J. Math. Soc. Japan
Vol. 63, No. 3 (2011) pp. 977–983
doi: 10.2969/jmsj/06330977

Addendum to: Characterizations of topological dimension by use of normal sequences of finite open covers and Pontrjagin-Schnirelmann theorem

[The original paper is in this journal, Vol. 63 (2011), 919–976]

By Hisao Kato

(Received July 2, 2010)

Abstract. In our recent paper [**5**] in this journal, we have studied strong relations between metrics of spaces and box-counting dimensions by use of Alexandroff-Urysohn metrics d induced by normal sequences. In this addendum, we intend to improve the main theorems given in [**5**, Theorem 0.1 and 0.2] and give the complete solution for a problem of metrics and two box-counting dimensions.

1. Introduction.

In this addendum we improve the main theorems given in [**5**] and give the complete solution for a problem of metrics d and box-counting dimensions $\underline{\dim}_B(X,d)$ and $\overline{\dim}_B(X,d)$.

We follow directly the notations of [**5**]. For a topological space X, we denote by $\dim X$ the topological (covering) dimension of X (see [**4**], [**6**], [**7**], [**9**]). For a totally bounded metric d on X and $\epsilon > 0$, let

$$N(\epsilon, d) = \min\{|\mathscr{U}| \mid \mathscr{U} \text{ is a finite open cover of } X \text{ with } \mathrm{mesh}_d(\mathscr{U}) \leq \epsilon\},$$

where $|A|$ denotes the cardinality of a set A. Then the lower and upper box-counting dimensions of (X,d) (see [**10**]) are given by

$$\underline{\dim}_B(X,d) = \liminf_{\epsilon \to 0} \frac{\log N(\epsilon,d)}{|\log \epsilon|}$$
$$\overline{\dim}_B(X,d) = \limsup_{\epsilon \to 0} \frac{\log N(\epsilon,d)}{|\log \epsilon|}.$$

2000 *Mathematics Subject Classification.* Primary 54F45; Secondary 28A78, 37C45, 54E35, 28A80.

Key Words and Phrases. normal sequence of finite open covers, topological dimension, lower (upper) box-counting dimensions, Menger compacta.

We obtain the following result which is the complete solution for a problem of metrics and two box-counting dimensions.

THEOREM 1.1 (cf. [**5**, Theorem 0.2]). *Let X be an infinite separable metric space. For any $\alpha, \beta \in [\dim X, \infty]$ with $\alpha \leq \beta$, there is a totally bounded metric $d = d_{\alpha\beta}$ on X such that*

$$[\alpha, \beta] = \left\{ \liminf_{k\to\infty} \frac{\log N(\epsilon_k, d)}{|\log \epsilon_k|} \;\middle|\; \{\epsilon_k\}_{k=1}^{\infty} \textit{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0 \right\}.$$

In particular, $\underline{\dim}_B(X, d) = \alpha$ and $\overline{\dim}_B(X, d) = \beta$.

To prove Theorem 1.1, we need the following theorem which is more precise result than [**5**, Theorem 0.1]. To prove it, we extend the technique of Banakh and Tuncali (see [**2**, Theorem 6.1]).

THEOREM 1.2 (cf. [**5**, Theorem 0.1]). *Let X be a nonempty separable metric space. Then*

$$\begin{aligned}\dim X &= \min\left\{ \liminf_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} \;\middle|\; \{\mathscr{U}_i\}_{i=1}^{\infty} \textit{ is a normal star-sequence of finite open covers of } X \textit{ and a development of } X \right\} \\ &= \min\left\{ \liminf_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i} \;\middle|\; \{\mathscr{U}_i\}_{i=1}^{\infty} \textit{ is a normal delta-sequence of finite open covers of } X \textit{ and a development of } X \right\}.\end{aligned}$$

Moreover, there exists a normal star-sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ of finite open covers of X which is a development of X such that

$$\dim X = \lim_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i}.$$

Also, there exists a normal delta-sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ of finite open covers of X which is a development of X such that

$$\dim X = \lim_{i\to\infty} \frac{\log_2 |\mathscr{U}_i|}{i}.$$

PROOF. We can suppose that $\dim X = n < \infty$. Let $M = [0,1]^{2n+1}$ be the unit cube in the $(2n+1)$-dimensional Euclidean space $\boldsymbol{R}^{2n+1}$. For $a \in \boldsymbol{N}$, we divide the edges of $M = [0,1]^{2n+1}$ into a equal subintervals and we obtain the collection $\mathscr{C}(1/a)$ of all a^{2n+1} subcubes of M with edge $1/a$. For each $i = 0, 1, 2, \dots$ we obtain the collection $\mathscr{C}(1/3^i)$ of all $3^{i(2n+1)}$ subcubes of M with edge $1/3^i$. Let $\boldsymbol{a} = \{a_i\}_{i=1}^{\infty}$ be any increasing sequence of natural numbers, i.e., $a_0 = 0 < a_1 < a_2 < \cdots < a_i < a_{i+1} < \cdots$. We shall construct an n-dimensional Menger universal compactum $M_{\boldsymbol{a}}$ as follows. First we put

$$\mathscr{M}_{\boldsymbol{a}}(a_0) = \{[0,1]^{2n+1}\}.$$

For each $a_0 + 1 = 1 \le i \le a_1$, we put

$$\mathscr{M}_{\boldsymbol{a}}(i) = \left\{ D \in \mathscr{C}\left(\frac{1}{3^i}\right) \,\middle|\, D \text{ intersects an } n\text{-dimensional face of } [0,1]^{2n+1} \right\}.$$

For each $a_1 + 1 \le i \le a_2$, we put

$$\mathscr{M}_{\boldsymbol{a}}(i) = \left\{ D \in \mathscr{C}\left(\frac{1}{3^i}\right) \,\middle|\, \begin{array}{l}\text{there is } C \in \mathscr{M}_{\boldsymbol{a}}(a_1) \text{ such that } D \subset C \text{ and } D \\ \qquad\text{intersects an } n\text{-dimensional face of } C\end{array} \right\}.$$

For each $a_2 + 1 \le i \le a_3$, we put

$$\mathscr{M}_{\boldsymbol{a}}(i) = \left\{ D \in \mathscr{C}\left(\frac{1}{3^i}\right) \,\middle|\, \begin{array}{l}\text{there is } C \in \mathscr{M}_{\boldsymbol{a}}(a_2) \text{ such that } D \subset C \text{ and } D \\ \qquad\text{intersects an } n\text{-dimensional face of } C\end{array} \right\}.$$

We iterate this procedure with respect to the sequence $\boldsymbol{a} = \{a_i\}_{i=1}^{\infty}$ and we obtain the collection $\mathscr{M}_{\boldsymbol{a}}(i)$ of subcubes with edges $1/3^i$ for each $i \in \boldsymbol{N}$. Put

$$M_{\boldsymbol{a}}(i) = \bigcup \{C \mid C \in \mathscr{M}_{\boldsymbol{a}}(i)\} \subset \boldsymbol{R}^{2n+1}.$$

Then $M_{\boldsymbol{a}}(i) \supset M_{\boldsymbol{a}}(i+1)$ for each $i \in \boldsymbol{N}$. We put

$$M_{\boldsymbol{a}} = \bigcap_{i=1}^{\infty} M_{\boldsymbol{a}}(i).$$

By use of Anderson-Bestvina's Characterization Theorem of Menger compacta (see [**1**] and [**3**]), we see that $M_{\boldsymbol{a}}$ is homeomorphic to the n-dimensional Menger universal compactum. Note that $\mathscr{M}_{\boldsymbol{a}}(i) \subset \mathscr{C}(1/3^i)$ and $M_{\boldsymbol{a}}$ is a subset of the standard n-dimensional Menger compactum $M_{\dot{\boldsymbol{a}}}$, where $\dot{\boldsymbol{a}} = \{\dot{a}_i\}_{i=1}^{\infty}$ and $\dot{a}_i = i$ for $i \in \boldsymbol{N}$.

We shall construct a normal star-sequence $\{\mathscr{W}(\boldsymbol{a})_i\}_{i=1}^{\infty}$ of finite open covers of $M_{\boldsymbol{a}}$ as follows. For each $i \in \boldsymbol{N}$ we put

$$\mathscr{W}(\boldsymbol{a})_i = \{\mathrm{Int}_{M_{\boldsymbol{a}}} St(C, \mathscr{M}_{\boldsymbol{a}}(i)) \mid C \in \mathscr{M}_{\boldsymbol{a}}(i)\},$$

where $St(C, \mathscr{M}_{\boldsymbol{a}}(i)) = \bigcup\{D \in \mathscr{M}_{\boldsymbol{a}}(i) \mid D \cap C \neq \phi\} \cap M_{\boldsymbol{a}}$. Then we see that $\{\mathscr{W}(\boldsymbol{a})_i\}_{i=1}^{\infty}$ is a normal star-sequence of finite open covers and a development of the space $M_{\boldsymbol{a}}$. Also, we see that $|\mathscr{W}(\boldsymbol{a})_i| = |\mathscr{M}_{\boldsymbol{a}}(i)|$.

Now, we consider the special sequence $\ddot{\boldsymbol{a}} = \{\ddot{a}_i\}_{i=1}^{\infty}$ of natural numbers, where $\ddot{a}_i = 2^i$ for $i \in \boldsymbol{N}$ and we obtain a desired n-dimensional Menger universal compactum $Y = M_{\ddot{\boldsymbol{a}}}$. We shall prove that the normal sequence $\{\mathscr{W}(\ddot{\boldsymbol{a}})_i\}_{i=1}^{\infty}$ of Y satisfies the condition

$$\lim_{i \to \infty} \frac{\log_3 |\mathscr{W}(\ddot{\boldsymbol{a}})_i|}{i} = n.$$

For each natural number k, we consider the sequence $\boldsymbol{k} = \{k_i\}_{i=1}^{\infty}$ such that $k_i = i$ for each $1 \leq i \leq 2^k$ and $k_i = 2^k(i+1-2^k)$ for each $i > 2^k$. Note that $Y = M_{\ddot{\boldsymbol{a}}} \subset M_{\boldsymbol{k}}$ and $|\mathscr{W}(\ddot{\boldsymbol{a}})_i| \leq |\mathscr{W}(\boldsymbol{k})_i|$ for $i \in \boldsymbol{N}$. We can calculate $\lim_{i\to\infty} \log_3 |\mathscr{W}(\boldsymbol{k})_i|/i$ as follows (see [**2**, Theorem 6.1]). For $a \in \boldsymbol{N}$, we put

$$\begin{aligned} H(a) &= \sum_{j=0}^{n} 2^{2n+1-j} C_j^{2n+1} (a-2)^j \\ &= 2^{2n+1} \cdot a^n \left[\sum_{j=0}^{n} C_j^{2n+1} \left(\frac{a-2}{2a} \right)^j \cdot a^{j-n} \right], \end{aligned}$$

where $C_p^q = q!/(p!(q-p)!)$. Note that

$$H(a) = \left| \left\{ D \in \mathscr{C}\left(\frac{1}{a}\right) \,\middle|\, D \text{ intersects an } n\text{-dimensional face of } [0,1]^{2n+1} \right\} \right|.$$

Note that there is a number $T > 0$ such that for any $a \in \boldsymbol{N}$

$$H(a) \leq 2^{2n+1} a^n T.$$

Let i be a sufficiently large natural number with $i \geq 2^k$. Put $i = 2^k + 2^k p + q$, where $p, q \in \boldsymbol{N} \cup \{0\}$ and $0 \leq q < 2^k$. Then

$$|\mathscr{W}(\boldsymbol{k})_i| = H(3)^{2^k} H(3^{2^k})^p H(3^q).$$

Then

$$\begin{aligned}\lim_{i\to\infty} \frac{\log_3 |\mathscr{W}(\boldsymbol{k})_i|}{i} &= \lim_{p\to\infty} \frac{2^k \log_3 H(3) + p \log_3 H(3^{2^k}) + \log_3 H(3^q)}{2^k + 2^k p + q} \\ &= \frac{\log_3 H(3^{2^k})}{2^k} \leq n + \frac{(2n+1)\log_3 2 + \log_3 T}{2^k}.\end{aligned}$$

Since

$$\limsup_{i\to\infty} \frac{\log_3 |\mathscr{W}(\ddot{\boldsymbol{a}})_i|}{i} \leq \lim_{i\to\infty} \frac{\log_3 |\mathscr{W}(\boldsymbol{k})_i|}{i}$$

for any $k \in \boldsymbol{N}$, we see that

$$\limsup_{i\to\infty} \frac{\log_3 |\mathscr{W}(\ddot{\boldsymbol{a}})_i|}{i} \leq n.$$

By [**5**, Theorem 0.1], we can conclude that

$$\lim_{i\to\infty} \frac{\log_3 |\mathscr{W}(\ddot{\boldsymbol{a}})_i|}{i} = n.$$

Let X be an n-dimensional compactum. Since Y is a universal space of the class of n-dimensional spaces, we may assume that $X \subset Y$. Put $\mathscr{U}_i = \mathscr{W}(\ddot{\boldsymbol{a}})_i \mid X$ for each $i \in \boldsymbol{N}$. Also by [**5**, Theorem 0.1], the normal sequence $\{\mathscr{U}_i\}_{i=1}^{\infty}$ of X satisfies the desired condition

$$\lim_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i} = n.$$

The existence of desired normal delta-sequence can be proved similarly.

By the proof of [**5**, Theorem 5.1], we have the following corollary.

COROLLARY 1.3. *Let X be an infinite separable metric space. For any $\alpha, \beta \in [\dim X, \infty]$ with $\alpha \le \beta$, there is a normal star (resp. delta)-sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of finite open covers of X which is a development of X such that*

$$[\alpha,\beta] = \left\{ \liminf_{k\to\infty} \frac{\log_3 |\mathscr{U}_{i_k}|}{i_k} \,\middle|\, \{i_k\}_{k=1}^\infty \text{ is an increasing subsequence of natural numbers} \right\}$$

$$\left(\text{resp. } [\alpha,\beta] = \left\{ \liminf_{k\to\infty} \frac{\log_2 |\mathscr{U}_{i_k}|}{i_k} \,\middle|\, \{i_k\}_{k=1}^\infty \text{ is an increasing subsequence of natural numbers} \right\} \right).$$

PROOF OF THEOREM 1.1. By Theorem 1.2, there is a normal star-sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of finite open covers of X which is a development of X such that

$$\dim X = \lim_{i\to\infty} \frac{\log_3 |\mathscr{U}_i|}{i}.$$

By [**5**, Theorem 6.2 and Theorem 6.4], we see that there is a totally bounded metric $d = d_{\alpha\beta}$ on X such that

$$[\alpha,\beta] = \left\{ \liminf_{k\to\infty} \frac{\log N(\epsilon_k, d)}{|\log \epsilon_k|} \,\middle|\, \{\epsilon_k\}_{k=1}^\infty \text{ is a decreasing sequence of positive numbers with } \lim_{k\to\infty} \epsilon_k = 0 \right\}.$$

REMARK. In this paper, "normal" sequence of open covers is essential. We can easily see that for any separable metric space X, there is a sequence $\{\mathscr{U}_i\}_{i=1}^\infty$ of finite open covers of X which is a development of X such that $\overline{\mathscr{U}}_{i+1} = \{\overline{U} \mid U \in \mathscr{U}_{i+1}\}$ is a refinement of $\mathscr{U}_i$ for each i and

$$\lim_{i\to\infty} \frac{\log |\mathscr{U}_i|}{i} = 0.$$

References

[1] R. D. Anderson, One-dimensional continuous curves and a homogeneity theorem, Ann. of Math. (2), **68** (1958), 1–16.

[2] T. Banakh and M. Tuncali, Controlled Hahn-Mazurkiewicz Theorem and some new dimension functions of Peano continua, Topology and its Applications, **154** (2007), 1286–1297.

[3] M. Bestvina, Characterizing k-dimensional universal Menger compacta, Mem. Amer. Math. Soc., **71** (1988), no. 380.

[4] W. Hurewicz and H. Wallman, Dimension Theory, Princeton Univ. Press, Princeton, New Jersey, 1948.

[5] H. Kato and M. Matsumoto, Characterizations of topological dimension by use of normal sequences of finite open covers and Pontrjagin-Schnirelmann theorem, J. Math. Soc. Japan, to appear.

[6] K. Nagami, Dimension Theory, Academic Press, New York-London, 1970.

[7] J. Nagata, Modern Dimension Theory, Heldermann Verlag, 1983.

[8] J. Nagata, Modern General Topology, North-Holland, Amsterdam, 1985.

[9] J. Nagata, Open problems left in my wake of research, Topology and its Applications, **146/147** (2005), 5–13.

[10] L. Pontrjagin and L. Schnirelmann, Sur une propriété métrique de la dimension, Ann. of Math. (2), **33** (1932), 156–162.

Hisao Kato
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571 Japan
E-mail: hisakato@sakura.cc.tsukuba.ac.jp