\varepsilon,\gamma,\xi}) - f'(\overline{U}_{\varepsilon,\gamma}))u]\big] \big\|_{H^2(\tilde{\Omega}_\varepsilon)} \\
&\quad + \big\| L_{\varepsilon,\gamma,\xi}^{-1} g \big\|_{H^2(\tilde{\Omega}_\varepsilon)} \\
&\le \frac{1}{2} \|u\|_{H^2(\tilde{\Omega}_\varepsilon)} + \frac{1}{\lambda} \|g\|_{L^2(\tilde{\Omega}_\varepsilon)}.
\end{aligned}
$$

Hence we have $\|u\|_{H^2(\tilde{\Omega}_\varepsilon)} \le 2\|g\|_{L^2(\tilde{\Omega}_\varepsilon)}/\lambda$. Thus we complete the proof. □

PROPOSITION 4.2. *Let $\phi_{\varepsilon,\gamma,\xi}$ be a fixed point of $T_{\varepsilon,\gamma,\xi}$ given by Proposition* 3.1. *For each $\varepsilon > 0$ sufficiently small and $\gamma \in [\gamma_1, \gamma_2]$, $\phi_{\varepsilon,\gamma,\xi}$ is of class C^1 in the space $H^2(\tilde{\Omega}_\varepsilon)$ with respect to $\xi \in I_\varepsilon$. The derivative is given by*

$$
\frac{\partial}{\partial \xi} \phi_{\varepsilon,\gamma,\xi} = e^{c_1/\varepsilon^{1-\alpha}} \frac{1}{\xi^2} \mathcal{L}_{\varepsilon,\gamma,\xi}^{-1} \big[\pi_{\varepsilon,\gamma}^{\perp} f(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}} \phi_{\varepsilon,\gamma,\xi})\big]. \tag{4.10}
$$

PROOF. We put

$$
\begin{aligned}
F(\xi, \phi) &:= \pi_{\varepsilon,\gamma}^{\perp} S\big[\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}} \phi; \xi\big] \\
&= \pi_{\varepsilon,\gamma}^{\perp} \bigg[(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}} \phi)'' - (\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}} \phi) \\
&\qquad + \frac{1}{\xi} f(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}} \phi) \bigg]
\end{aligned}
$$

for $(\xi, \phi) \in I_\varepsilon \times (B \cap K_{\varepsilon,\gamma}^{\perp})$, where B is defined by (3.28). We note that $F(\xi, \phi_{\varepsilon,\gamma,\xi}) = 0$. Moreover, the derivatives are given by

$$F_\xi(\xi,\phi_{\varepsilon,\gamma,\xi}) = -\frac{1}{\xi^2}\pi^\perp_{\varepsilon,\gamma} f\big(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi}\big),$$
$$F_\phi(\xi,\phi_{\varepsilon,\gamma,\xi}) = e^{-c_1/\varepsilon^{1-\alpha}}\mathcal{L}_{\varepsilon,\gamma,\xi}.$$

By Lemma 4.5, $F_\phi(\xi,\phi_{\varepsilon,\gamma,\xi})$ is bijective from $K^\perp_{\varepsilon,\gamma}\cap H^2_\nu(\tilde\Omega_\varepsilon)$ onto $C^\perp_{\varepsilon,\gamma}$. By the implicit function theorem and the uniqueness of $\phi_{\varepsilon,\gamma,\xi}$, we can find $\phi_{\varepsilon,\gamma,\xi}\in C^1(I_\varepsilon,H^2(\tilde\Omega_\varepsilon))$ and

$$\begin{aligned}\frac{\partial}{\partial\xi}\phi_{\varepsilon,\gamma,\xi} &= -F_\phi^{-1}(\xi,\phi_{\varepsilon,\gamma,\xi})\cdot F_\xi(\xi,\phi_{\varepsilon,\gamma,\xi})\\ &= e^{c_1/\varepsilon^{1-\alpha}}\frac{1}{\xi^2}\mathcal{L}^{-1}_{\varepsilon,\gamma,\xi}\big[\pi^\perp_{\varepsilon,\gamma} f(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi})\big].\end{aligned}$$

Thus we complete the proof. □

Remark 3. We can easily confirm that

$$\varepsilon^{1-\alpha}\int_{\tilde\Omega_\varepsilon} f^2\big(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi}\big)dz < C$$

holds for some $C>0$ independent of ε, γ and ξ. Therefore, we can estimate by (4.7) and (4.10) as follows:

$$\left\|\frac{\partial}{\partial\xi}\phi_{\varepsilon,\gamma,\xi}\right\|_{H^2(\tilde\Omega_\varepsilon)} \le C\varepsilon^{-(1-\alpha)/2}e^{c_1/\varepsilon^{1-\alpha}}, \tag{4.11}$$

the constant $C>0$ is independent of ε, γ and ξ. Hence, by applying Taylor's expansion theorem (see, e.g., Theorem 4.A in [**34**]), we obtain the following estimate:

$$\big\|\phi_{\varepsilon,\gamma,\xi_1} - \phi_{\varepsilon,\gamma,\xi_2}\big\|_{H^2(\tilde\Omega_\varepsilon)} \le C\varepsilon^{-(1-\alpha)/2}e^{c_1/\varepsilon^{1-\alpha}}|\xi_1-\xi_2|, \tag{4.12}$$

for any $\xi_1,\xi_2\in I_\varepsilon$.

5. Reduced problem.

In this section, we are going to carry out Step 2. For each ε sufficiently small and each $\gamma\in[\gamma_1,\gamma_2]$, we will find $\xi=\xi_{\varepsilon,\gamma}$ such that (1.43) holds, namely,

$$\int_{\tilde{\Omega}_\varepsilon} S\big[u_\varepsilon(z;\gamma,\xi_{\varepsilon,\gamma});\xi_{\varepsilon,\gamma}\big]\overline{U}'_{\varepsilon,\gamma}(z)dz = 0, \tag{5.1}$$

where

$$u_\varepsilon(z;\gamma,\xi) = \overline{U}_{\varepsilon,\gamma}(z) + e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi}(z). \tag{5.2}$$

Moreover, for the next step, we would like to show the continuity of $\xi_{\varepsilon,\gamma}$ with respect to γ. For the purpose, we reduce (5.1) into the problem: find the fixed point of $T_\gamma[\xi]$ defined by

$$T_\gamma[\xi] := \sigma_\varepsilon(\gamma)\int_{\tilde{\Omega}_\varepsilon} S[u_\varepsilon(z;\gamma,\xi);\xi]\overline{U}'_{\varepsilon,\gamma}(z)dz + \xi, \tag{5.3}$$

where $\sigma_\varepsilon(\gamma) \neq 0$ is a constant defined by (5.5) below.

Proposition 5.1. *For sufficiently small $\varepsilon > 0$, T_γ is a contraction mapping on I_ε for any $\gamma \in [\gamma_1,\gamma_2]$, and hence there exists a unique fixed point $\xi_{\varepsilon,\gamma} \in I_\varepsilon$ of T_γ. Moreover, $\xi_{\varepsilon,\gamma}$ is continuous with respect to $\gamma \in [\gamma_1,\gamma_2]$.*

Proof. As was done in (3.21), we can write as follows:

$$\begin{aligned}
&\int_{\tilde{\Omega}_\varepsilon} S\big[\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}}\phi_{\varepsilon,\gamma,\xi};\xi\big]\overline{U}'_{\varepsilon,\gamma}dz \\
&\quad= \int_{\tilde{\Omega}_\varepsilon} S[\overline{U}_{\varepsilon,\gamma};\xi]\overline{U}'_{\varepsilon,\gamma}dz + e^{-c_1/\varepsilon^{1-\alpha}}\int_{\tilde{\Omega}_\varepsilon}\tilde{L}_{\varepsilon,\gamma,\xi}[\phi_{\varepsilon,\gamma,\xi}]\overline{U}'_{\varepsilon,\gamma}dz \\
&\qquad+ \frac{1}{\xi}\int_{\tilde{\Omega}_\varepsilon} M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi}]\overline{U}'_{\varepsilon,\gamma}dz.
\end{aligned} \tag{5.4}$$

Then, by Lemma 2.3, for $\xi \in I_\varepsilon$,

$$\begin{aligned}
T_\gamma[\xi] - \xi_* &= \left\{\frac{\sigma_\varepsilon(\gamma)}{\xi\xi_*}\int_{-D_\varepsilon}^{D_\varepsilon} f(w_\gamma)w'_\gamma dz - 1\right\}(\xi_* - \xi) + \sigma_\varepsilon(\gamma)k(\varepsilon) \\
&\quad+ \sigma_\varepsilon(\gamma)e^{-c_1/\varepsilon^{1-\alpha}}\int_{\tilde{\Omega}_\varepsilon}\tilde{L}_{\varepsilon,\gamma,\xi}[\phi_{\varepsilon,\gamma,\xi}]\overline{U}'_{\varepsilon,\gamma}dz \\
&\quad+ \sigma_\varepsilon(\gamma)\frac{1}{\xi}\int_{\tilde{\Omega}_\varepsilon} M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi}]\overline{U}'_{\varepsilon,\gamma}dz.
\end{aligned}$$

Here, we set

$$\sigma_\varepsilon(\gamma) = \frac{5}{4}\xi_*^2\biggl(\int_{-D_\varepsilon}^{D_\varepsilon} f(w_\gamma)w_\gamma' dz\biggr)^{-1}. \tag{5.5}$$

Noting the exponential decay estimate stated in Lemma 1.1, we see that $|\sigma_\varepsilon(\gamma)|$ is bounded uniformly in $\gamma \in [\gamma_1, \gamma_2]$ for ε sufficiently small. Then we can estimate as follows:

$$\left|\left\{\frac{\sigma_\varepsilon(\gamma)}{\xi\xi_*}\int_{-D_\varepsilon}^{D_\varepsilon} f(w_\gamma)w_\gamma' dz - 1\right\}(\xi_* - \xi)\right| = \frac{1}{2}|\xi_* - \xi| \le \frac{1}{2}\varepsilon^{(1-\alpha)/2}e^{-c_1/\varepsilon^{1-\alpha}}.$$

From (2.4) and noting $c_1 < \overline{c}_3$, we have

$$|\sigma_\varepsilon(\gamma)k(\varepsilon)| \le C\overline{C}_3 e^{-\overline{c}_3/\varepsilon^{1-\alpha}} = \varepsilon^{(1-\alpha)/2}e^{-c_1/\varepsilon^{1-\alpha}}o(1), \quad \text{as } \varepsilon \to 0.$$

Moreover, by Lemma 2.2, we have

$$\begin{aligned}
&\left|\sigma_\varepsilon(\gamma)e^{-c_1/\varepsilon^{1-\alpha}}\int_{\tilde{\Omega}_\varepsilon} \tilde{L}_{\varepsilon,\gamma,\xi}[\phi_{\varepsilon,\gamma,\xi}]\overline{U}'_{\varepsilon,\gamma} dz\right| \\
&\quad = \left|\sigma_\varepsilon(\gamma)e^{-c_1/\varepsilon^{1-\alpha}}\int_{\tilde{\Omega}_\varepsilon} \phi_{\varepsilon,\gamma,\xi}\tilde{L}_{\varepsilon,\gamma,\xi}[\overline{U}'_{\varepsilon,\gamma}] dz\right| \\
&\quad \le Ce^{-c_1/\varepsilon^{1-\alpha}}\|\phi_{\varepsilon,\gamma,\xi}\|_{L^2(\tilde{\Omega}_\varepsilon)}\bigl\|\tilde{L}_{\varepsilon,\gamma,\xi}[\overline{U}'_{\varepsilon,\gamma}]\bigr\|_{L^2(\tilde{\Omega}_\varepsilon)} \\
&\quad \le C'e^{-c_1/\varepsilon^{1-\alpha}}\left\{\left|\frac{1}{\xi_*} - \frac{1}{\xi}\right|\bigl\|f'(w_\gamma)w_\gamma'\bigr\|_{L^2(\tilde{\Omega}_\varepsilon)} + \overline{C}_2 e^{-\overline{c}_2/\varepsilon^{1-\alpha}}\right\} \\
&\quad \le C''e^{-c_1/\varepsilon^{1-\alpha}}\bigl\{|\xi_* - \xi| + e^{-\overline{c}_3/\varepsilon^{1-\alpha}}\bigr\} \\
&\quad \le \varepsilon^{(1-\alpha)/2}e^{-c_1/\varepsilon^{1-\alpha}}o(1), \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Furthermore, by Lemma 3.2, we have

$$\begin{aligned}
\left|\sigma_\varepsilon(\gamma)\frac{1}{\xi}\int_{\tilde{\Omega}_\varepsilon} M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi}]\overline{U}'_{\varepsilon,\gamma} dz\right| &\le C\|M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi}]\|_{L^2(\tilde{\Omega}_\varepsilon)}\bigl\|\overline{U}'_{\varepsilon,\gamma}\bigr\|_{L^2(\tilde{\Omega}_\varepsilon)} \\
&\le C'e^{-2c_1/\varepsilon^{1-\alpha}}.
\end{aligned}$$

Thus, we have

$$|T_\gamma[\xi] - \xi_*| \le \frac{3}{4}\varepsilon^{(1-\alpha)/2}e^{-c_1/\varepsilon^{1-\alpha}}$$

for all $\gamma \in [\gamma_1, \gamma_2]$ and $\xi \in I_\varepsilon$ provided ε is small enough. Hence, T_γ is a mapping from I_ε into itself.

Next, for ξ_1, $\xi_2 \in I_\varepsilon$, Let

$$\begin{aligned}
T_\gamma[\xi_1] - T_\gamma[\xi_2] &= \sigma_\varepsilon(\gamma) \int_{\tilde{\Omega}_\varepsilon} \{S[\overline{U}_{\varepsilon,\gamma};\xi_1] - S[\overline{U}_{\varepsilon,\gamma};\xi_2]\}\overline{U}'_{\varepsilon,\gamma} dz + \xi_1 - \xi_2 \\
&\quad + \sigma_\varepsilon(\gamma) e^{-c_1/\varepsilon^{1-\alpha}} \int_{\tilde{\Omega}_\varepsilon} \{\tilde{L}_{\varepsilon,\gamma,\xi_1}\phi_{\varepsilon,\gamma,\xi_1} - \tilde{L}_{\varepsilon,\gamma,\xi_2}\phi_{\varepsilon,\gamma,\xi_2}\}\overline{U}'_{\varepsilon,\gamma} dz \\
&\quad + \sigma_\varepsilon(\gamma) \int_{\tilde{\Omega}_\varepsilon} \left\{\frac{1}{\xi_1} M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_2}] - \frac{1}{\xi_2} M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_2}]\right\}\overline{U}'_{\varepsilon,\gamma} dz.
\end{aligned}$$

By the definition of $S[\cdot;\xi]$, we have

$$\begin{aligned}
&\left|\sigma_\varepsilon(\gamma) \int_{\tilde{\Omega}_\varepsilon} \{S[\overline{U}_{\varepsilon,\gamma};\xi_1] - S[\overline{U}_{\varepsilon,\gamma};\xi_2]\}\overline{U}'_{\varepsilon,\gamma} dz + \xi_1 - \xi_2\right| \\
&= \left|\sigma_\varepsilon(\gamma) \int_{\tilde{\Omega}_\varepsilon} \left(\frac{1}{\xi_1} - \frac{1}{\xi_2}\right) f(\overline{U}_{\varepsilon,\gamma})\overline{U}'_{\varepsilon,\gamma} dz + \xi_1 - \xi_2\right| \\
&= \left|1 - \frac{\sigma_\varepsilon(\gamma)}{\xi_1\xi_2} \int_{\tilde{\Omega}_\varepsilon} f(\overline{U}_{\varepsilon,\gamma})\overline{U}'_{\varepsilon,\gamma} dz\right| |\xi_1 - \xi_2| \\
&\le \frac{1}{2}|\xi_1 - \xi_2|.
\end{aligned}$$

Here, we used

$$\frac{\sigma_\varepsilon(\gamma)}{\xi_1\xi_2} \int_{\tilde{\Omega}_\varepsilon} f(\overline{U}_{\varepsilon,\gamma})\overline{U}'_{\varepsilon,\gamma} dz = \frac{5}{4} + o(1)$$

as $\varepsilon \to 0$. Using Lemma 2.2 and (4.12), we have

$$\begin{aligned}
&\left| \int_{\tilde{\Omega}_\varepsilon} \{\tilde{L}_{\varepsilon,\gamma,\xi_1}\phi_{\varepsilon,\gamma,\xi_1} - \tilde{L}_{\varepsilon,\gamma,\xi_2}\phi_{\varepsilon,\gamma,\xi_2}\}\overline{U}'_{\varepsilon,\gamma} dz\right| \\
&\quad \le \left| \int_{\tilde{\Omega}_\varepsilon} (\phi_{\varepsilon,\gamma,\xi_1} - \phi_{\varepsilon,\gamma,\xi_2})\tilde{L}_{\varepsilon,\gamma,\xi_1}[\overline{U}'_{\varepsilon,\gamma}] dz\right| \\
&\qquad + \left| \int_{\tilde{\Omega}_\varepsilon} \phi_{\varepsilon,\gamma,\xi_2}\big(\tilde{L}_{\varepsilon,\gamma,\xi_1}[\overline{U}'_{\varepsilon,\gamma}] - \tilde{L}_{\varepsilon,\gamma,\xi_2}[\overline{U}'_{\varepsilon,\gamma}]\big) dz\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\phi_{\varepsilon,\gamma,\xi_1} - \phi_{\varepsilon,\gamma,\xi_2}\|_{L^2(\tilde{\Omega}_\varepsilon)} \big\|\tilde{L}_{\varepsilon,\gamma,\xi_1}[\overline{U}'_{\varepsilon,\gamma}]\big\|_{L^2(\tilde{\Omega}_\varepsilon)} \\
&\quad + \|\phi_{\varepsilon,\gamma,\xi_2}\|_{L^2(\tilde{\Omega}_\varepsilon)} \big\|f'(\overline{U}_{\varepsilon,\gamma})\overline{U}'_{\varepsilon,\gamma}\big\|_{L^2(\tilde{\Omega}_\varepsilon)} \left|\frac{1}{\xi_1} - \frac{1}{\xi_2}\right| \\
&\leq C\bigg\{\varepsilon^{-(1-\alpha)/2} e^{c_1/\varepsilon^{1-\alpha}} |\xi_1 - \xi_2| \bigg\{\left|\frac{1}{\xi_*} - \frac{1}{\xi_1}\right| \big\|f'(w_\gamma)w'_\gamma\big\|_{L^2(\tilde{\Omega}_\varepsilon)} + \overline{C}_2 e^{-\overline{c}_2/\varepsilon^{1-\alpha}}\bigg\} \\
&\qquad\quad + |\xi_1 - \xi_2|\bigg\} \\
&\leq C'\big\{\varepsilon^{-(1-\alpha)/2} e^{c_1/\varepsilon^{1-\alpha}} \big\{\varepsilon^{(1-\alpha)/2} e^{-c_1/\varepsilon^{1-\alpha}} + e^{-\overline{c}_2/\varepsilon^{1-\alpha}}\big\} + 1\big\} |\xi_1 - \xi_2|.
\end{aligned}
$$

Recall that $c_1 < \overline{c}_2$. Therefore, we have

$$
\begin{aligned}
&\left|\sigma_\varepsilon(\gamma) e^{-c_1/\varepsilon^{1-\alpha}} \int_{\tilde{\Omega}_\varepsilon} \big\{\tilde{L}_{\varepsilon,\gamma,\xi_1}\phi_{\varepsilon,\gamma,\xi_1} - \tilde{L}_{\varepsilon,\gamma,\xi_2}\phi_{\varepsilon,\gamma,\xi_2}\big\} \overline{U}'_{\varepsilon,\gamma} dz\right| \\
&\qquad \leq o(1)|\xi_1 - \xi_2| \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

Finally, by Lemma 3.2 and (4.12), we have

$$
\begin{aligned}
&\left|\sigma_\varepsilon(\gamma) \int_{\tilde{\Omega}_\varepsilon} \left\{\frac{1}{\xi_1} M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_2}] - \frac{1}{\xi_2} M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_2}]\right\} \overline{U}'_{\varepsilon,\gamma} dz\right| \\
&\quad \leq C\bigg\{\left|\frac{1}{\xi_1} - \frac{1}{\xi_2}\right| \left|\int_{\tilde{\Omega}_\varepsilon} M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_1}] \overline{U}'_{\varepsilon,\gamma} dz\right| \\
&\qquad\quad + \frac{1}{\xi_2}\left|\int_{\tilde{\Omega}_\varepsilon} \{M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_1}] - M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_2}]\} \overline{U}'_{\varepsilon,\gamma} dz\right|\bigg\} \\
&\quad \leq C'\big\{|\xi_1 - \xi_2| \|M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_1}]\|_{L^2(\tilde{\Omega}_\varepsilon)} + \|M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_1}] - M_{\varepsilon,\gamma}[\phi_{\varepsilon,\gamma,\xi_2}]\|_{L^2(\tilde{\Omega}_\varepsilon)}\big\} \\
&\quad \leq C'' e^{-2c_1/\varepsilon^{1-\alpha}} \big\{|\xi_1 - \xi_2| + \|\phi_{\varepsilon,\gamma,\xi_1} - \phi_{\varepsilon,\gamma,\xi_2}\|_{L^2(\tilde{\Omega}_\varepsilon)}\big\} \\
&\quad \leq C''' e^{-2c_1/\varepsilon^{1-\alpha}} |\xi_1 - \xi_2| \big\{1 + \varepsilon^{-(1-\alpha)/2} e^{c_1/\varepsilon^{1-\alpha}}\big\} \\
&\quad \leq o(1)|\xi_1 - \xi_2| \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

Thus we have

$$
|T_\gamma[\xi_1] - T_\gamma[\xi_2]| \leq \frac{3}{4}|\xi_1 - \xi_2|
$$

for all ξ_1, $\xi_2 \in I_\varepsilon$ and $\gamma \in [\gamma_1, \gamma_2]$ provided ε is small enough. Therefore, T_γ, $\gamma \in [\gamma_1, \gamma_2]$, is a contraction mapping on I_ε for ε sufficiently small. Then T_γ has a unique fixed point $\xi_{\varepsilon,\gamma} \in I_\varepsilon$. Moreover, it is easy to see that $T_\gamma[\xi]$ is continuous in $\gamma \in [\gamma_1, \gamma_2]$ for each $\xi \in I_\varepsilon$, and hence $\xi_{\varepsilon,\gamma}$ is continuous in $\gamma \in [\gamma_1, \gamma_2]$. Thus we complete the proof. □

6. Matching problem.

In this section, we will complete Step 3. That is, for each ε sufficiently small, we will find $\gamma = \gamma_\varepsilon \in [\gamma_1, \gamma_2]$ such that

$$\xi_{\varepsilon,\gamma_\varepsilon} = \varepsilon^{1-\alpha} \int_{\tilde{\Omega}_\varepsilon} u_\varepsilon^2(z; \gamma_\varepsilon, \xi_{\varepsilon,\gamma_\varepsilon}) dz \tag{6.1}$$

holds, where $u_\varepsilon(z; \gamma, \xi_{\varepsilon,\gamma}) = \overline{U}_{\varepsilon,\gamma}(z) + e^{-c_1/\varepsilon^{1-\alpha}} \phi_{\varepsilon,\gamma,\xi_{\varepsilon,\gamma}}(z)$. For the purpose, we first show the following lemma.

Lemma 6.1. *Let $\gamma \in [\gamma_1, \gamma_2]$. Then the following identity holds*:

$$\varepsilon^{1-\alpha} \int_{\tilde{\Omega}_\varepsilon} \overline{U}^2_{\varepsilon,\gamma}(z) dz = \xi_* + \varepsilon^{1-\alpha} I_\varepsilon(\gamma) + k_1(\varepsilon), \tag{6.2}$$

where

$$I_\varepsilon(\gamma) := \int_{-D_\varepsilon}^{D_\varepsilon} \left\{ w_\gamma^2(z) - \frac{1}{2}\beta^2 \right\} dz, \quad D_\varepsilon = \frac{1}{4\varepsilon^{1-\alpha}} y_c, \tag{6.3}$$

and the term $k_1(\varepsilon)$ satisfies

$$|k_1(\varepsilon)| \le \overline{C}_4 e^{-\overline{c}_4/\varepsilon^{1-\alpha}} \tag{6.4}$$

for some constants $\overline{C}_4$, $\overline{c}_4 > 0$ independent of $\gamma \in [\gamma_1, \gamma_2]$ and ε sufficiently small.

Proof. In this proof, it is convenient to treat the y-variable. Recall that $y = y_c + \varepsilon^{1-\alpha} z$ and

$$\varepsilon^{1-\alpha} \int_{\tilde{\Omega}_\varepsilon} \overline{U}^2_{\varepsilon,\gamma}(z) dz = \int_{\Omega_\varepsilon} U^2_{\varepsilon,\gamma}(y) dy, \quad \xi_* = \beta^2 y_c.$$

By the setting of χ_0 and χ_1, we can calculate as follows:

$$
\begin{aligned}
\int_{\Omega_\varepsilon} U_{\varepsilon,\gamma}^2(y)dy &= \int_{\Omega_\varepsilon} \left\{ w_\gamma\left(\frac{y-y_c}{\varepsilon^{1-\alpha}}\right)\chi_0(y) + \beta\chi_1(y) \right\}^2 dy \\
&= \int_0^{y_c/2} \beta^2 dy + \int_{y_c/2}^{3y_c/4} \left\{ w_\gamma\left(\frac{y-y_c}{\varepsilon^{1-\alpha}}\right)\chi_0(y) + \beta(1-\chi_0(y)) \right\}^2 dy \\
&\quad + \int_{3y_c/4}^{5y_c/4} w_\gamma^2\left(\frac{y-y_c}{\varepsilon^{1-\alpha}}\right) dy + \int_{5y_c/4}^{3y_c/2} w_\gamma^2\left(\frac{y-y_c}{\varepsilon^{1-\alpha}}\right)\chi_0^2(y) dy \\
&=: I + II + III + IV.
\end{aligned}
$$

Then, by Lemma 1.1, we have

$$
\begin{aligned}
I &= \frac{1}{2}\beta^2 y_c, \\
II &= \int_{y_c/2}^{3y_c/4} \left\{ \left(w_\gamma\left(\frac{y-y_c}{\varepsilon^{1-\alpha}}\right) - \beta \right)\chi_0(y) + \beta \right\}^2 dy \\
&= \frac{1}{4}\beta^2 y_c + e.s.t., \\
IV &= e.s.t.,
\end{aligned}
$$

where "e.s.t." means the exponentially small term and is estimated by $Ce^{-c/\varepsilon^{1-\alpha}}$ for some $C, c > 0$ independent of γ and ε. Therefore, we have

$$
\begin{aligned}
\int_{\Omega_\varepsilon} U_{\varepsilon,\gamma}^2(y)dy &= \frac{3}{4}\beta^2 y_c + \int_{3y_c/4}^{5y_c/4} w_\gamma^2\left(\frac{y-y_c}{\varepsilon^{1-\alpha}}\right) dy + e.s.t. \\
&= \beta^2 y_c + \int_{3y_c/4}^{5y_c/4} w_\gamma^2\left(\frac{y-y_c}{\varepsilon^{1-\alpha}}\right) dy - \frac{1}{4}\beta^2 y_c + e.s.t. \\
&= \xi_* + \int_{3y_c/4}^{5y_c/4} \left\{ w_\gamma^2\left(\frac{y-y_c}{\varepsilon^{1-\alpha}}\right) - \frac{1}{2}\beta^2 \right\} dy + e.s.t. \\
&= \xi_* + \varepsilon^{1-\alpha} I_\varepsilon(\gamma) + e.s.t.
\end{aligned}
$$

Thus we complete the proof. □

Secondly, we study the property of $I_\varepsilon(\gamma)$. We note that $I_\varepsilon(\gamma)$ is well-defined also for $\gamma \in (0, \beta)$ and is continuous and strictly monotone increasing on γ for each ε.

Lemma 6.2. *For each ε, there exists a unique $\tilde{\gamma}_\varepsilon \in (0,\beta)$ such that $I_\varepsilon(\tilde{\gamma}_\varepsilon) = 0$. Moreover, $\tilde{\gamma}_\varepsilon$ converges to a certain unique $\gamma_* \in (0,\beta)$ as $\varepsilon \to 0$.*

Proof. For each ε, it is easy to see that, if we take $\gamma \in (0,\beta)$ near 0, then $I_\varepsilon(\gamma) < 0$, and if we take $\gamma \in (0,\beta)$ near β, then $I_\varepsilon(\gamma) > 0$. Hence, by the continuity and monotonicity, there exists a unique $\tilde{\gamma}_\varepsilon \in (0,\beta)$ such that $I_\varepsilon(\tilde{\gamma}_\varepsilon) = 0$. Let $\gamma_* \in [0,\beta]$ be an accumulating point of γ_ε. Then, we can exclude the possibility of $\gamma_* = 0, \beta$. Let

$$\begin{aligned} I_\varepsilon(\gamma) &= \int_0^{D_\varepsilon} \Big(w_\gamma^2(z) - \frac{1}{2}\beta^2\Big)dz + \int_{-D_\varepsilon}^0 \Big(w_\gamma^2(z) - \frac{1}{2}\beta^2\Big)dz \\ &= \int_0^{D_\varepsilon} w_\gamma^2(z)dz - \frac{1}{2}\beta^2 D_\varepsilon + \int_{-D_\varepsilon}^0 w_\gamma^2(z)dz - \frac{1}{2}\beta^2 D_\varepsilon \\ &= \int_0^{D_\varepsilon} w_\gamma^2(z)dz + \int_{-D_\varepsilon}^0 \big(w_\gamma^2(z) - \beta^2\big)dz. \end{aligned}$$

From this, it holds that

$$\int_0^{D_\varepsilon} w_{\tilde{\gamma}_\varepsilon}^2(z)dz = \int_{-D_\varepsilon}^0 \big(\beta^2 - w_{\tilde{\gamma}_\varepsilon}^2(z)\big)dz. \tag{6.5}$$

If $\tilde{\gamma}_\varepsilon$ accumulates at 0, then there exists ε_n such that $\varepsilon_n \to 0$ as $n \to \infty$ and $\tilde{\gamma}_{\varepsilon_n} \to 0$ as $n \to \infty$. However, it is impossible because the left hand side of (6.5) remains bounded as $n \to \infty$, on the other hand, the right hand side of (6.5) tends to infinity as $n \to \infty$. Hence, $\gamma_* \neq 0$. Similarly, we can prove $\gamma_* \neq \beta$. Thus, $\tilde{\gamma}_\varepsilon$ accumulates neither at 0 nor at β. Hence, we may assume there exists $\underline{\gamma}, \overline{\gamma} \in (0,\beta)$, $\underline{\gamma} < \overline{\gamma}$, such that $\tilde{\gamma}_\varepsilon \in [\underline{\gamma}, \overline{\gamma}]$ for all ε sufficiently small. We claim that the accumulating point of $\tilde{\gamma}_\varepsilon$ is exactly one. Indeed, let $\tilde{\gamma}_\varepsilon$ possess two different accumulating points. Then, there exists $M > 0$ such that, for any $n \in \mathbb{N}$, there exist $\varepsilon_n, \varepsilon_n' \in (0, 1/n)$, $\varepsilon_n > \varepsilon_n'$, such that $|\tilde{\gamma}_{\varepsilon_n} - \tilde{\gamma}_{\varepsilon_n'}| \geq M$ holds. We see that

$$\begin{aligned} 0 &= I_{\varepsilon_n}(\tilde{\gamma}_{\varepsilon_n}) - I_{\varepsilon_n'}(\tilde{\gamma}_{\varepsilon_n'}) \\ &= \int_{-D_{\varepsilon_n'}}^{D_{\varepsilon_n'}} \big(w_{\tilde{\gamma}_{\varepsilon_n}}^2 - w_{\tilde{\gamma}_{\varepsilon_n'}}^2\big)dz + \int_{-D_{\varepsilon_n}}^{-D_{\varepsilon_n'}} w_{\tilde{\gamma}_{\varepsilon_n}}^2 dz + \int_{D_{\varepsilon_n'}}^{D_{\varepsilon_n}} \big(w_{\tilde{\gamma}_{\varepsilon_n}}^2 - \beta^2\big)dz \end{aligned}$$

holds. Because $\tilde{\gamma}_{\varepsilon_n}, \tilde{\gamma}_{\varepsilon_n'} \in [\underline{\gamma}, \overline{\gamma}]$ for all n sufficiently large, if we take $n \to \infty$, then the second term and the third term tend to 0. However, the first term cannot tend to 0 since $|\tilde{\gamma}_{\varepsilon_n} - \tilde{\gamma}_{\varepsilon_n'}| \geq M$. It is impossible. Therefore, the accumulating point of $\tilde{\gamma}_\varepsilon$ is unique. Thus, we complete the proof. $\square$

Remark 4. The number $\gamma_* \in (0, \beta)$ is characterized to be a number satisfying

$$\int_0^\infty w_{\gamma_*}^2(z)dz = \int_{-\infty}^0 \left(\beta^2 - w_{\gamma_*}^2(z)\right)dz \tag{6.6}$$

from the equation (6.5).

Hitherto, the constants $\gamma_1, \gamma_2 \in (0, \beta)$ have been arbitrarily fixed constants. From now on, let us fix the constants so that

$$\tilde{\gamma}_\varepsilon \in (\gamma_1, \gamma_2), \quad \gamma_* \in (\gamma_1, \gamma_2) \tag{6.7}$$

hold for all ε sufficiently small. We recall that there is only difference of translations among w_{γ_*}, w_{γ_1} and w_{γ_2}. Let $a_1, a_2 > 0$ be constants such that

$$w_{\gamma_1}(z) = w_{\gamma_*}(z + a_1), \quad w_{\gamma_2}(z) = w_{\gamma_*}(z - a_2). \tag{6.8}$$

Then we have the following lemma which will be needed for the problem (6.1).

Lemma 6.3. *Let $\gamma_1, \gamma_2 \in (0, \beta)$ be fixed so that* (6.7) *holds and $a_1, a_2 > 0$ be constants defined by* (6.8). *Then, $I_\varepsilon(\gamma_1)$ and $I_\varepsilon(\gamma_2)$ have the following asymptotic behavior*:

$$I_\varepsilon(\gamma_1) = -a_1\beta^2 + o(1), \tag{6.9}$$

$$I_\varepsilon(\gamma_2) = a_2\beta^2 + o(1), \tag{6.10}$$

as $\varepsilon \to 0$.

Proof. We can calculate as follows:

$$\begin{aligned}
I_\varepsilon(\gamma_1) &= \int_{-D_\varepsilon}^{D_\varepsilon} \left\{w_{\gamma_1}^2(z) - \frac{1}{2}\beta^2\right\}dz = \int_{-D_\varepsilon}^{D_\varepsilon} \left\{w_{\gamma_*}^2(z + a_1) - \frac{1}{2}\beta^2\right\}dz \\
&= \int_{-D_\varepsilon + a_1}^{D_\varepsilon + a_1} \left\{w_{\gamma_*}^2(z) - \frac{1}{2}\beta^2\right\}dz \\
&= \int_{-D_\varepsilon}^{D_\varepsilon} \left\{w_{\gamma_*}^2(z) - \frac{1}{2}\beta^2\right\}dz + \int_{D_\varepsilon}^{D_\varepsilon + a_1} \left\{w_{\gamma_*}^2(z) - \frac{1}{2}\beta^2\right\}dz \\
&\quad - \int_{-D_\varepsilon}^{-D_\varepsilon + a_1} \left\{w_{\gamma_*}^2(z) - \frac{1}{2}\beta^2\right\}dz
\end{aligned}$$

$$= \int_{-D_\varepsilon}^{D_\varepsilon} \Big\{ w_{\gamma_*}^2(z) - \frac{1}{2}\beta^2 \Big\} dz + \int_{D_\varepsilon}^{D_\varepsilon+a_1} w_{\gamma_*}^2(z)dz - \int_{-D_\varepsilon}^{-D_\varepsilon+a_1} w_{\gamma_*}^2(z)dz. \tag{6.11}$$

Then, we see that the first term tends to 0 as $\varepsilon \to 0$. Indeed, let

$$\int_{-D_\varepsilon}^{D_\varepsilon} \Big\{ w_{\gamma_*}^2(z) - \frac{1}{2}\beta^2 \Big\} dz = \int_{-D_\varepsilon}^{D_\varepsilon} \big\{ w_{\gamma_*}^2(z) - w_{\tilde{\gamma}_\varepsilon}^2(z) \big\} dz. \tag{6.12}$$

It is easy to see that the right hand side of (6.12) tends to 0 as $\varepsilon \to 0$ by using the estimates stated in Lemma 1.1 and noting $\tilde{\gamma}_\varepsilon \to \gamma_*$ as $\varepsilon \to 0$. Moreover, the third term of (6.11) clearly tends to 0 as $\varepsilon \to 0$. Therefore, we can see that

$$\begin{aligned} I_\varepsilon(\gamma_1) &= -\int_{-D_\varepsilon}^{-D_\varepsilon+a_1} w_{\gamma_*}^2(z)dz + o(1) \\ &= -\int_{-D_\varepsilon}^{-D_\varepsilon+a_1} \big\{ w_{\gamma_*}^2(z) - \beta^2 \big\} dz - a_1\beta^2 + o(1) \\ &= -a_1\beta^2 + o(1) \end{aligned}$$

as $\varepsilon \to 0$. Similarly, (6.10) can be proven. Thus we complete the proof. □

Now, we are ready to solve the problem (6.1).

PROPOSITION 6.1. *For each ε sufficiently small, there exists at least one number $\gamma_\varepsilon \in [\gamma_1, \gamma_2]$ such that* (6.1) *holds. Moreover, γ_ε accumulates at γ_* as $\varepsilon \to 0$.*

PROOF. Recall that $u_\varepsilon(z; \gamma, \xi_{\varepsilon,\gamma}) = \overline{U}_{\varepsilon,\gamma}(z) + e^{-c_1/\varepsilon^{1-\alpha}} \phi_{\varepsilon,\gamma,\xi_{\varepsilon,\gamma}}(z)$ and the $H^2(\tilde{\Omega}_\varepsilon)$-norm of $\phi_{\varepsilon,\gamma,\xi_{\varepsilon,\gamma}}$ is bounded uniformly on $\gamma \in [\gamma_1, \gamma_2]$ and ε sufficiently small. We write $\phi_{\varepsilon,\gamma} = \phi_{\varepsilon,\gamma,\xi_{\varepsilon,\gamma}}$ simply. Let

$$\begin{aligned} &\varepsilon^{1-\alpha} \int_{\tilde{\Omega}_\varepsilon} \big(\overline{U}_{\varepsilon,\gamma} + e^{-c_1/\varepsilon^{1-\alpha}} \phi_{\varepsilon,\gamma} \big)^2 dz \\ &= \varepsilon^{1-\alpha} \Big\{ \int_{\tilde{\Omega}_\varepsilon} \overline{U}_{\varepsilon,\gamma}^2 dz + 2e^{-c_1/\varepsilon^{1-\alpha}} \int_{\tilde{\Omega}_\varepsilon} \overline{U}_{\varepsilon,\gamma} \phi_{\varepsilon,\gamma} dz + e^{-2c_1/\varepsilon^{1-\alpha}} \int_{\tilde{\Omega}_\varepsilon} \phi_{\varepsilon,\gamma}^2 dz \Big\}. \end{aligned}$$

We notice that the leading term is $\varepsilon^{1-\alpha} \int_{\tilde{\Omega}_\varepsilon} \overline{U}_{\varepsilon,\gamma}^2 dz$. The problem (6.1) is equivalent to the following:

$$\int_{\tilde{\Omega}_\varepsilon} u_\varepsilon^2(z;\gamma,\xi_{\varepsilon,\gamma})dz - \frac{1}{\varepsilon^{1-\alpha}}\xi_* = \frac{1}{\varepsilon^{1-\alpha}}(\xi_{\varepsilon,\gamma}-\xi_*), \tag{6.13}$$

$$\begin{aligned}\int_{\tilde{\Omega}_\varepsilon} \overline{U}_{\varepsilon,\gamma}^2 dz - \frac{1}{\varepsilon^{1-\alpha}}\xi_* = \frac{1}{\varepsilon^{1-\alpha}}(\xi_{\varepsilon,\gamma}-\xi_*) - 2e^{-c_1/\varepsilon^{1-\alpha}}\int_{\tilde{\Omega}_\varepsilon} \overline{U}_{\varepsilon,\gamma}\phi_{\varepsilon,\gamma}dz \\ - e^{-2c_1/\varepsilon^{1-\alpha}}\int_{\tilde{\Omega}_\varepsilon}\phi_{\varepsilon,\gamma}^2 dz =: k(\varepsilon,\gamma).\end{aligned} \tag{6.14}$$

Because $\xi_{\varepsilon,\gamma}\in I_\varepsilon$, $|k(\varepsilon,\gamma)|$ is estimated by $C\varepsilon^{-(1-\alpha)/2}e^{-c_1/\varepsilon^{1-\alpha}}$ for some $C>0$ independent of ε and γ. Hence, we see that $k(\varepsilon,\gamma)$ converges to 0 as $\varepsilon\to 0$ uniformly on $\gamma\in[\gamma_1,\gamma_2]$. Moreover, we note that $k(\varepsilon,\gamma)$ is continuous in $\gamma\in[\gamma_1,\gamma_2]$. On the other hand, the left hand side of (6.14) is also continuous in $\gamma\in[\gamma_1,\gamma_2]$ and has the following behavior by Lemmas 6.1 and 6.3:

$$\int_{\tilde{\Omega}_\varepsilon} \overline{U}_{\varepsilon,\gamma}^2 dz - \frac{1}{\varepsilon^{1-\alpha}}\xi_* = I_\varepsilon(\gamma) + \frac{1}{\varepsilon^{1-\alpha}}k_1(\varepsilon) = \begin{cases} -a_1\beta^2 + o(1), & \gamma=\gamma_1, \\ a_2\beta^2+o(1), & \gamma=\gamma_2, \end{cases} \tag{6.15}$$

as $\varepsilon\to 0$. Hence, there exists a constant $\eta>0$ such that, for $\gamma=\gamma_1,\gamma_2$,

$$\int_{\tilde{\Omega}_\varepsilon} \overline{U}_{\varepsilon,\gamma_1}^2 dz - \frac{1}{\varepsilon^{1-\alpha}}\xi_* \le -\eta,$$
$$\int_{\tilde{\Omega}_\varepsilon} \overline{U}_{\varepsilon,\gamma_2}^2 dz - \frac{1}{\varepsilon^{1-\alpha}}\xi_* \ge \eta,$$

hold for all ε sufficiently small. Therefore, there exists at least one $\gamma=\gamma_\varepsilon\in[\gamma_1,\gamma_2]$ such that (6.14) holds by the intermediate value theorem. Moreover, we see that $I_\varepsilon(\gamma_\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence, γ_ε must be accumulated at γ_* as $\varepsilon\to 0$. □

Finally, we give the proofs of Theorem 1.1 and its corollary.

Proof of Theorem 1.1. Put

$$\tilde{\phi}_\varepsilon(y) = \phi_{\varepsilon,\gamma_\varepsilon,\xi_{\varepsilon,\gamma_\varepsilon}}\left(\frac{y-y_c}{\varepsilon^{1-\alpha}}\right),$$
$$a_\varepsilon(y) = U_{\varepsilon,\gamma_\varepsilon}(y) + e^{-c_1/\varepsilon^{1-\alpha}}\tilde{\phi}_\varepsilon(y),$$

and

$$\xi_\varepsilon = \xi_{\varepsilon,\gamma_\varepsilon}.$$

Then $a_\varepsilon(y)$ and ξ_ε solve the equation (1.5) and obviously satisfies (1.16) and (1.17). Moreover, we see that $a_\varepsilon(y) > 0$ for $y \in \Omega_\varepsilon$ by the usual maximum principle. Thus we complete the proof. □

Proof of Corollary 1.1. Put

$$A_\varepsilon(x) = \frac{1}{\varepsilon^\alpha} a_\varepsilon\left(\frac{x}{\varepsilon^\alpha}\right), \quad \hat{\xi}_\varepsilon = \frac{1}{\varepsilon^\alpha}\xi_\varepsilon,$$

where $(a_\varepsilon, \xi_\varepsilon)$ is a solution to (1.5) given by Theorem 1.1. Then we know that $(A_\varepsilon, \hat{\xi}_\varepsilon)$ gives a solution to (1.1). Omitting the hat of $\hat{\xi}_\varepsilon$, we complete the proof. □

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