J. Math. Soc. Japan
Vol. 62, No. 4 (2010) pp. 1219–1255
doi: 10.2969/jmsj/06241219

The reproducing formula with fractional orders on the parabolic Bloch space

Dedicated to Professor Yoshihiro Mizuta on the occasion of his sixtieth birthday

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(Received Mar. 3, 2009)
(Revised Sep. 9, 2009)

Abstract. In this paper, we study the reproducing formula with fractional orders on the parabolic Bloch space. As an application of the reproducing formula, we characterize the dual and pre-dual spaces of parabolic Bergman spaces. Furthermore, we generalize the integral pairing, which gives the duality between the parabolic Bloch space and the parabolic Bergman space.

1. Introduction.

Let H be the upper half-space of the $(n+1)$-dimensional Euclidean space $\boldsymbol{R}^{n+1}(n \geq 1)$, that is, $H = \{(x,t) \in \boldsymbol{R}^{n+1}; x \in \boldsymbol{R}^n, t > 0\}$. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} = \partial_t + (-\Delta_x)^\alpha,$$

where $\partial_t = \partial/\partial t$ and Δ_x is the Laplacian with respect to x. A continuous function u on H is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions (for details, see section 2). The parabolic Bloch space $\mathscr{B}_\alpha$ is the set of all C^1 class and $L^{(\alpha)}$-harmonic functions u on H which satisfy

$$\|u\|_{\mathscr{B}_\alpha} := \sup_{(x,t)\in H} \left\{ t^{\frac{1}{2\alpha}} |\nabla_x u(x,t)| + t|\partial_t u(x,t)| \right\} < \infty,$$

where ∇_x denotes the gradient operator with respect to x. Moreover, we denote by $\widetilde{\mathscr{B}}_\alpha$ the set of functions $u \in \mathscr{B}_\alpha$ such that $u(0,1) = 0$. We remark that $\widetilde{\mathscr{B}}_\alpha$ is a Banach space with the norm $\|\cdot\|_{\mathscr{B}_\alpha}$, and when $\alpha = 1/2$, $\widetilde{\mathscr{B}}_{1/2}$ coincides with the harmonic Bloch space of Ramey and Yi [**5**].

2000 *Mathematics Subject Classification.* Primary 35K05; Secondary 26A33, 26D10.

Key Words and Phrases. parabolic operator of fractional order, Bloch space, reproducing formula.

Our aim of this paper is the study the reproducing formula on the parabolic Bloch space. Ramey and Yi [**5**] studied basic properties of the harmonic Bloch space on H. In [**4**], Nishio, Shimomura, and Suzuki introduce the parabolic Bloch space and also study basic properties of the space. In this paper, we introduce the fractional derivatives of parabolic Bloch functions, and establish the reproducing formula with fractional orders on the parabolic Bloch space. As an application of the reproducing formula, we characterize the dual and pre-dual spaces of parabolic Bergman spaces.

To state our main results, we describe some notations. For a real number κ, a fractional differential operator is given by $\mathscr{D}_t^\kappa = (-\partial_t)^\kappa$, and $W^{(\alpha)}$ is the fundamental solution of the parabolic operator $L^{(\alpha)}$ (for the explicit definitions of $\mathscr{D}_t^\kappa$ and $W^{(\alpha)}$, see Section 2). First, we present the reproducing formula with fractional orders on parabolic Bergman spaces $\boldsymbol{b}_\alpha^p(\lambda)$ (for the explicit definitions, see also Section 2), which was studied in [**1**]. We note that $\boldsymbol{b}_\alpha^p(\lambda) = \{0\}$ whenever $\lambda \le -1$.

Theorem A (Theorem 5.2 of [**1**]). *Let $0 < \alpha \le 1$, $1 \le p < \infty$, and $\lambda > -1$. And let $\nu > -\frac{\lambda+1}{p}$ and $\kappa > \frac{\lambda+1}{p}$ be real numbers. Then, the reproducing formula*

$$u(x,t) = C_{\nu+\kappa} \int_H \mathscr{D}_t^\nu u(y,s) \mathscr{D}_t^\kappa W^{(\alpha)}(x-y, t+s) s^{\nu+\kappa-1} dV(y,s)$$

holds for all $u \in \boldsymbol{b}_\alpha^p(\lambda)$ and $(x,t) \in H$, where $C_\kappa = 2^\kappa/\Gamma(\kappa)$, $\Gamma(\cdot)$ is the gamma function, and dV is the Lebesgue volume measure on H.

We present our main result in this paper. A function ω_α^κ on $H \times H$ is defined by

$$\omega_\alpha^\kappa(x,t;y,s) = \mathscr{D}_t^\kappa W^{(\alpha)}(x-y, t+s) - \mathscr{D}_t^\kappa W^{(\alpha)}(-y, 1+s) \tag{1.1}$$

for all $(x,t), (y,s) \in H$. Theorem 1 gives the reproducing formula with fractional orders on the parabolic Bloch space.

Theorem 1. *Let $0 < \alpha \le 1$. And let $\nu \ge 0$ and $\kappa > 0$ be real numbers. Then, the reproducing formula*

$$u(x,t) = C_{\nu+\kappa} \int_H \mathscr{D}_t^\nu u(y,s) \omega_\alpha^\kappa(x,t;y,s) s^{\nu+\kappa-1} dV(y,s) \tag{1.2}$$

holds for all $u \in \widetilde{\mathscr{B}}_\alpha$ and $(x,t) \in H$, where C_κ is the constant defined in Theorem A.

As an application of Theorem 1, we show that the dual space of $\boldsymbol{b}_\alpha^1(\lambda)$ is isomorphic to the parabolic Bloch space.

THEOREM 2. *Let $0<\alpha\le 1$ and $\lambda>-1$. Then, $(\boldsymbol{b}_\alpha^1(\lambda))^*\cong\widetilde{\mathscr{B}}_\alpha$ under the pairing*

$$\langle u,v\rangle_\lambda = C_{\lambda+2}\int_H u(y,s)\mathscr{D}_t v(y,s)s^{\lambda+1}dV(y,s), \quad u\in \boldsymbol{b}_\alpha^1(\lambda),\ v\in\widetilde{\mathscr{B}}_\alpha.$$

We also show that a pre-dual space of $\boldsymbol{b}_\alpha^1(\lambda)$ is isomorphic to the subspace of the parabolic Bloch space. The parabolic little Bloch space $\mathscr{B}_{\alpha,0}$ is the set of all functions $u\in\mathscr{B}_\alpha$ such that

$$\lim_{(x,t)\to\partial H\cup\{\infty\}}\left\{t^{\frac{1}{2\alpha}}|\nabla_x u(x,t)|+t|\partial_t u(x,t)|\right\}=0.$$

Moreover, we denote by $\widetilde{\mathscr{B}}_{\alpha,0}$ the set of all functions $u\in\mathscr{B}_{\alpha,0}$ such that $u(0,1)=0$.

THEOREM 3. *Let $0<\alpha\le 1$ and $\lambda>-1$. Then, $\boldsymbol{b}_\alpha^1(\lambda)\cong(\widetilde{\mathscr{B}}_{\alpha,0})^*$ under the pairing*

$$\langle u,v\rangle_\lambda = C_{\lambda+2}\int_H u(y,s)\mathscr{D}_t v(y,s)s^{\lambda+1}dV(y,s), \quad u\in \boldsymbol{b}_\alpha^1(\lambda),\ v\in\widetilde{\mathscr{B}}_{\alpha,0}.$$

In the following theorem, we show that the integral pairing $\langle\cdot,\cdot\rangle_\lambda$ can be generalized.

THEOREM 4. *Let $0<\alpha\le 1$ and $\lambda>-1$. If $\nu>-(\lambda+1)$ and $\kappa>0$, then*

$$\Theta_\lambda^{\nu,\kappa}(u,v)=\langle u,v\rangle_\lambda, \quad u\in\boldsymbol{b}_\alpha^1(\lambda),\ v\in\widetilde{\mathscr{B}}_\alpha,$$

where

$$\Theta_\lambda^{\nu,\kappa}(u,v)=C_{\nu+\kappa+\lambda+1}\int_H \mathscr{D}_t^\nu u(x,t)\mathscr{D}_t^\kappa v(x,t)t^{\nu+\kappa+\lambda}dV(x,t).$$

This paper is constructed as follows. In Section 2, we present preliminary results. In Section 3, we study properties of fractional derivatives of parabolic Bloch functions. In Section 4, we show the reproducing formula on the parabolic Bloch space of Theorem 1. As an application of Theorem 1, we characterize the dual and pre-dual spaces of parabolic Bergman spaces $\boldsymbol{b}_\alpha^1(\lambda)$ in Section 5. In Section 6, we give a generalization of the integral pairing defined in Theorems 2 and 3. Furthermore, we discuss Banach space isomorphisms on parabolic Bergman spaces.

Throughout this paper, we will denote by C a positive constant whose value may not necessarily be the same at each occurrence.

2. Preliminaries.

First, we recall the definition of $L^{(\alpha)}$-harmonic functions. We describe the operator $(-\Delta_x)^\alpha$. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. Let $C_c^\infty(H)$ be the set of all infinitely differentiable functions on H with compact support. For $0 < \alpha < 1$, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(-\Delta_x)^\alpha \psi(x,t) = -c_{n,\alpha} \lim_{\delta \to 0^+} \int_{|y-x|>\delta} (\psi(y,t) - \psi(x,t))|y-x|^{-n-2\alpha} dy \tag{2.1}$$

for all $\psi \in C_c^\infty(H)$ and $(x,t) \in H$, where $c_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$. A continuous function u on H is said to be $L^{(\alpha)}$-harmonic on H if u satisfies the following condition: for every $\psi \in C_c^\infty(H)$,

$$\int_H |u \cdot \widetilde{L}^{(\alpha)} \psi| dV < \infty \quad \text{and} \quad \int_H u \cdot \widetilde{L}^{(\alpha)} \psi dV = 0, \tag{2.2}$$

where $\widetilde{L}^{(\alpha)} = -\partial_t + (-\Delta_x)^\alpha$ is the adjoint operator of $L^{(\alpha)}$. By (2.1) and the compactness of $\mathrm{supp}(\psi)$ (the support of ψ), there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ such that $\mathrm{supp}(\widetilde{L}^{(\alpha)}\psi) \subset S = \boldsymbol{R}^n \times [t_1, t_2]$ and $|\widetilde{L}^{(\alpha)}\psi(x,t)| \le C(1+|x|)^{-n-2\alpha}$ for all $(x,t) \in S$. Thus, the integrability condition $\int_H |u \cdot \widetilde{L}^{(\alpha)}\psi| dV < \infty$ is equivalent to the following: for any $0 < t_1 < t_2 < \infty$,

$$\int_{t_1}^{t_2} \int_{\boldsymbol{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dV(x,t) < \infty. \tag{2.3}$$

We introduce the fundamental solution of $L^{(\alpha)}$. For $x \in \boldsymbol{R}^n$, the fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ is defined by

$$W^{(\alpha)}(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\boldsymbol{R}^n} \exp(-t|\xi|^{2\alpha} + \sqrt{-1}\, x \cdot \xi)\, d\xi & t > 0 \\ 0 & t \le 0, \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\boldsymbol{R}^n$. It is known that $W^{(\alpha)}$ is $L^{(\alpha)}$-

harmonic on H and $W^{(\alpha)} \in C^\infty(H)$, where $C^\infty(H)$ is the set of all infinitely differentiable functions on H.

Next, we present definitions of fractional integral and differential operators. Let $C(\boldsymbol{R}_+)$ be the set of all continuous functions on $\boldsymbol{R}_+ = (0, \infty)$. For a positive real number κ, let $\mathscr{F}\mathscr{C}^{-\kappa}$ be the set of all functions $\varphi \in C(\boldsymbol{R}_+)$ such that there exists a constant $\varepsilon > 0$ with $\varphi(t) = O(t^{-\kappa-\varepsilon})$ as $t \to \infty$. We remark that $\mathscr{F}\mathscr{C}^{-\nu} \subset \mathscr{F}\mathscr{C}^{-\kappa}$ if $0 < \kappa \le \nu$. For $\varphi \in \mathscr{F}\mathscr{C}^{-\kappa}$, we can define the fractional integral of φ with order κ by

$$\mathscr{D}_t^{-\kappa}\varphi(t) = \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1}\varphi(t+\tau)d\tau, \quad t \in \boldsymbol{R}_+. \tag{2.4}$$

Furthermore, let $\mathscr{F}\mathscr{C}^{\kappa}$ be the set of all functions $\varphi \in C(\boldsymbol{R}_+)$ such that $d_t^{\lceil\kappa\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)}$, where $d_t = d/dt$ and $\lceil\kappa\rceil$ is the smallest integer greater than or equal to κ. In particular, we will write $\mathscr{F}\mathscr{C}^0 = C(\boldsymbol{R}_+)$. For $\varphi \in \mathscr{F}\mathscr{C}^{\kappa}$, we can also define the fractional derivative of φ with order κ by

$$\mathscr{D}_t^{\kappa}\varphi(t) = \mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)}(-d_t)^{\lceil\kappa\rceil}\varphi(t), \quad t \in \boldsymbol{R}_+. \tag{2.5}$$

Also, we define $\mathscr{D}_t^0\varphi = \varphi$. We may often call both (2.4) and (2.5) *the fractional derivative of φ with order κ*. Moreover, we call $\mathscr{D}_t^\kappa$ *the fractional differential operator with order κ*. Here, we give an example of the fractional derivative of an elementary function. Let $\kappa > 0$ be a real number. And let ν be a real number such that $-\kappa < \nu$. Then, we have

$$\mathscr{D}_t^{\nu} t^{-\kappa} = \frac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)} t^{-\kappa-\nu}. \tag{2.6}$$

The following proposition shows that fractional differential operators hold the commutative and exponential laws under some conditions.

Proposition 2.1 (Proposition 2.1 of [**1**]). *Let $\kappa, \nu > 0$ be real numbers. Then, the following statements hold.*

(1) *If $\varphi \in \mathscr{F}\mathscr{C}^{-\kappa}$, then $\mathscr{D}_t^{-\kappa}\varphi \in C(\boldsymbol{R}_+)$.*
(2) *If $\varphi \in \mathscr{F}\mathscr{C}^{-\kappa-\nu}$, then $\mathscr{D}_t^{-\kappa}\mathscr{D}_t^{-\nu}\varphi = \mathscr{D}_t^{-\kappa-\nu}\varphi$.*
(3) *If $d_t^k\varphi \in \mathscr{F}\mathscr{C}^{-\nu}$ for all integers $0 \le k \le \lceil\kappa\rceil - 1$ and $d_t^{\lceil\kappa\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)-\nu}$, then $\mathscr{D}_t^{\kappa}\mathscr{D}_t^{-\nu}\varphi = \mathscr{D}_t^{-\nu}\mathscr{D}_t^{\kappa}\varphi = \mathscr{D}_t^{\kappa-\nu}\varphi$.*
(4) *If $d_t^{k+\lceil\nu\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\nu\rceil-\nu)}$ for all integers $0 \le k \le \lceil\kappa\rceil - 1$, $d_t^{\lceil\kappa\rceil+\ell}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)}$ for all integers $0 \le \ell \le \lceil\nu\rceil - 1$, and $d_t^{\lceil\kappa\rceil+\lceil\nu\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-(\lceil\kappa\rceil-\kappa)-(\lceil\nu\rceil-\nu)}$, then $\mathscr{D}_t^{\kappa}\mathscr{D}_t^{\nu}\varphi = \mathscr{D}_t^{\kappa+\nu}\varphi$.*

We also need the following result.

Proposition 2.2. *Let $\kappa > 0$ be a real number. If $d_t^{\lceil\kappa\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-\lceil\kappa\rceil}$ and $\lim_{t\to\infty} d_t^k\varphi(t) = 0$ for all integers $0 \leq k \leq \lceil\kappa\rceil - 1$, then $\mathscr{D}_t^{-\kappa}\mathscr{D}_t^{\kappa}\varphi = \varphi$.*

Proof. First, we show the case $\kappa \in \boldsymbol{N}$ by the induction. If $\kappa = 1$, then by the condition $\lim_{t\to\infty}\varphi(t) = 0$, we have

$$\mathscr{D}_t^{-1}\mathscr{D}_t\varphi(t) = -\int_0^\infty d_t\varphi(t+\tau)d\tau = -\lim_{\tau\to\infty}\varphi(t+\tau) + \varphi(t) = \varphi(t) \tag{2.7}$$

for all $t \in \boldsymbol{R}_+$. Next, assume that the statement holds for some $\kappa \in \boldsymbol{N}$. Suppose $d_t^{\kappa+1}\varphi \in \mathscr{F}\mathscr{C}^{-(\kappa+1)}$ and $\lim_{t\to\infty} d_t^k\varphi(t) = 0$ for all integers $0 \leq k \leq \kappa$. Since $\lim_{t\to\infty} d_t^k\varphi(t) = 0$ for all integers $0 \leq k \leq \kappa - 1$, by the assumption of the induction κ, it is sufficient to show $d_t^\kappa\varphi \in \mathscr{F}\mathscr{C}^{-\kappa}$ and $\mathscr{D}_t^{-(\kappa+1)}\mathscr{D}_t^{\kappa+1}\varphi = \mathscr{D}_t^{-\kappa}\mathscr{D}_t^{\kappa}\varphi$. Since $d_t^{\kappa+1}\varphi \in \mathscr{F}\mathscr{C}^{-(\kappa+1)} \subset \mathscr{F}\mathscr{C}^{-1}$ and $\lim_{t\to\infty} d_t^\kappa\varphi(t) = 0$, (2.7) implies that

$$\mathscr{D}_t^{-1}\mathscr{D}_t^{\kappa+1}\varphi = \mathscr{D}_t^{-1}\mathscr{D}_t\mathscr{D}_t^{\kappa}\varphi = \mathscr{D}_t^{\kappa}\varphi. \tag{2.8}$$

Also by (2.6) and the condition $d_t^{\kappa+1}\varphi \in \mathscr{F}\mathscr{C}^{-(\kappa+1)}$, there exists $\varepsilon > 0$ such that

$$\begin{aligned}|d_t^\kappa\varphi(t)| = |\mathscr{D}_t^\kappa\varphi(t)| = |\mathscr{D}_t^{-1}\mathscr{D}_t^{\kappa+1}\varphi(t)| &\leq C\int_0^\infty |\mathscr{D}_t^{\kappa+1}\varphi(t+\tau)|d\tau \\ &\leq C\int_0^\infty (t+\tau)^{-(\kappa+1)-\varepsilon}d\tau = Ct^{-\kappa-\varepsilon}\end{aligned}$$

for t sufficiently large. Hence we have $d_t^\kappa\varphi \in \mathscr{F}\mathscr{C}^{-\kappa}$. Moreover, (2) of Proposition 2.1 and (2.8) imply that

$$\mathscr{D}_t^{-(\kappa+1)}\mathscr{D}_t^{\kappa+1}\varphi = \mathscr{D}_t^{-\kappa}\mathscr{D}_t^{-1}\mathscr{D}_t^{\kappa+1}\varphi = \mathscr{D}_t^{-\kappa}\mathscr{D}_t^{\kappa}\varphi.$$

Thus we obtain the statement in case $\kappa \in \boldsymbol{N}$.

Let $\kappa \in \boldsymbol{R}_+ \setminus \boldsymbol{N}$. Suppose that $d_t^{\lceil\kappa\rceil}\varphi \in \mathscr{F}\mathscr{C}^{-\lceil\kappa\rceil}$ and $\lim_{t\to\infty} d_t^k\varphi(t) = 0$ for all integers $0 \leq k \leq \lceil\kappa\rceil - 1$. Then (2) of Proposition 2.1 implies that

$$\mathscr{D}_t^{-\kappa}\mathscr{D}_t^{\kappa}\varphi = \mathscr{D}_t^{-\kappa}\mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)}\mathscr{D}_t^{\lceil\kappa\rceil}\varphi = \mathscr{D}_t^{-\lceil\kappa\rceil}\mathscr{D}_t^{\lceil\kappa\rceil}\varphi.$$

Since $\lceil\kappa\rceil$ is a positive integer, we get the desired result. This completes the proof. □

The following proposition gives basic properties of fractional derivatives of the fundamental solution $W^{(\alpha)}$. Let $\boldsymbol{N}_0 = \boldsymbol{N} \cup \{0\}$. For a multi-index $\beta = (\beta_1, \dots, \beta_n) \in \boldsymbol{N}_0^n$, let $\partial_x^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}$, where $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$.

PROPOSITION 2.3 (Theorems 1 and 3.1 of [**1**]). *Let $0 < \alpha \le 1$, $\beta \in \boldsymbol{N}_0^n$, and $\kappa > -n/2\alpha$ be a real number. Then the following statements hold.*

(1) *The derivative $\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}$ is well-defined. Moreover, there exists a constant $C > 0$ such that*

$$\left|\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t)\right| \le C(t + |x|^{2\alpha})^{-\frac{n+|\beta|}{2\alpha} - \kappa}$$

for all $(x,t) \in H$.

(2) *If $0 < q < \infty$ and $\theta > -1$ satisfy the condition $n/2\alpha + \theta + 1 - ((n + |\beta|)/2\alpha + \kappa)q < 0$, then there exists a constant $C > 0$ such that*

$$\int_H \left|\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x - y, t + s)\right|^q s^\theta dV(y,s) \le C t^{\frac{n}{2\alpha} + \theta + 1 - (\frac{n+|\beta|}{2\alpha} + \kappa)q}$$

for all $(x,t) \in H$.

(3) *Let ν be a real number such that $\kappa + \nu > -n/2\alpha$. Then,*

$$\mathscr{D}_t^\nu \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x,t) = \partial_x^\beta \mathscr{D}_t^{\kappa+\nu} W^{(\alpha)}(x,t)$$

for all $(x,t) \in H$.

(4) *$\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic on H.*

Here, we present the following lemma, which is frequently used in later argument.

LEMMA 2.4 (Lemma 3.3 of [**3**]). *Let θ and c be real numbers such that $\theta > -1$ and $n/2\alpha + \theta + 1 - c < 0$. Then, there exists a constant $C > 0$ such that*

$$\int_H \frac{s^\theta}{(t + s + |x - y|^{2\alpha})^c} dV(y,s) = C t^{\frac{n}{2\alpha} + \theta + 1 - c}$$

for all $(x,t) \in H$.

Finally, we present the definition of the parabolic Bergman space $\boldsymbol{b}_\alpha^p(\lambda)$. And we also present properties about parabolic Bergman functions. For $1 \le p < \infty$ and $\lambda > -1$, $L^p(\lambda)$ is the set of all Lebesgue measurable functions f on H which satisfy

$$\|f\|_{L^p(\lambda)} = \left(\int_H |f(x,t)|^p t^\lambda dV(x,t) \right)^{\frac{1}{p}} < \infty.$$

The parabolic Bergman space $\boldsymbol{b}_\alpha^p(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions u on H which belong to $L^p(\lambda)$. Here we denote $L^p = L^p(0)$ and $\boldsymbol{b}_\alpha^p = \boldsymbol{b}_\alpha^p(0)$. As a remark, $\boldsymbol{b}_\alpha^p(\lambda) = \{0\}$ whenever $\lambda \leq -1$ (see Proposition 4.3 of [**2**]). And let $\boldsymbol{b}_\alpha^\infty$ be the set of all $L^{(\alpha)}$-harmonic functions u on H which satisfy

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{(x,t)\in H} |f(x,t)| < \infty.$$

We remark that $\boldsymbol{b}_\alpha^\infty \subset \mathscr{B}_\alpha$ by Theorem 5.4 of [**4**]. In order to prove our reproducing formula, we need the following lemma.

LEMMA 2.5 (Lemma 3.1 of [**6**] and Theorem 7.4 of [**4**]). *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. If $u \in \boldsymbol{b}_\alpha^p(\lambda)$, then u satisfies the Huygens property, that is,*

$$u(x,t) = \int_{\boldsymbol{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s) dy$$

holds for all $x \in \boldsymbol{R}^n$ and $0 < s < t < \infty$. If $u \in \mathscr{B}_\alpha$, then u also satisfies the Huygens property.

Proposition 2.6 gives the basic properties of fractional derivatives of parabolic Bergman functions.

PROPOSITION 2.6 (Proposition 4.1 of [**1**]). *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, $\beta \in \boldsymbol{N}_0^n$, and $\kappa > -(n/2\alpha + \lambda + 1)1/p$ be a real number. If $u \in \boldsymbol{b}_\alpha^p(\lambda)$, then the following statements hold.*

(1) *The derivatives $\partial_x^\beta \mathscr{D}_t^\kappa u$ and $\mathscr{D}_t^\kappa \partial_x^\beta u$ are well-defined, and*

$$\partial_x^\beta \mathscr{D}_t^\kappa u(x,t) = \mathscr{D}_t^\kappa \partial_x^\beta u(x,t)$$

for all $(x,t) \in H$. Moreover, there exists a constant $C > 0$ such that

$$\left|\partial_x^\beta \mathscr{D}_t^\kappa u(x,t)\right| \leq C t^{-\frac{|\beta|}{2\alpha} - \kappa - (\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}} \|u\|_{L^p(\lambda)}$$

for all $(x,t) \in H$.

(2) *Let ν be a real number such that $\kappa + \nu > -(n/2\alpha + \lambda + 1)1/p$. Then,*

$$\mathscr{D}_t^{\nu}\partial_x^{\beta}\mathscr{D}_t^{\kappa}u(x,t) = \partial_x^{\beta}\mathscr{D}_t^{\kappa+\nu}u(x,t)$$

for all $(x,t) \in H$.

(3) $\partial_x^{\beta}\mathscr{D}_t^{\kappa}u$ *is* $L^{(\alpha)}$*-harmonic on* H.

In order to characterize the dual and pre-dual spaces of $\boldsymbol{b}_{\alpha}^{1}(\lambda)$, we need the following proposition.

Proposition 2.7 (Corollary 3.2 of [**2**]). *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda > -1$. *Then, the following statements hold.*

(1) *For* $\kappa > (\lambda+1)/p$, *the operator* P_{α}^{κ} *defined by*

$$P_{\alpha}^{\kappa}f(x,t) = \int_H f(y,s)\mathscr{D}_t^{\kappa}W^{(\alpha)}(x-y,t+s)s^{\kappa-1}dV(y,s)$$

is a bounded projection from $L^p(\lambda)$ *onto* $\boldsymbol{b}_{\alpha}^{p}(\lambda)$.

(2) *Let* $1 < p < \infty$ *and* q *be the exponent conjugate to* p. *Then,* $(\boldsymbol{b}_{\alpha}^{p}(\lambda))^* \cong \boldsymbol{b}_{\alpha}^{q}(\lambda)$ *under the pairing*

$$\langle u, v\rangle_{\lambda} = \int_H u(x,t)v(x,t)t^{\lambda}dV(x,t), \quad u \in \boldsymbol{b}_{\alpha}^{p}(\lambda),\ v \in \boldsymbol{b}_{\alpha}^{q}(\lambda).$$

(3) *For a real number* $\nu > -(\lambda+1)/p$, *there exists a constant* $C = C(n,p,\alpha,\lambda,\nu) > 0$ *such that*

$$C^{-1}\|u\|_{L^p(\lambda)} \le \left\|t^{\nu}\mathscr{D}_t^{\nu}u\right\|_{L^p(\lambda)} \le C\|u\|_{L^p(\lambda)}$$

for all $u \in \boldsymbol{b}_{\alpha}^{p}(\lambda)$.

3. Fractional derivatives of parabolic Bloch functions.

In this section, we study fractional derivative of parabolic Bloch functions. First, we generalize the definition of the function ω_{α}^{κ}, and give basic properties. For a multi-index $\beta \in \boldsymbol{N}_0^n$ and a real number $\kappa > -n/2\alpha$, a function $\omega_{\alpha}^{\beta,\kappa}$ on $H \times H$ is defined by

$$\omega_{\alpha}^{\beta,\kappa}(x,t;y,s) = \partial_x^{\beta}\mathscr{D}_t^{\kappa}W^{(\alpha)}(x-y,t+s) - \partial_x^{\beta}\mathscr{D}_t^{\kappa}W^{(\alpha)}(-y,1+s)$$

for all $(x,t),(y,s) \in H$. Here we remark that $\omega_{\alpha}^{\kappa} = \omega_{\alpha}^{0,\kappa}$.

Proposition 3.1. *Let* $0 < \alpha \le 1$, $\beta \in \boldsymbol{N}_0^n$, *and* $\kappa > -n/2\alpha$ *be a real* *number.*

(1) *For any compact set $K \subset \boldsymbol{R}^n$ and $T > 1$, there exist constants $C_1, C_2 > 0$ such that*

$$\left|\omega_\alpha^{\beta,\kappa}(x,t;y,s)\right| \le \frac{C_1|x|}{(1+s+|y|^{2\alpha})^{\frac{n+|\beta|+1}{2\alpha}+\kappa}} + \frac{C_2|t-1|}{(1+s+|y|^{2\alpha})^{\frac{n+|\beta|}{2\alpha}+\kappa+1}}$$

for all $(x,t) \in K \times [T^{-1}, T]$ and $(y,s) \in H$.

(2) *Let $(x,t) \in H$ be fixed. Then, there exists a constant $C > 0$ such that*

$$\left|\omega_\alpha^{\beta,\kappa}(x,t;y,s)\right| \le C(1+s+|y|^{2\alpha})^{-\frac{n+|\beta|}{2\alpha}-\kappa-\sigma}$$

for all $(y,s) \in H$, where $\sigma = \min\{1, 1/2\alpha\}$.

(3) *Let $(x,t) \in H$ be fixed. Then, $\omega_\alpha^\kappa(x,t;\cdot,\cdot) \in \widetilde{\mathscr{B}}_{\alpha,0}$.*

(4) *Moreover, let $|\beta|/2\alpha + \kappa > 0$. Then, there exists a constant $C > 0$ such that*

$$\int_H \left|\omega_\alpha^{\beta,\kappa}(x,t;y,s)\right| s^{\frac{|\beta|}{2\alpha}+\kappa-1} dV(y,s) \le C(1+\log(1+|x|)+|\log t|)$$

for all $(x,t) \in H$.

Proof.

(1) The chain rule implies that

$$\begin{aligned}
\left|\omega_\alpha^{\beta,\kappa}(x,t;y,s)\right| &\le \left|\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x-y,t+s) - \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(-y,t+s)\right| \\
&\quad + \left|\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(-y,t+s) - \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(-y,1+s)\right| \\
&= \left|\int_0^1 x\cdot\nabla_x \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(rx-y,t+s)\,dr\right| \\
&\quad + \left|\int_1^t \partial_x^\beta \mathscr{D}_t^{\kappa+1} W^{(\alpha)}(-y,\tau+s)\,d\tau\right|. \qquad (3.1)
\end{aligned}$$

Since $x \in K$, there exists a constant $M > 0$ such that $|rx| \le M$ for all $0 < r < 1$. If $|y| \ge 2M$, then we have

$$|y| \le |rx-y| + |rx| \le |rx-y| + M \le |rx-y| + \frac{1}{2}|y|.$$

Therefore, we have $|y| \le 2|rx-y|$. If $|y| < 2M$, then since $T^{-1} \le t \le T$, we have

$$|y| < 2MTT^{-1} \le 2MTt \le 2MT(t + |rx - y|).$$

Therefore, we have $t + |y| \le (2MT + 1)(t + |rx - y|)$. Hence by (1) of Proposition 2.3, there exist constants $C_1, C_2 > 0$ such that the right-hand side of (3.1) is less than or equal to

$$\begin{aligned}
&C_1|x| \int_0^1 (t + s + |rx - y|^{2\alpha})^{-\frac{n+|\beta|+1}{2\alpha} - \kappa} dr + C_2 \left| \int_1^t (\tau + s + |y|^{2\alpha})^{-\frac{n+|\beta|}{2\alpha} - \kappa - 1} d\tau \right| \\
&\quad \le \frac{C_1|x|}{(1 + s + |y|^{2\alpha})^{\frac{n+|\beta|+1}{2\alpha} + \kappa}} \\
&\qquad + \frac{C_2|t-1|}{(t + s + |y|^{2\alpha})^{\frac{n+|\beta|}{2\alpha} + \kappa}(1 + s + |y|^{2\alpha})} \left\{ 1 + \left(\frac{t + s + |y|^{2\alpha}}{1 + s + |y|^{2\alpha}} \right)^{\frac{n+|\beta|}{2\alpha} + \kappa - 1} \right\}.
\end{aligned}$$

Since $T^{-1} \le t \le T$, we get the desired result.

(2) Let $(x, t) \in H$ be fixed. Then, we can get the desired result by (1), directly.

(3) Let $(x, t) \in H$ be fixed. Then by (4) of Proposition 2.3, $\omega_\alpha^\kappa(x, t; \cdot, \cdot)$ is $L^{(\alpha)}$-harmonic on H. And by (2) of Proposition 3.1, there exists a constant $C > 0$ such that

$$\begin{aligned}
\left| \partial_{y_j} \omega_\alpha^\kappa(x, t; y, s) \right| &\le C(1 + s + |y|^{2\alpha})^{-\frac{n+1}{2\alpha} - \kappa - \sigma}, \\
\left| \mathscr{D}_s \omega_\alpha^\kappa(x, t; y, s) \right| &\le C(1 + s + |y|^{2\alpha})^{-\frac{n}{2\alpha} - \kappa - 1 - \sigma}
\end{aligned}$$

for all $(y, s) \in H$ and $1 \le j \le n$, where $\sigma = \min\{1, 1/2\alpha\}$. Hence we have

$$s^{\frac{1}{2\alpha}} \left| \nabla_y \omega_\alpha^\kappa(x, t; y, s) \right| \to 0, \quad s \left| \mathscr{D}_s \omega_\alpha^\kappa(x, t; y, s) \right| \to 0$$

as $(y, s) \to \partial H \cup \{\infty\}$.

(4) Let $\kappa > 0$ be a real number. And put $\rho = ((1 + |x|)/(1 + \log(1 + |x|)))^{2\alpha}$. Then,

$$\begin{aligned}
&\int_H \left| \omega_\alpha^{\beta,\kappa}(x, t; y, s) \right| s^{\frac{|\beta|}{2\alpha} + \kappa - 1} dV(y, s) \\
&\quad \le \int_H \left| \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x - y, t + s) - \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x - y, \rho + s) \right| s^{\frac{|\beta|}{2\alpha} + \kappa - 1} dV(y, s) \\
&\qquad + \int_H \left| \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(x - y, \rho + s) - \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(-y, \rho + s) \right| s^{\frac{|\beta|}{2\alpha} + \kappa - 1} dV(y, s)
\end{aligned}$$

$$+\int_H \left|\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(-y,\rho+s)-\partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(-y,1+s)\right| s^{\frac{|\beta|}{2\alpha}+\kappa-1} dV(y,s). \tag{3.2}$$

By (2) of Proposition 2.3, the first term of the right-hand side in (3.2) is less than or equal to

$$\begin{aligned}&\left|\int_\rho^t \int_H \left|\partial_x^\beta \mathscr{D}_t^{\kappa+1} W^{(\alpha)}(x-y,\tau+s)\right| s^{\frac{|\beta|}{2\alpha}+\kappa-1} dV(y,s)d\tau\right| \\ &\quad \le C\left|\int_t^\rho \tau^{-1}d\tau\right| \le C(|\log t|+\log\rho).\end{aligned}$$

By the chain rule and (2) of Proposition 2.3, the second term is less than or equal to

$$\int_0^1 |x| \int_H \left|\nabla_x \partial_x^\beta \mathscr{D}_t^\kappa W^{(\alpha)}(rx-y,\rho+s)\right| s^{\frac{|\beta|}{2\alpha}+\kappa-1} dV(y,s)dr \le C|x|\rho^{-\frac{1}{2\alpha}}.$$

Also, by (2) of Proposition 2.3, the third term is less than or equal to

$$\int_1^\rho \int_H \left|\partial_x^\beta \mathscr{D}_t^{\kappa+1} W^{(\alpha)}(-y,\tau+s)\right| s^{\frac{|\beta|}{2\alpha}+\kappa-1} dV(y,s)d\tau \le C\log\rho.$$

Thus by the definition ρ, we complete the proof. □

Next, we give basic properties of fractional derivatives of the parabolic Bloch functions. We present properties of ordinary derivatives of parabolic Bloch functions. Let $(\beta,k)\in(\boldsymbol{N}_0^n\times\boldsymbol{N}_0)\setminus\{(0,0)\}$. The following estimate is established in Theorem 7.3 of [**4**]: if $u\in\mathscr{B}_\alpha$, then $u\in C^\infty(H)$, $\partial_x^\beta\mathscr{D}_t^k u$ is $L^{(\alpha)}$-harmonic on H, and there exists a constant $C>0$ such that

$$|u(x,t)| \le C(1+\log(1+|x|)+|\log t|), \tag{3.3}$$

$$\left|\partial_x^\beta \mathscr{D}_t^k u(x,t)\right| \le Ct^{-\frac{|\beta|}{2\alpha}-k}\|u\|_{\mathscr{B}_\alpha} \tag{3.4}$$

for all $(x,t)\in H$. In the following proposition, we give basic properties of fractional derivatives of parabolic Bloch functions.

PROPOSITION 3.2. *Let $0<\alpha\le 1$, $\beta\in\boldsymbol{N}_0^n$, and $\kappa\ge 0$ be a real number. If $u\in\mathscr{B}_\alpha$, then the following statements hold.*

(1) *The derivatives* $\partial_x^\beta \mathscr{D}_t^\kappa u$ *and* $\mathscr{D}_t^\kappa \partial_x^\beta u$ *are well-defined, and*

$$\partial_x^\beta \mathscr{D}_t^\kappa u(x,t) = \mathscr{D}_t^\kappa \partial_x^\beta u(x,t) \tag{3.5}$$

for all $(x,t) \in H$. *Moreover, for* $(\beta,\kappa) \neq (0,0)$, *there exists a constant* $C > 0$ *such that*

$$\left|\partial_x^\beta \mathscr{D}_t^\kappa u(x,t)\right| \leq C t^{-\frac{|\beta|}{2\alpha}-\kappa} \|u\|_{\mathscr{B}_\alpha} \tag{3.6}$$

for all $(x,t) \in H$.

(2) *Let* $\nu \geq 0$ *be a real number. Then,*

$$\mathscr{D}_t^\nu \partial_x^\beta \mathscr{D}_t^\kappa u(x,t) = \partial_x^\beta \mathscr{D}_t^{\kappa+\nu} u(x,t) \tag{3.7}$$

for all $(x,t) \in H$. *If* $\nu < 0$ *is a real number such that* $\kappa + \nu \geq 0$ *and* $(\beta, \kappa+\nu) \neq (0,0)$, *then* (3.7) *also holds.*

(3) $\partial_x^\beta \mathscr{D}_t^\kappa u$ *is* $L^{(\alpha)}$*-harmonic on* H.

Proof.

(1) Let $u \in \mathscr{B}_\alpha$. Since $u \in C^\infty(H)$, we have $\partial_x^\beta \mathscr{D}_t^{\lceil\kappa\rceil} u = \mathscr{D}_t^{\lceil\kappa\rceil} \partial_x^\beta u$. And by (3.4), we have $\partial_x^\beta \mathscr{D}_t^{\lceil\kappa\rceil} u(x,\cdot) \in \mathscr{FC}^{-(\lceil\kappa\rceil-\kappa)}$ for all $x \in \boldsymbol{R}^n$. These imply that derivatives $\partial_x^\beta \mathscr{D}_t^\kappa u$ and $\mathscr{D}_t^\kappa \partial_x^\beta u$ are well-defined. Also, since $u \in C^\infty(H)$, differentiating through the integral, (3.4) implies that

$$\begin{aligned} \partial_x^\beta \mathscr{D}_t^\kappa u &= \partial_x^\beta \mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)} \mathscr{D}_t^{\lceil\kappa\rceil} u = \mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)} \partial_x^\beta \mathscr{D}_t^{\lceil\kappa\rceil} u \\ &= \mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)} \mathscr{D}_t^{\lceil\kappa\rceil} \partial_x^\beta u = \mathscr{D}_t^\kappa \partial_x^\beta u. \end{aligned}$$

We show the estimate of $\partial_x^\beta \mathscr{D}_t^\kappa u$. By (2.5), (2.6), and (3.4), there exists a constant $C > 0$ such that

$$\begin{aligned} \left|\mathscr{D}_t^\kappa \partial_x^\beta u(x,t)\right| &= \left|\mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)} \mathscr{D}_t^{\lceil\kappa\rceil} \partial_x^\beta u(x,t)\right| \\ &\leq C\big(\mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)} t^{-\frac{|\beta|}{2\alpha}-\lceil\kappa\rceil}\big) \|u\|_{\mathscr{B}_\alpha} = C t^{-\frac{|\beta|}{2\alpha}-\kappa} \|u\|_{\mathscr{B}_\alpha}. \end{aligned}$$

(2) First, we assume $\nu \geq 0$. By (3.5) and (4) of Proposition 2.1, we have

$$\mathscr{D}_t^\nu \partial_x^\beta \mathscr{D}_t^\kappa u = \mathscr{D}_t^\nu \mathscr{D}_t^\kappa \partial_x^\beta u = \mathscr{D}_t^{\kappa+\nu} \partial_x^\beta u = \partial_x^\beta \mathscr{D}_t^{\kappa+\nu} u. \tag{3.8}$$

Let $\nu < 0$ such that $\kappa + \nu \geq 0$ and $(\beta, \kappa+\nu) \neq (0,0)$. This implies that $|\beta|/2\alpha +$

$\kappa+\nu>0$. Therefore, (3.6) implies that $\partial_x^\beta \mathscr{D}_t^\kappa u(x,\cdot)\in \mathscr{F}\mathscr{C}^\nu$ for all $x\in \boldsymbol{R}^n$. By (3.8), we have

$$\mathscr{D}_t^\nu \partial_x^\beta \mathscr{D}_t^\kappa u=\mathscr{D}_t^\nu \partial_x^\beta \mathscr{D}_t^{\kappa+\nu-\nu} u=\mathscr{D}_t^\nu \mathscr{D}_t^{-\nu}\partial_x^\beta \mathscr{D}_t^{\kappa+\nu} u.$$

Thus, Proposition 2.2 implies that

$$\mathscr{D}_t^\nu \mathscr{D}_t^{-\nu}\partial_x^\beta \mathscr{D}_t^{\kappa+\nu} u=\partial_x^\beta \mathscr{D}_t^{\kappa+\nu} u.$$

(3) Let $(\beta,\kappa)\neq(0,0)$. For any $0<t_1<t_2<\infty$, it is easy to show that

$$\int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1}\int_{t_1}^{t_2}\int_{\boldsymbol{R}^n}(t+\tau)^{-\frac{|\beta|}{2\alpha}-\lceil\kappa\rceil}(1+|x|)^{-n-2\alpha}dxdtd\tau<\infty.$$

Thus by (2.6), (3.6), and (2.3), $\partial_x^\beta \mathscr{D}_t^\kappa u$ satisfies the integrability condition (2.2). Moreover, for all $\psi\in C_c^\infty(H)$, the Fubini theorem implies that

$$\begin{aligned}&\int_H \partial_x^\beta \mathscr{D}_t^\kappa u(x,t)\widetilde{L}^{(\alpha)}\psi(x,t)dV(x,t)\\&=\frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)}\int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1}\int_H \partial_x^\beta \mathscr{D}_t^{\lceil\kappa\rceil}u(x,t+\tau)\widetilde{L}^{(\alpha)}\psi(x,t)dV(x,t)d\tau=0.\end{aligned}$$

This completes the proof. □

The following estimate is used in the proof of the reproducing formula for parabolic Bloch functions.

Proposition 3.3. *Let $0<\alpha\le 1$, $\beta\in \boldsymbol{N}_0^n$, and $\kappa\ge 0$ be a real number.*

(1) *For any $T>1$, there exists a constant $C>0$ such that*

$$\begin{aligned}&\left|\partial_x^\beta \mathscr{D}_t^\kappa u(x,t+s)-\partial_x^\beta \mathscr{D}_t^\kappa u(0,1+s)\right|\\&\le C\|u\|_{\mathscr{B}_\alpha}\left\{\frac{|x|}{(1+s)^{\frac{|\beta|+1}{2\alpha}+\kappa}}+\frac{|t-1|}{(1+s)^{\frac{|\beta|}{2\alpha}+\kappa+1}}\right\}\end{aligned}\tag{3.9}$$

for all $u\in\mathscr{B}_\alpha$, $(x,t)\in \boldsymbol{R}^n\times[T^{-1},T]$, and $s\ge 0$.

(2) *Let $(x,t)\in H$ be fixed. Then there exists a constant $C>0$ such that*

$$\left|\partial_x^\beta \mathscr{D}_t^\kappa u(x,t+s)-\partial_x^\beta \mathscr{D}_t^\kappa u(0,1+s)\right|\le C(1+s)^{-\frac{|\beta|}{2\alpha}-\kappa-\sigma},$$

for all $u \in \mathscr{B}_\alpha$ *and* $s \geq 0$, *where* $\sigma = \min\{1, 1/2\alpha\}$.

PROOF.

(1) By (1) of Proposition 3.2, there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
&\left|\partial_x^\beta \mathscr{D}_t^\kappa u(x,t+s) - \partial_x^\beta \mathscr{D}_t^\kappa u(0,1+s)\right| \\
&\quad \leq \left|\partial_x^\beta \mathscr{D}_t^\kappa u(x,t+s) - \partial_x^\beta \mathscr{D}_t^\kappa u(0,t+s)\right| \\
&\qquad + \left|\partial_x^\beta \mathscr{D}_t^\kappa u(0,t+s) - \partial_x^\beta \mathscr{D}_t^\kappa u(0,1+s)\right| \\
&\quad \leq \int_0^1 |x|\left|\nabla_x \partial_x^\beta \mathscr{D}_t^\kappa u(rx,t+s)\right| dr + \left|\int_1^t \partial_x^\beta \mathscr{D}_t^{\kappa+1} u(0,\tau+s)\, d\tau\right| \\
&\quad \leq C_1 \int_0^1 |x|(t+s)^{-\frac{|\beta|+1}{2\alpha}-\kappa} dr + C_2 \left|\int_1^t (\tau+s)^{-\frac{|\beta|}{2\alpha}-\kappa-1} d\tau\right| \\
&\quad \leq C_1|x|(t+s)^{-\frac{|\beta|+1}{2\alpha}-\kappa} + C_2|t-1|(t+s)^{-\frac{|\beta|}{2\alpha}-\kappa-1}.
\end{aligned}$$

Since $T^{-1} < t < T$, we get the estimate (3.9).

(2) Let $(x,t) \in H$ be fixed. Then, we can get the desired result by (1). This completes the proof. $\square$

4. The reproducing formula on the parabolic Bloch space.

In this section, we give the reproducing formula with fractional orders on the parabolic Bloch space. First, we prepare the following lemma, which is an important tool for the proof of the reproducing formula.

LEMMA 4.1. *Let* $0 < \alpha \leq 1$, $\sigma > 0$, $0 \leq \delta < 1$, *and* $c_1, c_2 > 0$. *Then the following statements hold.*

(1) *If* $u \in \mathscr{B}_\alpha$, *then for each* $\varepsilon > 0$, *there exists a constant* $C > 0$ *such that*

$$|u(y,s)| \leq C(1 + s^\varepsilon + s^{-\varepsilon} + |y|^{2\alpha\varepsilon})$$

for all $(y,s) \in H$.

(2) *If* $\kappa \geq 0$ *and* $0 < \varepsilon < \sigma$, *then for every* $y \in \boldsymbol{R}^n$,

$$\frac{(1 + (c_1 s + \delta)^\varepsilon + (c_1 s + \delta)^{-\varepsilon} + |y|^{2\alpha\varepsilon}) s^\kappa}{(1 + c_2 s + |y|^{2\alpha})^{\frac{n}{2\alpha}+\kappa+\sigma}} \to 0$$

as $s \to \infty$.

(3) *If* $\kappa > 0$ *and* $0 < \varepsilon < \min\{\kappa, \sigma\}$, *then*

$$\int_H \frac{(1 + (c_1 s + \delta)^{\varepsilon} + (c_1 s + \delta)^{-\varepsilon} + |y|^{2\alpha\varepsilon}) s^{\kappa - 1}}{(1 + c_2 s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \kappa + \sigma}} dV(y,s) < \infty.$$

Proof.

(1) By (3.3), there exists a constant $C > 0$ such that

$$|u(y,s)| \le C(1 + \log(1 + |y|) + |\log s|)$$

for all $(y,s) \in H$. Also, for each $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\log(1 + |y|) \le C|y|^{2\alpha\varepsilon}, \quad |\log s| \le C(s^{\varepsilon} + s^{-\varepsilon}). \tag{4.1}$$

(2) Since $0 \le \delta < 1$, there exists a constant $C = C(c_1, c_2, n, \alpha, \kappa, \sigma) > 0$ such that

$$\begin{aligned}
&\frac{1 + (c_1 s + \delta)^{\varepsilon} + (c_1 s + \delta)^{-\varepsilon} + |y|^{2\alpha\varepsilon}}{(1 + c_2 s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \kappa + \sigma}} \\
&\quad \le C \frac{1 + (s + \delta)^{\varepsilon} + (s + \delta)^{-\varepsilon} + |y|^{2\alpha\varepsilon}}{(1 + s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \kappa + \sigma}} \\
&\quad = \frac{1 + s^{\varepsilon} + |y|^{2\alpha\varepsilon}}{(1 + s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \kappa + \sigma}} + \frac{s^{-\varepsilon}}{(1 + s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \kappa + \sigma}} \\
&\quad \le C \frac{1}{(1 + s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \kappa + \sigma - \varepsilon}} + \frac{s^{-\varepsilon}}{(1 + s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \kappa + \sigma}}.
\end{aligned} \tag{4.2}$$

Since $0 < \varepsilon < \sigma$, we have

$$\begin{aligned}
&\frac{(1 + (c_1 s + \delta)^{\varepsilon} + (c_1 s + \delta)^{-\varepsilon} + |y|^{2\alpha\varepsilon}) s^{\kappa}}{(1 + c_2 s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \kappa + \sigma}} \\
&\quad \le C \left\{ \frac{1}{(1 + s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \sigma - \varepsilon}} + \frac{s^{-\varepsilon}}{(1 + s + |y|^{2\alpha})^{\frac{n}{2\alpha} + \sigma}} \right\} \to 0
\end{aligned}$$

as $s \to \infty$.

(3) By (4.2), there exists a constant $C > 0$ such that

$$\int_H \frac{(1+(c_1s+\delta)^{\varepsilon}+(c_1s+\delta)^{-\varepsilon}+|y|^{2\alpha\varepsilon})s^{\kappa-1}}{(1+c_2s+|y|^{2\alpha})^{\frac{n}{2\alpha}+\kappa+\sigma}}dV(y,s)$$
$$\leq C\int_H \left\{\frac{s^{\kappa-1}}{(1+s+|y|^{2\alpha})^{\frac{n}{2\alpha}+\kappa+\sigma-\varepsilon}}+\frac{s^{\kappa-\varepsilon-1}}{(1+s+|y|^{2\alpha})^{\frac{n}{2\alpha}+\kappa+\sigma}}\right\}dV(y,s).$$

Since $\kappa-1>\kappa-\varepsilon-1>-1$ and $-\sigma-\varepsilon<-\sigma+\varepsilon<0$, Lemma 2.4 implies that

$$\int_H \left\{\frac{s^{\kappa-1}}{(1+s+|y|^{2\alpha})^{\frac{n}{2\alpha}+\kappa+\sigma-\varepsilon}}+\frac{s^{\kappa-\varepsilon-1}}{(1+s+|y|^{2\alpha})^{\frac{n}{2\alpha}+\kappa+\sigma}}\right\}dV(y,s)<\infty.$$

This completes the proof. □

For $\delta>0$ and a function u on H, we define u_δ by $u_\delta(x,t)=u(x,t+\delta)$. First, we show the reproducing formula for fractional derivatives of u_δ.

Proposition 4.2. *Let $0<\alpha\leq 1$ and $\delta>0$. And let $\nu,\kappa\geq 0$ be real numbers with $\nu+\kappa>0$. Then,*

$$u_\delta(x,t)-u_\delta(0,1)=C_{\nu+\kappa}\int_H \mathscr{D}_t^{\nu}u_\delta(y,s)\omega_\alpha^{\kappa}(x,t;y,s)s^{\nu+\kappa-1}dV(y,s) \tag{4.3}$$

holds for all $u\in\mathscr{B}_\alpha$ and $(x,t)\in H$, where C_κ is the constant defined in Theorem A.

Proof. Let $u\in\mathscr{B}_\alpha$ and $\delta>0$. And let $m,k\in\boldsymbol{N}_0$ such that $m+k>0$. First, we show that

$$\begin{aligned}&u_\delta(x,t)-u_\delta(0,1)\\&\quad=\frac{(c_1+c_2)^{m+k}}{\Gamma(m+k)}\int_H \mathscr{D}_t^{m}u_\delta(y,c_1s)\omega_\alpha^{k}(x,t;y,c_2s)s^{m+k-1}dV(y,s)\end{aligned} \tag{4.4}$$

for all $(x,t)\in H$ and real numbers $c_1,c_2>0$. We prove the equality (4.4) for $m\in\boldsymbol{N}$ and $k=0$. By (2) of Proposition 3.1, (1) of Proposition 3.2, and Lemma 2.4, we obtain

$$\int_H \left|\mathscr{D}_t^{m}u_\delta(y,c_1s)\right|\left|\omega_\alpha^{0}(x,t;y,c_2s)\right|s^{m-1}dV(y,s)<\infty.$$

Since $\mathscr{D}_t^{m}u_\delta\in\boldsymbol{b}_\alpha^{\infty}\subset\mathscr{B}_\alpha$ by (1) and (3) of Proposition 3.2, the Fubini theorem and Lemma 2.5 imply that

$$
\begin{aligned}
&\int_H \mathscr{D}_t^m u_\delta(y,s)\omega_\alpha^0(x,t;y,c_2 s)s^{m-1}dV(y,s)\\
&\quad = \int_0^\infty \int_{\boldsymbol{R}^n} \mathscr{D}_t^m u_\delta(y,c_1 s)\big(W^{(\alpha)}(x-y,t+c_2 s)-W^{(\alpha)}(-y,1+c_2 s)\big)dy s^{m-1}ds\\
&\quad = \int_0^\infty \big(\mathscr{D}_t^m u_\delta(x,t+(c_1+c_2)s)-\mathscr{D}_t^m u_\delta(0,1+(c_1+c_2)s)\big)s^{m-1}ds. \qquad (4.5)
\end{aligned}
$$

Integrating by parts $m-1$ times, (2) of Proposition 3.3 implies that the right-hand side of (4.5) is equal to

$$
\begin{aligned}
&-\frac{1}{c_1+c_2}\big[\big(\mathscr{D}_t^{m-1}u_\delta(x,t+(c_1+c_2)s)-\mathscr{D}_t^{m-1}u_\delta(0,1+(c_1+c_2)s)\big)s^{m-1}\big]_0^\infty\\
&\qquad +\frac{m-1}{c_1+c_2}\int_0^\infty \big(\mathscr{D}_t^{m-1}u_\delta(x,t+(c_1+c_2)s)\\
&\qquad\qquad\qquad -\mathscr{D}_t^{m-1}u_\delta(0,1+(c_1+c_2)s)\big)s^{m-2}ds\\
&\quad = \frac{\Gamma(m)}{(c_1+c_2)^{m-1}}\int_0^\infty \mathscr{D}_t u_\delta(x,t+(c_1+c_2)s)-\mathscr{D}_t u_\delta(0,1+(c_1+c_2)s)ds\\
&\quad = \frac{\Gamma(m)}{(c_1+c_2)^m}[u_\delta(x,t+(c_1+c_2)s)-u_\delta(0,1+(c_1+c_2)s)]_0^\infty\\
&\quad = \frac{\Gamma(m)}{(c_1+c_2)^m}(u_\delta(x,t)-u_\delta(0,1)).
\end{aligned}
$$

Therefore, we obtain (4.4) in the case of $m\in\boldsymbol{N}$ and $k=0$.

We show (4.4) in the case of $m\in\boldsymbol{N}_0$ and $k\in\boldsymbol{N}$ by induction on k. Let $k=1$. If $m=0$, then by (1) of Lemma 4.1, for each $0<\varepsilon<\sigma$, there exists a constant $C>0$ such that

$$
|u_\delta(y,c_1 s)|\le C\big(1+(c_1 s+\delta)^\varepsilon+(c_1 s+\delta)^{-\varepsilon}+|y|^{2\alpha\varepsilon}\big) \qquad (4.6)
$$

for all $(y,s)\in H$, where $\sigma=\min\{1,1/2\alpha\}$. By (3) of Lemma 4.1, we obtain

$$
\int_H |u_\delta(y,c_1 s)|\big|\omega_\alpha^1(x,t;y,c_2 s)\big|dV(y,s)<\infty.
$$

Hence, the Fubini theorem implies that

$$\begin{aligned}
&\int_H u_\delta(y,c_1s)\omega_\alpha^1(x,t;y,c_2s)dV(y,s)\\
&\quad=-\frac{1}{c_2}\int_{\boldsymbol{R}^n}\left[u_\delta(y,c_1s)\big(W^{(\alpha)}(x-y,t+c_2s)-W^{(\alpha)}(-y,1+c_2s)\big)\right]_0^\infty dy\\
&\qquad-\frac{c_1}{c_2}\int_H \mathscr{D}_t u_\delta(y,c_1s)\big(W^{(\alpha)}(x-y,t+c_2s)-W^{(\alpha)}(-y,1+c_2s)\big)dV(y,s).
\end{aligned}$$

By (4.6) and (2) of Proposition 3.1, (1) of Lemma 4.1 and Lemma 2.5 imply that

$$\begin{aligned}
&-\frac{1}{c_2}\int_{\boldsymbol{R}^n}\left[u_\delta(y,c_1s)\big(W^{(\alpha)}(x-y,t+c_2s)-W^{(\alpha)}(-y,1+c_2s)\big)\right]_0^\infty dy\\
&=\frac{1}{c_2}\int_{\boldsymbol{R}^n}u_\delta(y,0)\big(W^{(\alpha)}(x-y,t)-W^{(\alpha)}(-y,1)\big)dy\\
&=\frac{1}{c_2}(u_\delta(x,t)-u_\delta(0,1)).
\end{aligned}$$

Therefore, (4.5) implies that

$$\begin{aligned}
&\int_H u_\delta(y,c_1s)\omega_\alpha^1(x,t;y,c_2s)dV(y,s)\\
&\quad=\frac{1}{c_2}(u_\delta(x,t)-u_\delta(0,1))-\frac{c_1}{c_2(c_1+c_2)}(u_\delta(x,t)-u_\delta(0,1))\\
&\quad=\frac{1}{c_1+c_2}(u_\delta(x,t)-u_\delta(0,1)).
\end{aligned}$$

If $m\geq 1$, then (2) of Proposition 3.1, (1) of Proposition 3.2, and (4.5) imply that

$$\begin{aligned}
&\int_H \mathscr{D}_t^m u_\delta(y,c_1s)\omega_\alpha^1(x,t;y,c_2s)s^m dV(y,s)\\
&\quad=-\frac{1}{c_2}\int_{\boldsymbol{R}^n}\left[\mathscr{D}_t^m u_\delta(y,c_1s)\omega_\alpha^0(x,t;y,c_2s)s^m\right]_0^\infty dy\\
&\qquad-\frac{c_1}{c_2}\int_H \mathscr{D}_t^{m+1}u_\delta(y,c_1s)\omega_\alpha^0(x,t;y,c_2s)s^m dV(y,s)\\
&\qquad+\frac{m}{c_2}\int_H \mathscr{D}_t^m u_\delta(y,c_1s)\omega_\alpha^0(x,t;y,c_2s)s^{m-1}dV(y,s)\\
&\quad=-\frac{c_1\Gamma(m+1)}{c_2(c_1+c_2)^{m+1}}(u_\delta(x,t)-u_\delta(0,1))+\frac{\Gamma(m+1)}{c_2(c_1+c_2)^m}(u_\delta(x,t)-u_\delta(0,1))
\end{aligned}$$

$$= \frac{\Gamma(m+1)}{(c_1+c_2)^{m+1}}(u_\delta(x,t) - u_\delta(0,1)).$$

Therefore, (4.4) holds for all $m \in \boldsymbol{N}_0$ whenever $k = 1$.

Let $k \geq 1$ and suppose that (4.4) holds for all $m \in \boldsymbol{N}_0$. Then by (2) of Proposition 3.1, (4.6), (1) of Proposition 3.2, and (2) of Lemma 4.1, we have

$$\mathscr{D}_t^m u_\delta(y, c_1 s)\omega_\alpha^k(x,t;y,c_2 s)s^{m+k} \to 0$$

as $s \to \infty$. Therefore, the assumption of the induction implies that

$$\begin{aligned}
&\int_H \mathscr{D}_t^m u_\delta(y,c_1 s)\omega_\alpha^{k+1}(x,t;y,c_2 s)s^{m+k}dV(y,s) \\
&\quad = -\frac{1}{c_2}\int_{\boldsymbol{R}^n} \left[\mathscr{D}_t^m u_\delta(y,c_1 s)\omega_\alpha^k(x,t;y,c_2 s)s^{m+k}\right]_0^\infty dy \\
&\qquad - \frac{c_1}{c_2}\int_H \mathscr{D}_t^{m+1} u_\delta(y,c_1 s)\omega_\alpha^k(x,t;y,c_2 s)s^{m+k}dV(y,s) \\
&\qquad + \frac{m+k}{c_2}\int_H \mathscr{D}_t^m u_\delta(y,c_1 s)\omega_\alpha^k(x,t;y,c_2 s)s^{m+k-1}dV(y,s) \\
&\quad = -\frac{c_1\Gamma(m+k+1)}{c_2(c_1+c_2)^{m+k+1}}(u_\delta(x,t)-u_\delta(0,1)) + \frac{\Gamma(m+k+1)}{c_2(c_1+c_2)^{m+k}}(u_\delta(x,t)-u_\delta(0,1)) \\
&\quad = \frac{\Gamma(m+k+1)}{(c_1+c_2)^{m+k+1}}(u_\delta(x,t)-u_\delta(0,1)).
\end{aligned}$$

Therefore, we obtain (4.4) in case $m, k \in \boldsymbol{N}_0$ with $m + k > 0$.

Next, we show the equality (4.3). Let $\nu, \kappa \in \boldsymbol{R}_+ \setminus \boldsymbol{N}_0$ with $\nu + \kappa > 0$. Then, by (2.4) and (2.5), we have

$$\begin{aligned}
&\int_H \mathscr{D}_t^\nu u_\delta(y,s)\omega_\alpha^\kappa(x,t;y,s)s^{\nu+\kappa-1}dV(y,s) \\
&\quad = \int_H \frac{1}{\Gamma(\lceil\nu\rceil-\nu)}\int_0^\infty \tau_1^{\lceil\nu\rceil-\nu-1}\mathscr{D}_t^{\lceil\nu\rceil}u_\delta(y,s+\tau_1)d\tau_1 \\
&\qquad \times \frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)}\int_0^\infty \tau_2^{\lceil\kappa\rceil-\kappa-1}\omega_\alpha^{\lceil\kappa\rceil}(x,t;y,s+\tau_2)d\tau_2 s^{\nu+\kappa-1}dV(y,s).
\end{aligned}$$

By (4.4), the Fubini theorem implies that

$$\begin{aligned}
&\int_H \mathscr{D}_t^\nu u_\delta(y,s)\omega_\alpha^\kappa(x,t;y,s)s^{\nu+\kappa-1}dV(y,s) \\
&= \frac{1}{\Gamma(\lceil\nu\rceil-\nu)}\frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)}\int_0^\infty \tau_1^{\lceil\nu\rceil-\nu-1}\int_0^\infty \tau_2^{\lceil\kappa\rceil-\kappa-1} \\
&\quad \times \int_H \mathscr{D}_t^{\lceil\nu\rceil} u_\delta(y,(1+\tau_1)s)\omega_\alpha^{\lceil\kappa\rceil}(x,t;y,(1+\tau_2)s)s^{\lceil\nu\rceil+\lceil\kappa\rceil-1}dV(y,s)d\tau_2 d\tau_1 \\
&= (u_\delta(x,t)-u_\delta(0,1)) \\
&\quad \times \frac{\Gamma(\lceil\nu\rceil+\lceil\kappa\rceil)}{\Gamma(\lceil\nu\rceil-\nu)\Gamma(\lceil\kappa\rceil-\kappa)}\int_0^\infty \tau_1^{\lceil\nu\rceil-\nu-1}\int_0^\infty \frac{\tau_2^{\lceil\kappa\rceil-\kappa-1}}{(\tau_1+\tau_2+2)^{\lceil\nu\rceil+\lceil\kappa\rceil}}d\tau_2 d\tau_1.
\end{aligned}$$

Furthermore, it is easy to show that

$$\frac{\Gamma(\lceil\nu\rceil+\lceil\kappa\rceil)}{\Gamma(\lceil\nu\rceil-\nu)\Gamma(\lceil\kappa\rceil-\kappa)}\int_0^\infty \tau_1^{\lceil\nu\rceil-\nu-1}\int_0^\infty \frac{\tau_2^{\lceil\kappa\rceil-\kappa-1}}{(\tau_1+\tau_2+2)^{\lceil\nu\rceil+\lceil\kappa\rceil}}d\tau_2 d\tau_1 = \frac{\Gamma(\nu+\kappa)}{2^{\nu+\kappa}}.$$

Therefore, we obtain (4.3) in case $\nu,\kappa \in \boldsymbol{R}_+ \setminus \boldsymbol{N}$. The remaining cases can be proved similarly. This completes the proof. □

Now, we give the reproducing formula with fractional orders on the parabolic Bloch space. Theorem 1 is an immediate consequence of Theorem 4.3.

THEOREM 4.3. *Let $0<\alpha\le 1$. And let $\nu\ge 0$ and $\kappa>0$ be real numbers. Then, the reproducing formula*

$$u(x,t)-u(0,1) = C_{\nu+\kappa}\int_H \mathscr{D}_t^\nu u(y,s)\omega_\alpha^\kappa(x,t;y,s)s^{\nu+\kappa-1}dV(y,s) \tag{4.7}$$

holds for all $u\in\mathscr{B}_\alpha$ and $(x,t)\in H$, where C_κ is the constant defined in Theorem A.

PROOF. Let $u\in\mathscr{B}_\alpha$ and $(x,t)\in H$. For $0<\delta<1$, Proposition 4.2 implies that

$$u_\delta(x,t)-u_\delta(0,1) = C_{\nu+\kappa}\int_H \mathscr{D}_t^\nu u_\delta(y,s)\omega_\alpha^\kappa(x,t;y,s)s^{\nu+\kappa-1}dV(y,s).$$

First, we show (4.7) in the case of $\nu=0$. Let $\sigma=\min\{1,1/2\alpha\}$ and $0<\varepsilon<\min\{\kappa,\sigma\}$. By (1) of Lemma 4.1, there exists a constant $C>0$ (independent of δ) such that

$$|u_\delta(y,s)| \le C(1+(s+\delta)^{\varepsilon}+(s+\delta)^{-\varepsilon}+|y|^{2\alpha\varepsilon}) \le C(1+s^{\varepsilon}+s^{-\varepsilon}+|y|^{2\alpha\varepsilon}).$$

Therefore (3) of Lemma 4.1 implies that

$$\begin{aligned} C&\int_H |u_\delta(y,s)|\cdot\left|\omega_\alpha^\kappa(x,t;y,s)\right|s^{\kappa-1}dV(y,s) \\ &\le C\int_H \frac{(1+s^{\varepsilon}+s^{-\varepsilon}+|y|^{2\alpha\varepsilon})s^{\kappa-1}}{(1+s+|y|^{2\alpha})^{\frac{n}{2\alpha}+\kappa+\sigma}}dV(y,s)<\infty. \end{aligned}$$

Thus, the Lebesgue dominated convergence theorem implies the equality (4.7) with $\nu=0$.

Next, we show (4.7) in the case of $\nu>0$. By (1) of Proposition 3.2, there exists a constant $C>0$ (independent of δ) such that

$$s^{\nu}|\mathscr{D}_t^{\nu}u_\delta(y,s)| \le C\|u\|_{\mathscr{B}_\alpha}$$

for all $(y,s)\in H$. Therefore by (4) of Proposition 3.1, the Lebesgue dominated convergence theorem implies the equality (4.7) with $\nu>0$. This completes the proof. □

In the rest of this section, we give the estimate of the normal derivative norm with fractional orders on the parabolic Bloch space. Let $0<\alpha\le 1$ and $\kappa>0$ be a real number. For $f\in L^\infty$, the integral operator $\widetilde{P}_\alpha^\kappa$ is defined by

$$\widetilde{P}_\alpha^\kappa f(x,t)=\int_H f(y,s)\omega_\alpha^\kappa(x,t;y,s)s^{\kappa-1}dV(y,s) \tag{4.8}$$

for all $(x,t)\in H$. In the following proposition, we show that $\widetilde{P}_\alpha^\kappa$ is a bounded linear operator from L^∞ onto $\widetilde{\mathscr{B}}_\alpha$.

Proposition 4.4. *Let $0<\alpha\le 1$ and $\kappa>0$ be a real number. Then the following statements hold.*

(1) *For each $(\beta,k)\in(\boldsymbol{N}_0^n\times\boldsymbol{N}_0)\setminus\{(0,0)\}$,*

$$\partial_x^\beta\mathscr{D}_t^k\widetilde{P}_\alpha^\kappa f(x,t)=\int_H f(y,s)\partial_x^\beta\mathscr{D}_t^{\kappa+k}W^{(\alpha)}(x-y,t+s)s^{\kappa-1}dV(y,s)$$

for all $f\in L^\infty$ and $(x,t)\in H$.

(2) *$\widetilde{P}_\alpha^\kappa$ is a bounded linear operator from L^∞ onto $\widetilde{\mathscr{B}}_\alpha$.*

PROOF.
(1) Let $f \in L^\infty$ and $(x,t) \in H$. By (4) of Proposition 3.1, we obtain

$$\int_H |f(y,s)| \left|\omega_\alpha^\kappa(x,t;y,s)\right| s^{\kappa-1} dV(y,s) < \infty.$$

Also by (2) of Proposition 2.3 and the condition $W^{(\alpha)} \in C^\infty(H)$, differentiating through the integral, (3) of Proposition 2.3 implies that

$$\begin{aligned}\partial_x^\beta \mathscr{D}_t^k \widetilde{P}_\alpha^\kappa f(x,t) &= \int_H f(y,s) \left(\partial_x^\beta \mathscr{D}_t^k \omega_\alpha^\kappa(x,t;y,s)\right) s^{\kappa-1} dV(y,s) \\ &= \int_H f(y,s) \partial_x^\beta \mathscr{D}_t^{\kappa+k} W^{(\alpha)}(x-y,t+s) s^{\kappa-1} dV(y,s).\end{aligned}$$

(2) First, we show that $\widetilde{P}_\alpha^\kappa$ is a bounded linear operator from L^∞ to $\widetilde{\mathscr{B}}_\alpha$. Let $f \in L^\infty$. By the definition of ω_α^κ, we have $\widetilde{P}_\alpha^\kappa f(0,1) = 0$. Also by (2) of Proposition 2.3 and (1) of Proposition 4.4, for each $(\beta,k) \in (\boldsymbol{N}_0^n \times \boldsymbol{N}_0) \setminus \{(0,0)\}$, we obtain

$$\left|\partial_x^\beta \mathscr{D}_t^k \widetilde{P}_\alpha^\kappa f(x,t)\right| \le C t^{-\frac{|\beta|}{2\alpha}-k} \|f\|_{L^\infty}.$$

Therefore we have $\|\widetilde{P}_\alpha^\kappa f\|_{\mathscr{B}_\alpha} \le C\|f\|_{L^\infty}$. We show that $\widetilde{P}_\alpha^\kappa f$ is $L^{(\alpha)}$-harmonic on H. By (4) of Proposition 3.1, for any $0 < t_1 < t_2 < \infty$, there exists a constant $C > 0$ such that

$$\begin{aligned}&\int_{t_1}^{t_2} \int_{\boldsymbol{R}^n} \int_H |f(y,s)| \left|\omega_\alpha^\kappa(x,t;y,s)\right| s^{\kappa-1} dV(y,s) (1+|x|)^{-n-2\alpha} dx dt \\ &\le C\|f\|_{L^\infty} \int_{t_1}^{t_2} \int_{\boldsymbol{R}^n} (1+\log(1+|x|) + |\log t|)(1+|x|)^{-n-2\alpha} dx dt \\ &\le C \int_0^\infty \frac{r^{n-1}(1+\log(1+r))}{(1+r)^{n+2\alpha}} dr < \infty.\end{aligned}$$

Therefore $\widetilde{P}_\alpha^\kappa f$ holds the integrability condition (2.3) for all $f \in L^\infty$. Hence $\widetilde{P}_\alpha^\kappa f$ is $L^{(\alpha)}$ -harmonic on H. Thus, we obtain $\widetilde{P}_\alpha^\kappa$ is a bounded linear operator from L^∞ to $\widetilde{\mathscr{B}}_\alpha$.

Next, we show that $\widetilde{P}_\alpha^\kappa$ is onto. In fact, for each $u \in \widetilde{\mathscr{B}}_\alpha$, (1) of Proposition 3.2 implies that $t\mathscr{D}_t u \in L^\infty$. Furthermore by Theorem 4.3, we have $u = C_{\kappa+1} \widetilde{P}_\alpha^\kappa (t\mathscr{D}_t u)$. This completes the proof. □

Now, we give the estimates of the normal derivative norm of parabolic Bloch functions.

THEOREM 4.5. *Let $0 < \alpha \leq 1$. And let $\nu > 0$ be a real number. Then there exists a constant $C > 0$ such that*

$$C^{-1}\|u\|_{\mathscr{B}_\alpha} \leq \left\|t^\nu \mathscr{D}_t^\nu u\right\|_{L^\infty} \leq C\|u\|_{\mathscr{B}_\alpha}$$

for all $u \in \widetilde{\mathscr{B}}_\alpha$.

PROOF. Let $\nu > 0$ be a real number and $u \in \widetilde{\mathscr{B}}_\alpha$. By (1) of Proposition 3.2, there exists a constant $C > 0$ such that

$$\left\|t^\nu \mathscr{D}_t^\nu u\right\|_{L^\infty} \leq C\|u\|_{\mathscr{B}_\alpha}.$$

On the other hand, Theorem 4.3 implies that $u = C_{\nu+1}\widetilde{P}_\alpha^1(t^\nu \mathscr{D}_t^\nu u)$. Hence by Proposition 4.4, there exists a constant $C > 0$ such that

$$\|u\|_{\mathscr{B}_\alpha} = C\left\|\widetilde{P}_\alpha^1(t^\nu \mathscr{D}_t^\nu u)\right\|_{\mathscr{B}_\alpha} \leq C\left\|t^\nu \mathscr{D}_t^\nu u\right\|_{L^\infty}.$$

This completes the proof. □

5. Dual and pre-dual spaces of parabolic Bergman spaces.

In this section, we give the proofs of Theorems 2 and 3, that is, we show dual and pre-dual spaces of $\boldsymbol{b}_\alpha^1(\lambda)$ are isomorphic to $\widetilde{\mathscr{B}}_\alpha$ and $\widetilde{\mathscr{B}}_{\alpha,0}$, respectively. For $u \in \boldsymbol{b}_\alpha^1(\lambda)$ and $v \in \widetilde{\mathscr{B}}_\alpha$, we define

$$\langle u, v\rangle_\lambda = C_{\lambda+2}\int_H u(x,t)\mathscr{D}_t v(x,t)t^{\lambda+1}dV(x,t). \tag{5.1}$$

Then, we clearly have

$$|\langle u, v\rangle_\lambda| \leq C\|u\|_{L^1(\lambda)}\|v\|_{\mathscr{B}_\alpha}. \tag{5.2}$$

Now, we show that the dual space of $\boldsymbol{b}_\alpha^1(\lambda)$ is isomorphic to $\widetilde{\mathscr{B}}_\alpha$.

THEOREM 5.1. *Let $0 < \alpha \leq 1$ and $\lambda > -1$. Then $(\boldsymbol{b}_\alpha^1(\lambda))^* \cong \widetilde{\mathscr{B}}_\alpha$ under the pairing*

$$\Phi_v(u) = \langle u, v\rangle_\lambda, \quad u \in \boldsymbol{b}_\alpha^1(\lambda),$$

where Φ_v is a linear functional on $\boldsymbol{b}_\alpha^1(\lambda)$ induced by $v \in \widetilde{\mathscr{B}}_\alpha$. Moreover, there exists a constant $C > 0$ such that

$$C^{-1}\|v\|_{\mathscr{B}_\alpha} \le \|\Phi_v\| \le C\|v\|_{\mathscr{B}_\alpha}$$

for all $v \in \widetilde{\mathscr{B}}_\alpha$.

Proof. We define a map $\iota = \iota_\lambda : \widetilde{\mathscr{B}}_\alpha \to (\boldsymbol{b}_\alpha^1(\lambda))^*$ by $\iota(v) = \Phi_v$. Then by (5.2), ι is bounded, that is, $\|\Phi_v\| \le C\|v\|_{\mathscr{B}_\alpha}$ for some $C > 0$.

We show that ι is injective. Let $\Phi_v = \iota(v) = 0$. We show $v = 0$. In fact, by (2) of Proposition 3.1, for each $(x,t) \in H$, $\omega_\alpha^{\lambda+1}(x,t;\cdot,\cdot) \in \boldsymbol{b}_\alpha^1(\lambda)$. Therefore Theorem 4.3 implies that

$$v(x,t) = C_{\lambda+2}\int_H \mathscr{D}_t v(y,s)\omega_\alpha^{\lambda+1}(x,t;y,s)s^{\lambda+1}dV(y,s) = \Lambda_v(\omega_\alpha^{\lambda+1}(x,t;\cdot,\cdot)) = 0.$$

Thus ι is injective.

We show that for $\Phi \in (\boldsymbol{b}_\alpha^1(\lambda))^*$, there exists $v \in \widetilde{\mathscr{B}}_\alpha$ such that $\iota(v) = \Phi$ and $\|v\|_{\mathscr{B}_\alpha} \le C\|\Phi\|$ for some $C > 0$. Let $\Phi \in (\boldsymbol{b}_\alpha^1(\lambda))^*$. Since $(L^1(\lambda))^* \cong L^\infty$, by the Hahn-Banach extension theorem and the Riesz representation theorem, there exists $f \in L^\infty$ such that

$$\Phi(u) = \int_H u(x,t)f(x,t)t^\lambda dV(x,t)$$

for all $u \in \boldsymbol{b}_\alpha^1(\lambda)$ with $\|\Phi\| = \|f\|_{L^\infty}$. Let $v = \widetilde{P}_\alpha^{\lambda+1}f$, where $\widetilde{P}_\alpha^{\lambda+1}$ is the operator defined in (4.8). Then by (2) of Proposition 4.4, $v \in \widetilde{\mathscr{B}}_\alpha$ and there exists a constant $C > 0$ such that $\|v\|_{\mathscr{B}_\alpha} \le C\|f\|_{L^\infty} = C\|\Phi\|$. We claim $\iota(v) = \Phi$. By (1) of Proposition 4.4, we obtain

$$\mathscr{D}_t\widetilde{P}_\alpha^{\lambda+1}f(x,t) = \int_H f(y,s)\mathscr{D}_t^{\lambda+2}W^{(\alpha)}(x-y,t+s)s^\lambda dV(y,s).$$

Therefore, the Fubini theorem and Theorem A imply that

$$\begin{aligned}\Phi_v(u) &= C_{\lambda+2}\int_H u(x,t)\mathscr{D}_t v(x,t)t^{\lambda+1}dV(x,t)\\ &= C_{\lambda+2}\int_H u(x,t)\big(\mathscr{D}_t\widetilde{P}_\alpha^{\lambda+1}f(x,t)\big)t^{\lambda+1}dV(x,t)\end{aligned}$$

$$
\begin{aligned}
&= C_{\lambda+2}\int_H u(x,t)\int_H f(y,s)\mathscr{D}_t^{\lambda+2}W^{(\alpha)}(x-y,t+s)s^\lambda dV(y,s)t^{\lambda+1}dV(x,t) \\
&= \int_H C_{\lambda+2}\int_H u(x,t)\mathscr{D}_t^{\lambda+2}W^{(\alpha)}(y-x,s+t)t^{\lambda+1}dV(x,t)f(y,s)s^\lambda dV(y,s) \\
&= \int_H u(y,s)f(y,s)s^\lambda dV(y,s) = \Phi(u)
\end{aligned}
$$

for all $u \in \boldsymbol{b}_\alpha^1(\lambda)$. Thus we obtain $\iota(v) = \Phi$. This completes the proof. $\square$

In the rest of this section, we show that the pre-dual space of $\boldsymbol{b}_\alpha^1(\lambda)$ is isomorphic to $\widetilde{\mathscr{B}}_{\alpha,0}$. The following lemma is useful to the proof of Theorem 5.3. Let $C_0(H)$ be the set of all continuous functions on H which vanish continuously at $\partial H \cup \{\infty\}$.

Lemma 5.2. *Let $0 < \alpha \le 1$ and $\kappa > 0$ be a real number. Then,*

$$
\widetilde{\mathscr{B}}_{\alpha,0} = \bigl\{u \in \widetilde{\mathscr{B}}_\alpha; t\mathscr{D}_t u \in C_0(H)\bigr\} = \bigl\{\widetilde{P}_\alpha^\kappa f; f \in C_0(H)\bigr\}.
$$

Proof. The first equality is established in Lemma 9.2 of [**4**]. Therefore we only show the second equality. Let $f \in C_0(H)$. Then we show that $u \in \widetilde{\mathscr{B}}_\alpha$ and $t\mathscr{D}_t u \in C_0(H)$ where $u = \widetilde{P}_\alpha^\kappa f$. Indeed, by (2) of Proposition 4.4, we have $u = \widetilde{P}_\alpha^\kappa f \in \widetilde{\mathscr{B}}_\alpha$. For any $\varepsilon > 0$, there is a compact set K in H such that $|f(x,t)| < \varepsilon$ for all $(x,t) \in H \setminus K$. By (1) of Proposition 4.4, we have

$$
\begin{aligned}
t\bigl|\mathscr{D}_t\widetilde{P}_\alpha^\kappa f(x,t)\bigr| &= t\left|\int_H f(y,s)\mathscr{D}_t^{\kappa+1}W^{(\alpha)}(x-y,t+s)s^{\kappa-1}dV(y,s)\right| \\
&< t\varepsilon\int_{H\setminus K}\bigl|\mathscr{D}_t^{\kappa+1}W^{(\alpha)}(x-y,t+s)\bigr|s^{\kappa-1}dV(y,s) \\
&\quad + t\int_K\bigl|f(y,s)\mathscr{D}_t^{\kappa+1}W^{(\alpha)}(x-y,t+s)\bigr|s^{\kappa-1}dV(y,s). \qquad (5.3)
\end{aligned}
$$

The first term of the right-hand side of (5.3) is less than $C\varepsilon$ by (2) of Proposition 2.3. Also, since K is compact, (1) of Proposition 2.3 implies that the second term of the right-hand side of (5.3) tends to 0 as $(x,t) \to \partial H \cup \{\infty\}$. Therefore we obtain $t\mathscr{D}_t u \in C_0(H)$. On the other hand, we can easily prove the converse inclusion by Theorem 4.3. This completes the proof. $\square$

Finally, we show that the pre-dual space of $\boldsymbol{b}_\alpha^1(\lambda)$ is isomorphic to $\widetilde{\mathscr{B}}_{\alpha,0}$.

THEOREM 5.3. *Let $0 < \alpha \le 1$ and $\lambda > -1$. Then, $\boldsymbol{b}_\alpha^1(\lambda) \cong (\widetilde{\mathscr{B}}_{\alpha,0})^*$ under the pairing*

$$\Psi_u(v) = \langle u, v\rangle_\lambda, \quad v \in \widetilde{\mathscr{B}}_{\alpha,0},$$

where Ψ_u is a linear functional on $\widetilde{\mathscr{B}}_{\alpha,0}$ induced by $u \in \boldsymbol{b}_\alpha^1(\lambda)$. Moreover, there exists a constant $C > 0$ such that

$$C^{-1}\|u\|_{L^1(\lambda)} \le \|\Psi_u\| \le C\|u\|_{L^1(\lambda)}.$$

PROOF. We define a map $\pi = \pi_\lambda : \boldsymbol{b}_\alpha^1(\lambda) \to (\widetilde{\mathscr{B}}_{\alpha,0})^*$ by $\pi(u) = \Psi_u$. Then by (5.2), π is bounded, that is, $\|\Psi_u\| \le C\|u\|_{L^1(\lambda)}$.

We show that π is injective. Let $\Psi_u = \pi(u) = 0$. We show $u = 0$. By Theorem A and (3) of Proposition 3.1, we obtain

$$\begin{aligned} u(x,t) &= C_{\lambda+2}\int_H u(y,s)\mathscr{D}_t^{\lambda+2}W^{(\alpha)}(x-y,t+s)s^{\lambda+1}dV(y,s) \\ &= C_{\lambda+2}\int_H u(y,s)\mathscr{D}_t\big(\omega_\alpha^{\lambda+1}(x,t;y,s)\big)s^{\lambda+1}dV(y,s) \\ &= \Psi_u\big(\omega_\alpha^{\lambda+1}(x,t;\cdot,\cdot)\big) = 0 \end{aligned}$$

for all $(x,t) \in H$. Thus π is injective.

We show that for each $\Psi \in (\widetilde{\mathscr{B}}_{\alpha,0})^*$, there exists $u \in \boldsymbol{b}_\alpha^1(\lambda)$ such that $\Psi = \pi(u)$ and $\|u\|_{L^1(\lambda)} \le C\|\Psi\|$ for some $C > 0$. Let $\Psi \in (\widetilde{\mathscr{B}}_{\alpha,0})^*$. And let Λ be a functional on $C_0(H)$ with $\Lambda(f) = C_{\lambda+2}\Psi(\widetilde{P}_\alpha^{\lambda+1}f)$. Then by (2) of Proposition 4.4 and Lemma 5.2, Λ is a bounded linear functional on $C_0(H)$. Therefore by the Riesz representation theorem, there exists a bounded signed measure μ on H such that

$$\Lambda(f) = \int_H f(x,t)d\mu(x,t)$$

for all $f \in C_0(H)$ and $\|\mu\| = \|\Lambda\| \le C\|\Psi\|$ for some $C > 0$. We define a function u on H such that

$$u(x,t) = \int_H \mathscr{D}_t^{\lambda+2}W^{(\alpha)}(x-y,t+s)s\,d\mu(y,s).$$

Then by (2) of Proposition 2.3 and the Fubini theorem, we have

$$\begin{aligned}\|u\|_{L^1(\lambda)} &\le \int_H \int_H \left|\mathscr{D}_t^{\lambda+2} W^{(\alpha)}(x-y,t+s)\right| sd|\mu|(y,s) t^\lambda dV(x,t) \\ &= \int_H \int_H \left|\mathscr{D}_t^{\lambda+2} W^{(\alpha)}(x-y,t+s)\right| t^\lambda dV(x,t) sd|\mu|(y,s) \\ &\le C \int_H s^{-1} \cdot sd|\mu|(y,s) = C\|\mu\|.\end{aligned}$$

Therefore we obtain $\|u\|_{L^1(\lambda)} \le C\|\Psi\|$. Furthermore, since $\mathscr{D}_t^{\lambda+2} W^{(\alpha)}(x-y,t+s)$ is $L^{(\alpha)}$-harmonic with respect to (x,t) by (4) of Proposition 2.3, u is also $L^{(\alpha)}$-harmonic on H, that is, $u \in \boldsymbol{b}_\alpha^1(\lambda)$. We claim $\Psi = \pi(u)$. Let $v \in \widetilde{\mathscr{B}}_{\alpha,0}$. Then by Theorem 4.3, we have $v = C_{\lambda+2}\widetilde{P}_\alpha^{\lambda+1}(s\mathscr{D}_s v)$. Therefore by the definition of Λ, we obtain

$$\Psi(v) = C_{\lambda+2}\Psi(\widetilde{P}_\alpha^{\lambda+1}(s\mathscr{D}_s v)) = \Lambda(s\mathscr{D}_s v) = \int_H \mathscr{D}_s v(y,s) sd\mu(y,s). \tag{5.4}$$

Also since $v = C_{\lambda+2}\widetilde{P}_\alpha^{\lambda+1}(t\mathscr{D}_t v)$ by Theorem 4.3, (1) of Proposition 4.4 implies that the right-hand side of (5.4) is equal to

$$C_{\lambda+2} \int_H \int_H \mathscr{D}_t v(x,t) \mathscr{D}_t^{\lambda+2} W^{(\alpha)}(y-x,s+t) t^{\lambda+1} dV(x,t) sd\mu(y,s).$$

Therefore by (2) of Proposition 2.3, the Fubini theorem implies that

$$\begin{aligned}\Psi(v) &= C_{\lambda+2} \int_H \int_H \mathscr{D}_t^{\lambda+2} W^{(\alpha)}(x-y,t+s) sd\mu(y,s) \mathscr{D}_t v(x,t) t^{\lambda+1} dV(x,t) \\ &= C_{\lambda+2} \int_H u(x,t) \mathscr{D}_t v(x,t) t^{\lambda+1} dV(x,t) = \Psi_u(v).\end{aligned}$$

Thus we obtain $\Psi = \pi(u)$. This completes the proof. □

6. Bilinear forms on $\boldsymbol{b}_\alpha^1(\lambda) \times \widetilde{\mathscr{B}}_\alpha$.

In this section, we generalize the integral pairing $\langle \cdot, \cdot \rangle_\lambda$ of (5.1). Let $0 < \alpha \le 1$ and $\lambda > -1$. For real numbers ν and κ, a bilinear form $\Theta_\lambda^{\nu,\kappa}$ on $\boldsymbol{b}_\alpha^1(\lambda) \times \widetilde{\mathscr{B}}_\alpha$ is defined by

$$\Theta_\lambda^{\nu,\kappa}(u,v) = C_{\nu+\kappa+\lambda+1} \int_H \mathscr{D}_t^\nu u(x,t) \mathscr{D}_t^\kappa v(x,t) t^{\nu+\kappa+\lambda} dV(x,t), \qquad u \in \boldsymbol{b}_\alpha^1(\lambda),\ v \in \widetilde{\mathscr{B}}_\alpha. \tag{6.1}$$

Here we remark that $\langle u,v\rangle_\lambda = \Theta_\lambda^{0,1}(u,v)$. In the following theorem, we show that the integral pairing $\langle\cdot,\cdot\rangle_\lambda$ can be generalized.

THEOREM 6.1. *Let $0<\alpha\le 1$ and $\lambda>-1$. If $\nu>-(\lambda+1)$ and $\kappa>0$, then*

$$\Theta_\lambda^{\nu,\kappa}(u,v) = \langle u,v\rangle_\lambda, \quad u\in \boldsymbol{b}_\alpha^1(\lambda),\ v\in\widetilde{\mathscr{B}}_\alpha.$$

PROOF. Let $u\in\boldsymbol{b}_\alpha^1(\lambda)$ and $v\in\widetilde{\mathscr{B}}_\alpha$. Then by Theorem 4.3 and (1) of Proposition 4.4, we have

$$\begin{aligned}\mathscr{D}_t^{\lceil\kappa\rceil}v(x,t) &= C_{\lambda+2}\mathscr{D}_t^{\lceil\kappa\rceil}\widetilde{P}_\alpha^{\lambda+1}(s\mathscr{D}_t v)(x,t)\\ &= C_{\lambda+2}\int_H \mathscr{D}_t v(y,s)\mathscr{D}_t^{\lceil\kappa\rceil+\lambda+1}W^{(\alpha)}(x-y,t+s)s^{\lambda+1}dV(y,s).\end{aligned}$$

Furthermore, (2) of Proposition 2.3 and (2.6) imply that

$$\begin{aligned}&\int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1}\int_H |\mathscr{D}_t v(y,s)|\big|\mathscr{D}_t^{\lceil\kappa\rceil+\lambda+1}W^{(\alpha)}(x-y,t+\tau+s)\big|s^{\lambda+1}dV(y,s)d\tau\\ &\quad\le C\|v\|_{\mathscr{B}_\alpha}\int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1}(t+\tau)^{-\lceil\kappa\rceil}d\tau = Ct^{-\kappa}.\end{aligned}$$

Therefore by the Fubini theorem and (3) of Proposition 2.3, we obtain

$$\begin{aligned}\mathscr{D}_t^{\kappa}v(x,t) &= \mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)}\mathscr{D}_t^{\lceil\kappa\rceil}v(x,t)\\ &= \frac{C_{\lambda+2}}{\Gamma(\lceil\kappa\rceil-\kappa)}\int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1}\\ &\quad\times\int_H \mathscr{D}_t v(y,s)\mathscr{D}_t^{\lceil\kappa\rceil+\lambda+1}W^{(\alpha)}(x-y,t+\tau+s)s^{\lambda+1}dV(y,s)d\tau\\ &= C_{\lambda+2}\int_H \mathscr{D}_t v(y,s)\mathscr{D}_t^{-(\lceil\kappa\rceil-\kappa)}\mathscr{D}_t^{\lceil\kappa\rceil+\lambda+1}W^{(\alpha)}(x-y,t+s)s^{\lambda+1}dV(y,s)\\ &= C_{\lambda+2}\int_H \mathscr{D}_t v(y,s)\mathscr{D}_t^{\kappa+\lambda+1}W^{(\alpha)}(x-y,t+s)s^{\lambda+1}dV(y,s).\end{aligned}$$

Hence we have

$$\begin{aligned}\Theta_\lambda^{\nu,\kappa}(u,v) &= C_{\nu+\kappa+\lambda+1}\int_H \mathscr{D}_t^\nu u(x,t)\mathscr{D}_t^\kappa v(x,t)t^{\nu+\kappa+\lambda}dV(x,t)\\ &= C_{\nu+\kappa+\lambda+1}C_{\lambda+2}\int_H \mathscr{D}_t^\nu u(x,t)\\ &\quad\times\int_H \mathscr{D}_t v(y,s)\mathscr{D}_t^{\kappa+\lambda+1}W^{(\alpha)}(x-y,t+s)s^{\lambda+1}dV(y,s)t^{\nu+\kappa+\lambda}dV(x,t).\end{aligned}$$

By (1) of Proposition 3.2, (2) of Proposition 2.3, and (3) of Proposition 2.7, there exists a constant $C>0$ such that

$$\begin{aligned}&\int_H \left|\mathscr{D}_t^\nu u(x,t)\right|\int_H |\mathscr{D}_t v(y,s)|\\ &\qquad\times\left|\mathscr{D}_t^{\kappa+\lambda+1}W^{(\alpha)}(x-y,t+s)\right|s^{\lambda+1}dV(y,s)t^{\nu+\kappa+\lambda}dV(x,t)\\ &\quad\le C\|v\|_{\mathscr{B}_\alpha}\int_H \left|\mathscr{D}_t^\nu u(x,t)\right|t^{\nu+\lambda}dV(x,t)\le C\|v\|_{\mathscr{B}_\alpha}\|u\|_{L^1(\lambda)}<\infty.\end{aligned}$$

Therefore the Fubini theorem and Theorem A imply that

$$\begin{aligned}\Theta_\lambda^{\nu,\kappa}(u,v) &= C_{\lambda+2}\int_H\Bigg(C_{\nu+\kappa+\lambda+1}\int_H \mathscr{D}_t^\nu u(x,t)\mathscr{D}_t^{\kappa+\lambda+1}\\ &\qquad\times W^{(\alpha)}(y-x,s+t)t^{\nu+\kappa+\lambda}dV(x,t)\Bigg)\mathscr{D}_t v(y,s)s^{\lambda+1}dV(y,s)\\ &= C_{\lambda+2}\int_H u(y,s)\mathscr{D}_t v(y,s)s^{\lambda+1}dV(y,s)=\langle u,v\rangle_\lambda.\end{aligned}$$

This completes the proof. $\square$

By Theorem 6.1, we give a generalization of the integral pairing $\langle\cdot,\cdot\rangle_\lambda$. By (2) of Proposition 2.7, the relation of the duality $(\boldsymbol{b}_\alpha^p(\lambda))^*\cong\boldsymbol{b}_\alpha^q(\lambda)$ is given by the pairing (6.1) with $\nu=\kappa=0$. However, we do not know whether our generalization (6.1) holds in case $\kappa=0$. In Theorem 6.5 below, we show the generalization of the integral pairing (6.1) holds for $\kappa=0$ on a dense subspace of $\boldsymbol{b}_\alpha^1(\lambda)$. We introduce a function space $\mathscr{S}(\eta)$, which will be a dense subspace of $\boldsymbol{b}_\alpha^p(\lambda)$. For a real number η, let

$$\mathscr{S}(\eta):=\left\{u\in\boldsymbol{b}_\alpha^\infty;(1+t+|x|^{2\alpha})^{\frac{n}{2\alpha}+\eta}u(x,t)\text{ is bounded on }H\right\}.$$

PROPOSITION 6.2. *Let $0<\alpha\le1$, $1\le p<\infty$ and $\lambda>-1$. If $\eta>(\lambda+1)/p$, then $\mathscr{S}(\eta)$ is dense in $\boldsymbol{b}_\alpha^p(\lambda)$.*

PROOF. First, we show that $\mathscr{S}(\eta) \subset \boldsymbol{b}_\alpha^p(\lambda)$. Let $u \in \mathscr{S}(\eta)$. Then, u is $L^{(\alpha)}$-harmonic on H. Moreover by Lemma 2.4, there exists a constant $C > 0$ such that

$$\int_H |u(x,t)|^p t^\lambda dV(x,t) \leq C \int_H \frac{t^\lambda}{(1+t+|x|^{2\alpha})^{(\frac{n}{2\alpha}+\eta)p}} dV(x,t) < \infty.$$

Thus we have $u \in \boldsymbol{b}_\alpha^p(\lambda)$.

Next, we show that $\mathscr{S}(\eta)$ is dense in $\boldsymbol{b}_\alpha^p(\lambda)$. Let $u \in \boldsymbol{b}_\alpha^p(\lambda)$. Taking an exhaustion $\{K_j\}$ of H, we define a function u_j such that

$$u_j(x,t) := C_\eta \int_H u \cdot \chi_{K_j}(y,s) \mathscr{D}_t^\eta W^{(\alpha)}(x-y, t+s) s^{\eta-1} dV(y,s),$$

where χ_{K_j} is the characteristic function of K_j. Then by (1) of Proposition 2.7, we obtain

$$\|u - u_j\|_{L^p(\lambda)} = \left\| P_\alpha^\eta (u - u \cdot \chi_{K_j}) \right\|_{L^p(\lambda)} \leq C \|u - u \cdot \chi_{K_j}\|_{L^p(\lambda)} \to 0, \quad (j \to \infty).$$

Finally, we show $u_j \in \mathscr{S}(\eta)$. By (1) of Proposition 2.6 and (1) of Proposition 2.3, there exists a constant $C > 0$ such that

$$\begin{aligned}
&\left| \int_H u \cdot \chi_{K_j}(y,s) \mathscr{D}_t^\eta W^{(\alpha)}(x-y,t+s) s^{\eta-1} dV(y,s) \right| \\
&\quad \leq \int_{K_j} |u(y,s)| \left| \mathscr{D}_t^\eta W^{(\alpha)}(x-y,t+s) \right| s^{\eta-1} dV(y,s) \\
&\quad \leq C \int_{K_j} \frac{s^{-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}+\eta-1}}{(t+s+|x-y|^{2\alpha})^{\frac{n}{2\alpha}+\eta}} dV(y,s).
\end{aligned}$$

Since K_j is compact, we obtain

$$\int_{K_j} \frac{s^{-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}+\eta-1}}{(t+s+|x-y|^{2\alpha})^{\frac{n}{2\alpha}+\eta}} dV(y,s) \leq \frac{C}{(1+t+|x|^{2\alpha})^{\frac{n}{2\alpha}+\eta}}.$$

This completes the proof. □

In order to prove Theorem 6.5, we need the following proposition. We remark that the proof of Proposition 6.3 is similar to that of Proposition 5.8 of [**4**]. Thus we omit the proof.

Proposition 6.3. *Let $0 < \alpha \le 1$ and $\lambda > -1$. Then for every $u \in \boldsymbol{b}_\alpha^1(\lambda)$ and $t > 0$,*

$$\int_{\boldsymbol{R}^n} u(x,t)dx = 0.$$

In order to prove Theorem 6.5, we also prepare the following lemma.

Lemma 6.4. *Let $0 < \alpha \le 1$, $\lambda > -1$, $\eta > \lambda + 1$, and $\nu > -(\lambda+1)$. If $u \in S(\eta)$, then*

$$\int_H \left|\mathscr{D}_t^\nu u(x,t)\right|(1+\log(1+|x|)+|\log t|)t^{\nu+\lambda}dV(x,t) < \infty. \tag{6.2}$$

Proof. Let $u \in S(\eta)$. First, we show the case $\nu \le 0$. By the definition of $S(\eta)$ and (2.6), there exists a constant $C > 0$ such that

$$\left|\mathscr{D}_t^\nu u(x,t)\right| \le C(1+t+|x|^{2\alpha})^{-\frac{n}{2\alpha}-\eta-\nu}.$$

Therefore by (4.1) and (3) of Lemma 4.1, we obtain (6.2) in the case of $\nu \le 0$.

Next, we show the case $\nu > 0$. Since $u \in S(\eta) \subset \boldsymbol{b}_\alpha^1(\lambda)$ by Proposition 6.2, differentiating through the integral, Theorem A and the Fubini theorem imply that

$$\mathscr{D}_t^\nu u(x,t) = C_{\lambda+2}\int_H u(y,s)\mathscr{D}_t^{\nu+\lambda+2}W^{(\alpha)}(x-y,t+s)s^{\lambda+1}dV(y,s).$$

Hence by the Fubini theorem and the definition of $S(\eta)$, there exists a constant $C > 0$ such that

$$\begin{aligned}
&\int_H \left|\mathscr{D}_t^\nu u(x,t)\right|(1+\log(1+|x|)+|\log t|)t^{\nu+\lambda}dV(x,t) \\
&\le C\int_H\int_H |u(y,s)|\left|\mathscr{D}_t^{\nu+\lambda+2}W^{(\alpha)}(x-y,t+s)\right| \\
&\quad \times s^{\lambda+1}dV(y,s)(1+\log(1+|x|)+|\log t|)t^{\nu+\lambda}dV(x,t) \\
&\le C\int_H \frac{s^{\lambda+1}}{(1+s+|y|^{2\alpha})^{\frac{n}{2\alpha}+\eta}}\int_H \left|\mathscr{D}_t^{\nu+\lambda+2}W^{(\alpha)}(x-y,t+s)\right| \\
&\quad \times (1+\log(1+|x|)+|\log t|)t^{\nu+\lambda}dV(x,t)dV(y,s).
\end{aligned} \tag{6.3}$$

Here by (4.1), we can choose $\varepsilon > 0$ small enough, and

$$1 + \log(1 + |x|) + |\log t| \leq C(1 + t^{-\varepsilon} + t^{\varepsilon} + |x|^{2\alpha\varepsilon})$$

for some $C > 0$. Therefore, (1) of Proposition 2.3 and Lemma 2.4 imply that

$$\begin{aligned}
&\int_H \left|\mathscr{D}_t^{\nu+\lambda+2} W^{(\alpha)}(x - y, t + s)\right|(1 + \log(1 + |x|) + |\log t|)t^{\nu+\lambda} dV(x, t) \\
&\leq C \int_H \frac{(1 + t^{-\varepsilon} + t^{\varepsilon} + |x|^{2\alpha\varepsilon})t^{\nu+\lambda}}{(t + s + |x - y|^{2\alpha})^{\frac{n}{2\alpha}+\nu+\lambda+2}} dV(x, t) \\
&\leq C\bigg(\int_H \frac{t^{\nu+\lambda} + t^{\nu+\lambda-\varepsilon} + t^{\nu+\lambda+\varepsilon}}{(t + s + |x - y|^{2\alpha})^{\frac{n}{2\alpha}+\nu+\lambda+2}} dV(x, t) \\
&\qquad + \int_H \frac{|x|^{2\alpha\varepsilon} t^{\lambda}}{(t + s + |x - y|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2}} dV(x, t)\bigg) \\
&\leq C\bigg(s^{-1} + s^{-1-\varepsilon} + s^{-1+\varepsilon} + \int_H \frac{|x|^{2\alpha\varepsilon} t^{\lambda}}{(t + s + |x - y|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2}} dV(x, t)\bigg). \qquad (6.4)
\end{aligned}$$

Furthermore, we consider

$$\begin{aligned}
&\int_H \frac{|x|^{2\alpha\varepsilon} t^{\lambda}}{(t + s + |x - y|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2}} dV(x, t) \\
&= \int_0^\infty \int_{|x|\leq 2|y|} \frac{|x|^{2\alpha\varepsilon} t^{\lambda}}{(t + s + |x - y|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2}} dV(x, t) \\
&\quad + \int_0^\infty \int_{|x|>2|y|} \frac{|x|^{2\alpha\varepsilon} t^{\lambda}}{(t + s + |x - y|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2}} dV(x, t).
\end{aligned}$$

If $|x| > 2|y|$, then we have $|x - y| \geq |x| - |y| > |x|/2$. Hence Lemma 2.4 implies that

$$\begin{aligned}
&\int_H \frac{|x|^{2\alpha\varepsilon} t^{\lambda}}{(t + s + |x - y|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2}} dV(x, t) \\
&\leq C \int_0^\infty \int_{|x|\leq 2|y|} \frac{|y|^{2\alpha\varepsilon} t^{\lambda}}{(t + s + |x - y|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2}} dV(x, t) \\
&\quad + C \int_0^\infty \int_{|x|>2|y|} \frac{|x|^{2\alpha\varepsilon} t^{\lambda}}{(t + s + |x|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2}} dV(x, t).
\end{aligned}$$

$$\leq C\bigg(|y|^{2\alpha\varepsilon}s^{-1} + \int_0^\infty \int_{|x|>2|y|} \frac{|x|^{2\alpha\varepsilon}t^\lambda}{(t+s+|x|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2}}dV(x,t)\bigg)$$
$$\leq C\bigg(|y|^{2\alpha\varepsilon}s^{-1} + \int_0^\infty \int_{|x|>2|y|} \frac{t^\lambda}{(t+s+|x|^{2\alpha})^{\frac{n}{2\alpha}+\lambda+2-\varepsilon}}dV(x,t)\bigg)$$
$$\leq C\big(|y|^{2\alpha\varepsilon}s^{-1} + s^{-1+\varepsilon}\big)$$

for some $C > 0$. Thus by (6.4), there exists a constant $C > 0$ such that

$$\int_H \big|\mathscr{D}_t^{\nu+\lambda+2}W^{(\alpha)}(x-y,t+s)\big|(1+\log(1+|x|)+|\log t|)t^\lambda dV(x,t)$$
$$\leq Cs^{-1}(1+s^{-\varepsilon}+s^\varepsilon+|y|^{2\alpha\varepsilon}).$$

Therefore by (6.3) and (3) of Lemma 4.1, we have

$$\int_H \big|\mathscr{D}_t^{\nu}u(x,t)\big|(1+\log(1+|x|)+|\log t|)t^{\nu+\lambda} dV(x,t)$$
$$\leq C\int_H \frac{s^\lambda(1+s^{-\varepsilon}+s^\varepsilon+|y|^{2\alpha\varepsilon})}{(1+s+|y|^{2\alpha})^{\frac{n}{2\alpha}+\eta}}dV(y,s) < \infty.$$

Thus we obtain (6.2) in the case of $\nu > 0$. This completes the proof. $\square$

Now, we show $\Theta_\lambda^{\nu,0}(\cdot,\cdot) = \langle\cdot,\cdot\rangle_\lambda$ on $S(\eta)\times\widetilde{\mathscr{B}}_\alpha$.

Theorem 6.5. *Let $0<\alpha\leq 1$, $\lambda > -1$, and $\eta > \lambda+1$. If $\nu > -(\lambda+1)$, then*

$$\Theta_\lambda^{\nu,0}(u,v) = \langle u,v\rangle_\lambda, \quad u\in S(\eta),\ v\in\widetilde{\mathscr{B}}_\alpha.$$

Proof. Let $v\in\widetilde{\mathscr{B}}_\alpha$. Then by Theorem 4.3, we have

$$v(x,t) = C_{\lambda+2}\int_H \mathscr{D}_t v(y,s)\omega_\alpha^{\lambda+1}(x,t;y,s)s^{\lambda+1}dV(y,s).$$

Therefore we obtain

$$\Theta_\lambda^{\nu,0}(u,v) = C_{\nu+\lambda+1}\int_H \mathscr{D}_t^\nu u(x,t)v(x,t)t^{\nu+\lambda}dV(x,t)$$

$$= C_{\nu+\lambda+1}C_{\lambda+2}\int_H \mathscr{D}_t^\nu u(x,t)\int_H \mathscr{D}_t v(y,s) \\ \times \omega_\alpha^{\lambda+1}(x,t;y,s)s^{\lambda+1}dV(y,s)t^{\nu+\lambda}dV(x,t).$$

Here by (3.6) and (4) of Proposition 3.1, there exists a constant $C>0$ such that

$$\int_H |\mathscr{D}_t v(y,s)||\omega_\alpha^{\lambda+1}(x,t;y,s)|s^{\lambda+1}dV(y,s) \le C(1+\log(1+|x|)+|\log t|).$$

Hence Lemma 6.4 and the Fubini theorem imply that

$$\Theta_\lambda^{\nu,0}(u,v) = C_{\nu+\lambda+1}C_{\lambda+2}\int_H\int_H \mathscr{D}_t^\nu u(x,t)\omega_\alpha^{\lambda+1}(x,t;y,s) \\ \times t^{\nu+\lambda}dV(x,t)\mathscr{D}_t v(y,s)s^{\lambda+1}dV(y,s).$$

By (3) of Proposition 2.6 and (3) of Proposition 2.7, we have $\mathscr{D}_t^\nu u \in \boldsymbol{b}_\alpha^1(\nu+\lambda)$. Therefore Proposition 6.3 implies that

$$\int_H \mathscr{D}_t^\nu u(x,t)\mathscr{D}_t^{\lambda+1}W^{(\alpha)}(-y,1+s)t^{\nu+\lambda}dV(x,t) = 0.$$

Thus by Theorem A, we obtain

$$C_{\nu+\lambda+1}\int_H \mathscr{D}_t^\nu u(x,t)\omega_\alpha^{\lambda+1}(x,t;y,s)t^{\nu+\lambda}dV(x,t) \\ = C_{\nu+\lambda+1}\int_H \mathscr{D}_t^\nu u(x,t)\mathscr{D}_t^{\lambda+1}W^{(\alpha)}(y-x,s+t)t^{\nu+\lambda}dV(x,t) = u(y,s).$$

Therefore we have

$$\Theta_\lambda^{\nu,0}(u,v) = C_{\lambda+2}\int_H u(y,s)\mathscr{D}_t v(y,s)s^{\lambda+1}dV(y,s) = \langle u,v\rangle_\lambda.$$

This completes the proof. $\square$

Finally, we discuss isomorphisms on parabolic Bergman spaces. We recall the definition of the map π from $\boldsymbol{b}_\alpha^1(\lambda)$ to $(\widetilde{\mathscr{B}}_{\alpha,0})^*$ in the proof of Theorem 5.3. For $\lambda>-1$, $\pi=\pi_\lambda : \boldsymbol{b}_\alpha^1(\lambda)\to(\widetilde{\mathscr{B}}_{\alpha,0})^*$ is defined by $\pi(u)=\Psi_u$, where

$$\Psi_u(v) = \langle u,v\rangle_\lambda, \quad v\in\widetilde{\mathscr{B}}_{\alpha,0}. \tag{6.5}$$

Hence, $\boldsymbol{b}_\alpha^1(\lambda_1) \cong \boldsymbol{b}_\alpha^1(\lambda_2)$ under the Banach space isomorphism

$$\pi_{\lambda_2}^{-1} \circ \pi_{\lambda_1} : \boldsymbol{b}_\alpha^1(\lambda_1) \to \boldsymbol{b}_\alpha^1(\lambda_2).$$

We study the isomorphism $\pi_{\lambda_2}^{-1} \circ \pi_{\lambda_1}$.

Theorem 6.6. *Let $0 < \alpha \leq 1$. And let $\lambda_1, \lambda_2 > -1$. Then, $\pi_{\lambda_2}^{-1} \circ \pi_{\lambda_1}(u) = \mathscr{D}_t^{\lambda_2-\lambda_1} u$ for all $u \in \boldsymbol{b}_\alpha^1(\lambda_1)$.*

Proof. Let $u \in \boldsymbol{b}_\alpha^1(\lambda_1)$. Then we have $\pi_{\lambda_1}(u) \in (\widetilde{\mathscr{B}}_{\alpha,0})^*$. Therefore there exists a function $U \in \boldsymbol{b}_\alpha^1(\lambda_2)$ such that $\pi_{\lambda_2}(U) = \pi_{\lambda_1}(u)$. It follows that

$$\langle U, v\rangle_{\lambda_2} = \langle u, v\rangle_{\lambda_1} \tag{6.6}$$

for all $v \in \widetilde{\mathscr{B}}_{\alpha,0}$. By Theorem 6.1 with $\nu = \lambda_2 - \lambda_1$, we obtain

$$\langle u, v\rangle_{\lambda_1} = \Theta_{\lambda_1}^{\lambda_2-\lambda_1,1}(u, v) = \left\langle \mathscr{D}_t^{\lambda_2-\lambda_1} u, v\right\rangle_{\lambda_2} \tag{6.7}$$

for all $v \in \widetilde{\mathscr{B}}_{\alpha,0}$. Hence by (6.6) and (6.7), we have

$$\left\langle U - \mathscr{D}_t^{\lambda_2-\lambda_1} u, v\right\rangle_{\lambda_2} = 0$$

for all $v \in \widetilde{\mathscr{B}}_{\alpha,0}$. Since $U - \mathscr{D}_t^{\lambda_2-\lambda_1} u \in \boldsymbol{b}_\alpha^1(\lambda_2)$ by (3) of Proposition 2.6 and (3) of Proposition 2.7, we obtain $U - \mathscr{D}_t^{\lambda_2-\lambda_1} u = 0$ on H. This completes the proof. □

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