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Complex bordism of the dihedral group

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Introduction.

Let G be a finite group. By a G-U-manifold we mean a weakly complex manifold with a free G-action preserving its weakly complex structure. The group of bordism classes of closed G-U-manifolds is isomorphic to the complex bordism group $MU_*(BG)$ of the classifying space BG [**C-F**]. If S is a Sylow p-subgroup of G, the inclusion map induces a splitting epimorphism $MU_*(BS)_{(p)} \to MU_*(BG)_{(p)}$. Hence we need to know first $MU_*(BG)$ for p-groups G. Moreover the Quillen isomorphism $MU_*(-)_{(p)} \cong MU_{*(p)} \otimes_{BP*} BP_*(-)$ shows that we need to know only $BP_*(BG)$.

When G is a cyclic or quaternion group, the graded module associated to the dimensional filtration gr $BP_*(BG)$ is isomorphic to $BP_* \otimes H_*(BG)$ since $H_{even}(BG) \cong 0$ [**M**]. By Johnson-Wilson [**J-W**], gr $BP_*(BG)$ is given for an elementary abelian p-group using arguments to generalize Künneth formula. In this paper we determine BP_*-module structure of $BP_*(BG) \bmod (p, v_1, \cdots)^2$ for nonabelian groups of the order p^3. For $p=2$, the new group is the dihedral group D_4. The bordism group $BP_*(BD_{2q})$, q : prime$\neq 2$, was studied by Kamata-Minami [**K-M**] in early seventies.

Recall the Milnor primitive operation $Q_0=\beta$, $Q_1=p^1\beta-\beta p^1$ $(=Sq^2Sq^1-Sq^1Sq^2$ for $p=2)$. For the above groups, we can extend the operation Q_1 on $H^*(BG)$ so that $Q_1|H^{even}(BG)=0$. Let us write by $H(- ; Q_1)$ the homology with the differential Q_1. Then we know (compare [**T-Y**])

$$\mathrm{gr}\, BP^*(BG) \cong BP^* \otimes H(H^*(BG) ; Q_1) \oplus BP^*/(p, v_1) \otimes \mathrm{Im}\, Q_1$$

since $d_{2p-1}=v_1 \otimes Q_1$ is the only non zero differential in the Atiyah-Hirzebruch spectral sequence. The similar fact occurs for the BP_*-homology

$$\mathrm{gr}\, BP_*(BG) \cong BP_* s^{-1} H(H^*(BG) ; Q_1) \oplus BP_*/(p, v_1) s^{-1} H^{odd}(BG)$$

where s^{-1} is the shift map which decreases degree by one. Here we use the spectral sequence $E_2^{*,*}=\mathrm{Ext}_{BP*}(BP^*(BG), BP^*) \Rightarrow BP_*(BG)$. In particular, generators and relations are given explicitely for $BP_*(BD_4)$ in the last section.

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§ 1. Bordism and cobordism.

Assume always G is a p-group. Let us write by H^* (resp. HZ/p^*, H^{even}, H^{odd}) for the cohomology $H^*(BG)$ (resp. $H^*(BG; Z/p)$, $H^{even}(BG)$, $H^{odd}(BG)$). Recall that $Q_{n+1}=Q_n p^{p^n}-p^{p^n}Q_n$ $(=Q_n Sq^{2^{n+1}}+Sq^{2^{n+1}}Q_n$ for $p=2)$. In this section we consider only groups which satisfy the following assumption.

ASSUMPTION 1.1. (i) $p\cdot HZ^{odd}=0$, hence $H^{odd}\subset HZ/p^{odd}$. So we may define Q_n on H^{odd}.

(ii) $Q_n|H^{odd}$ is injective for each $n\geqq 1$,

(iii) $Q_n(H^{odd})\subset Q_1(H^{odd})$ for each $n\geqq 1$.

Define $Q_n|H^{even}=0$ such that $Q_n^2=0$.

LEMMA 1.2. gr $BP^*(BG)\cong BP^*\otimes H(H^*; Q_1)\oplus BP^*/(p, v_1)\otimes \mathrm{Im}\, Q_1$.

PROOF. Consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*}\cong H^*(BG; BP^*)\cong BP^*\otimes H^*\Longrightarrow BP^*(BG).$$

The first nonzero differential is $d_{2p-1}(x)=v_1\otimes Q_1(x)$ for $x\in H^{odd}$ since it is so for H/p^*. For $x\in H^{even}$, $d_{2p-1}(x)=0$ otherwise $d_{2p-1}{}^2(x)\neq 0$ from the injectivity of $d_{2p-1}|H^{odd}$. Hence we get that E_{2p} is isomorphic to the right hand side of the module in the lemma. Since $\mathrm{Ker}\, Q_1\cong \mathrm{Im}\, Q_1\oplus H(H^*; Q_1)$ is even dimensionally generated, and so is $E_{2p}{}^{*,*}$. Therefore $E_{2p}\cong E_\infty$. q.e.d.

Given $Z_{(p)}$-module A, let us write by FA the $Z_{(p)}$-free module generated by $Z_{(p)}$-module generators of A. Let $F(x)$ be a generator which corresponds x in A.

THEOREM 1.3. *There is a BP^*-module isomorphism*

$$BP^*(BG)\cong BP^*\otimes(FH(H^*; Q_1)\oplus F\,\mathrm{Im}\, Q_1)/R$$

where R is generated, modulo $(p, v_1, \cdots)^2$, by $\sum_{n=0} v_n F(Q_n Q_i{}^{-1}(x))=0$ for $i=0, 1$, and $x\in \mathrm{Ker}\, Q_1$.

PROOF. If $x_1\in \mathrm{Im}\, Q_1$, then there is a relation $v_1\bar{x}_1+v_2\bar{x}_2+\cdots=0$ from Lemma 1.2, for $\rho(\bar{x}_1)=x_1$ where $\rho: BP\to HZ_{(p)}$ is the Thom map. From Lemma 2.1 in [**Y**] there is y in HZ/p^* such that $Q_n(y)=\rho(\bar{x}_n)$, and $y=Q_1{}^{-1}x_1$. Since $BP^*(BG)\otimes_{BP^*}Z_{(p)}\cong H^{even}$ we have the relation in the lemma. For $x_0\in \mathrm{Im}\, Q_0$, we also have the relation by the same arguments. q.e.d.

Now we consider the bordism theory. We also write by H_* the homology $H_*(BG)$. Since H_* is a torsion module, there is an isomorphism

$$H_{*-1} \cong s^{-1}H^* \qquad \text{for } *\geqq 2\,,$$

where s^{-1} is the operation decreasing degree by one. Note that if $px=0$, $s^{-1}x=Q_0{}^{-1}x$ for $x\in H^*$.

Consider the spectral sequence

$$E^2{}_{*,*} \cong H_*(BG\,;BP_*) \Longrightarrow BP_*(BG)\,. \tag{1.4}$$

LEMMA 1.5. $E^{2p}{}_{*,*}\cong BP_*s^{-1}H(H^*\,;Q_1)\oplus BP_*/(p,\,v_1)s^{-1}H^{odd}$.

PROOF. First note $HZ/p_*=\mathrm{Hom}\,(HZ/p^*\,;Z/p)$. Hence we can define the dual operation Q_{1*} in HZ/p_*. Since $Q_1Q_0=-Q_0Q_1$, we see easily

$$Q_{1*}s^{-1}(\mathrm{Im}\,Q_1)=s^{-1}H^{odd}\,.$$

The first nonzero differential in (1.4) is $d_{2p-1}=v_1\otimes Q_{1*}$. Hence we get the lemma. q.e.d.

We use here arguments by Ravenel and Johnson-Wilson [**J-W**]. Recall the universal coefficient spectral sequence

$$E_2{}^{*,*}=\mathrm{Ext}_{BP^*}(BP^*(BG),\,BP^*)\Longrightarrow BP_*(BG)\,. \tag{1.6}$$

Given BP^*-filtration in $BP^*(BG)$, we can construct a spectral sequence

$$G_2{}^{*,*}=\mathrm{Ext}_{BP^*}(\mathrm{gr}\,BP^*(BG),\,BP^*)\Longrightarrow E_2{}^{*,*}\,. \tag{1.7}$$

It is easily seen

LEMMA 1.8. ([**J-W**] Lemma 6.5.) $\mathrm{Ext}_{BP^*}(BP^*/(p^k),\,BP^*)\cong s^{-1}BP_*/(p^k)$,

$$\mathrm{Ext}_{BP^*}(BP^*/(p,\,v_1),\,BP^*)\cong s^{-2p}BP_*/(p,\,v_1)\,.$$

Therefore from Lemma 1.2, Lemma 1.5 and Lemma 1.8, we know

$$\mathrm{Ext}_{BP^*}(\mathrm{gr}\,BP^*(EG),\,BP^*)\cong(E^{2p}{}_{*,*}\text{ in Lemma 1.5})\,.$$

Since $G_2{}^{i,*}=0$ for $i\neq 1,2$, so $E_2{}^{i,*}=0$ for $i\neq 1,2$. Hence $d_r=0$ for $r\geqq 2$ in $E_r{}^{*,*}$. Therefore if we can prove

$$G_2{}^{*,*}\cong G_\infty{}^{*,*}\,, \tag{1.9}$$

then we get

$$(E^{2p}{}_{*,*}\text{ in Lemma 1.5})\cong G_2{}^{*,*}\cong \mathrm{gr}\,E_2{}^{*,*}\cong \mathrm{gr}\,E_\infty{}^{*,*}\cong \mathrm{gr}\,BP_*(BG).$$

Thus we can show

THEOREM 1.10. *There is a BP_*-module isomorphism*

$$BP_*(BG)\cong BP_*\otimes Fs^{-1}(H(H^*\,;Q_1)\oplus H^{odd})/R$$

where the relation R *is generated, modulo* $(p, v_1, \cdots)^2$, *by*

$$\sum v_n F(Q_{n*}Q_{i*}^{-1}s^{-1}(x))=0 \qquad \text{for } i=0,1,\ x\in(H(H^*;Q_1)\oplus H^{odd}).$$

PROOF OF (1.9). Since $G_2^{1,*}$ is v_1-torsion free but $G_2^{2,*}$ is v_1-torsion, we only need to prove that each element in $G_2^{1,*}$ is permanent. By the map induced from the inclusion $BP_*(BG^r)\to BP_*(BG)$, we may prove the above facts for the BP-homology of an r-skeleton BG^r.

Let us write a BP^*-free resolution of $BP^*(BG^r)$

$$0 \longleftarrow BP^*(BG^r) \longleftarrow \oplus BP^*b_i \oplus BP^*b_j$$

$$\overset{d_1}{\longleftarrow} \oplus BP^*r_i \oplus BP^*r_j \oplus BP^*s_j \overset{d_2}{\longleftarrow} \oplus BP^*$$

where b_j (resp. b_i) are $Z_{(p)}$-basis for $\operatorname{Im} Q_1$ (resp. $H(H^*, Q_1)$), and

$$d_1(r_t) = R_t = pb_t + \cdots \qquad t=i \text{ or } j$$

$$d_1(s_j) = S_j = v_1 b_j + \cdots .$$

Let $|b_N|=\max(|b_i|, |b_j|)$ and $b_N\in H(H^*, Q_1)$. We will prove that we can take new base s_j', r_j', r_i' such that $BP^*r_N'\cap\operatorname{Image} d_2=0$. Then the dual base $r_N'^*$ is a cocycle because $\delta(r_N'^*(c))=r_N'^*(d_2(c))=0$ for all $c\in\oplus BP^*$. Hence by induction we can see that all the elements in $G_2^{1,*}$ are permanent.

Suppose that there is a relation in $\oplus BP^*b_i \oplus BP^*b_j$

$$pS_n+v_1R_n+\sum a_jS_j+a_j'R_j = cR_N \tag{1.11}$$

with $a, c\in BP^*$ and $|b_n|\leqq|b_j|\leqq|b_N|$, $n\neq j$. If $c=pc'+v_1c''$, then put $S_n'=S_n-c'R_N$ and $R_n'=R_n-c''R_N$; hence the relation (1.11) is reduced to a relation without R_N. Therefore $BP^*r_N\cap\operatorname{Im} d_2=0$ in this case.

Thus we may assume $c=\lambda v_2^s \bmod (v_3, v_4, \cdots)$. Let $b_n=Q_1b$ and $b_q=Q_2b\neq0$ from Assumption 1.1 (ii). Moreover from the Assumption 1.1 (iii), $b_q\in\operatorname{Im} Q_1$, so $b_q\neq b_N\in H(H^*, Q_1)$. Then with the modulo $(v_1, v_3, v_4, \cdots)$ the relation (1.11) is written

$$p(v_2b_q+\cdots)+\sum a_jS_j+a_j'R_j = \lambda v_2^s R_N.$$

Hence $a_j'R_j$ contains the term $-v_2R_q$. Take $R_q'=(R_q-\lambda v_2^{s-1}R_N)$ and we can deduce the case $c=0 \bmod (v_3, \cdots)$. Continue these arguments and by dimensional reasons we get the relation $pS_n'+v_1R_n'+\cdots=0$ which does not contain R_N. q.e.d.

§ 2. Q_n-operation.

We give examples 2.1-2.3 satisfying Assumption 1.1.

2.1. $G=Z/p\times Z/p$. The cohomology $H^{even}=Z/p[y_1, y_2]$ and $H^{odd}=H^{even}e$

where $|y_i|=2$, $|e|=3$ and $Q_n e=y_1^{p^n}y_2-y_1y_2^{p^n}$.

2.2. G is a non abelian p-group of the order p^3. Then G is isomorphic to one of D, Q, E, M; the dihedral group, the quaternion group, the p exponent group for odd prime and the metacyclic group for odd prime (see Lewis [**L**] or [**T-Y**]). The cohomology H^{even} is generated by elements $c_1, \cdots, c_p, y_1, y_2$, and H^{odd} is generated as a H^{even}-module by e (resp. 0, d_1 and d_2, e) for D (resp. Q, E, M). Then we can take ring generators such that the Q_n-operation is given by $Q_n e=c_2y_2^{2^{n-1}} \bmod (c_2^2y_2^2)$ (resp. 0, $Q_n d_i=c_py_i^{p^{n-1}} \bmod (c_p^2y_i)$, $Q_n e=c_py_2^{p^{n-1}} \bmod (c_p^2y_2)$). Hence we can prove that Assumption 1.1 is satisfied for these cases.

2.3. The semi-dihedral groups SD_2. $HZ/2^*$ is detected by (D, Q) (see [**E-P**]). Hence we get the assumption.

§ 3. Relation to other theories.

Recall that $BP\langle n\rangle_*(-)$ is the homology theory with the coefficient $BP\langle n\rangle_* =Z_{(p)}[v_1, \cdots, v_n]$. Then similar arguments work for this theory.

PROPOSITION 3.1. *If Assumption* 1.1 *holds, then for* $n\geqq 1$,

$BP\langle n\rangle_*(BG)\cong BP\langle n\rangle_*\otimes_{BP*}BP_*(BG)$ *and we get (see* [**J-W 2**]*)*
$\operatorname{homdim}_{BP*}BP_*(BG)=2$.

Let us write by $\tilde{P}(n)_*(-)$ the homology theory with the coefficient $\tilde{P}(n)_*\cong BP_*/(v_1, \cdots, v_{n-1})\cong Z_{(p)}[v_n, \cdots]$.

PROPOSITION 3.2. *For groups in* § 2,

$$\tilde{P}(n)_*(BG)\cong \tilde{P}(n)_*Fs^{-1}(H(H^*; Q_n)\oplus H^{odd})/R$$

where R is the same relation in Theorem 1.10.

Recall that $\tilde{P}(n)_*(-)$ is the bordism theory of manifolds with singularities of type $(v_1, \cdots, v_{n-1})$ and there is the natural map $\rho: \tilde{P}(n-1)_*(-)\to\tilde{P}(n)_*(-)$. Hence $H(H^*; Q_{n-1})\subset H(H^*; Q_n)$ and each element in

$$s^{-1}(H(H^*; Q_n)-H(H^*; Q_{n-1}))=s^{-1}(\operatorname{Im} Q_{n-1}/\operatorname{Im} Q_n)$$

is represented by a manifold with singularities of type $(v_1, \cdots, v_{n-1})$ but not of type $(v_1, \cdots, v_{n-2})$.

§ 4. Explicit description of $BP_*(BD)$.

In this section we write down $BP_*(BD)$ more explicitely. Recall $D=\langle a, b\,|\,a^4=b^2=1, [a, b]=a^2\rangle$. The cohomology is given ([**E**], [**L**], [**T-Y**])

(4.1) $$H^{even}=Z[y_1, y_2, c_2]/(y_1^2+y_1y_2, 2y_1, 2y_2, 4c_2)$$
$$H^{odd}=(Z/2[y_1, y_2, c_2]/(y_1^2+y_1y_2))e$$
$$HZ/2^*=Z/2[x_1, x_2, u]/(x_1^2+x_1x_2)$$

where $x_i^2=y_i$, $c_2=u^2$ and $e=x_2u$ in $HZ/2^*$.

Since $Q_0u=ux_2$ and $Q_1e=y_2c_2$, we get

(4.2) $$H(H^*; Q_1)\cong H^{even}/(\text{Ideal}\,(y_2c_2))$$
$$\cong Z\{1\}\oplus Z/2\{y_1^i, y_2^i, y_1c_2^i\,|\,i\geqq 1\}\oplus Z/4\{c_2^i\,|\,i\geqq 1\}$$

where $Z/a\{b_1, \cdots, b_s\}$ is the free Z/a-module generated by $b_1, \cdots, b_s$.

From Lemma 1.5 and Theorem 1.10, we have

(4.3) $$\operatorname{gr} BP_*(BD)\cong BP_*\{1\}\oplus BP_*/2s^{-1}\{y_1^i, y_2^i, y_1c_2^i\}$$
$$\oplus BP/4s^{-1}\{c_2^i\}\oplus BP_*/(2, v_1)s^{-1}\{y_1^kc_2^je, y_2^kc_2^je\,|\,(k, j)\neq(0, 0)\}.$$

We will construct D-U-manifolds which represent elements in (4.3). Before doing this, we see how these generators in HZ_* are defined. Consider the extension

(4.4) $$0\longrightarrow\langle a\rangle=Z/4\longrightarrow D\longrightarrow\langle b\rangle=Z/2\longrightarrow 0$$

and the induced spectral sequence (see Lewis p. 510 [**L**])

(4.5) $$E^2_{*,*}=H_*(Z/2, H_*(Z/4))\Longrightarrow H_*(D).$$

The action b^* on $H^*(BZ/4)\cong Z[u]/(4u)$ is given by $b^*u=3u=-u$. Let us write $T=(1-b^*)$ and $N=(1+b^*)$. Then if $i|2$, $b^*u^i=u^i$ and $T=0$ and $N=2$, otherwise $T=2$ and $N=0$. Thus we get

(4.6) (i) for $*=odd>0$
$$\begin{cases} E^2_{0,*}\cong H_*(BZ/4)/\operatorname{Im}T\cong\begin{cases} Z/4\{s^{-1}u^i\} & \text{if } i|2\\ Z/2\{s^{-1}u^i\} & \text{otherwise}\end{cases}\\ E^2_{2j+1,*}\cong\operatorname{Ker}T/\operatorname{Im}N\cong\begin{cases} Z/2\{s^{-1}u^i\} & \text{if } i|2\\ Z/2\{s^{-1}2u^i\} & \text{otherwise}\end{cases}\\ E^2_{2j+2,*}\cong\operatorname{Ker}N/\operatorname{Im}T\cong\begin{cases} Z/2\{s^{-1}2u^i\} & \text{if } i|2\\ Z/2\{s^{-1}u^i\} & \text{otherwise}\end{cases}\end{cases}$$

for $*=even>0$ and all j, $E^2_{j,*}\cong 0$.

(ii) $E^2_{2i+1,0}\cong Z/2\{1\}$, $E^2_{even,0}\cong 0$.

By the universal coefficient theorem and (4.1) this spectral sequence collapses (compare Lewis p. 510). The elements $s^{-1}u$, $s^{-1}u^2$ in $E^2_{0,*}$ correspond to $s^{-1}y_1$, $s^{-1}c_2$, the element $s^{-1}2u\in E^2_{1,1}$ corresponds to $s^{-1}e$, and $s^{-1}u\in E^2_{2j-1,2}$ corresponds to $s^{-1}(y_1y_2^j)$. Moreover $1\in E^2_{2j-1,0}$ corresponds to $s^{-1}y_2^j$.

We define D-U-manifolds

(4.7) (i) $X(j, 0)=S^{2j-1}\times_{\langle a\rangle} D$, $X(0, i)=D\times_{\langle b\rangle} S^{2i-1}$

(ii) $X(2j, i)=S^{4j-1}\times S^{2i-1}$ for $ij>0$

(iii) $X(2j-1, i)=(S^{4j-3}\times Z/2)\times S^{2i-1}$ for $ij>0$.

Here the D-actions are given as follows. For (i) $a(z)=(\sqrt{-1}z)$ and $b(z)=(-z)$ identifying $z\in S^{2k-1}\subset \boldsymbol{C}^k$ for $k=j, i$ respectively. In the case (ii), think of S^{2i-1} as a D-manifold by $a^tb(z)=(-z)$ for all t, and the D-action on S^{4j-1} is the induced representation $\mathrm{Ind}_{\langle a\rangle}{}^D(\eta)$ of the usual 1-dimensional representation η of $\langle a\rangle$, that is, $a(z_1, z_2)=(\sqrt{-1}z_1, -\sqrt{-1}z_2)$ and $b(z_1, z_2)=(z_2, z_1)$ in $\boldsymbol{C}^j\times \boldsymbol{C}^j=\boldsymbol{C}^{2j}$. For case (iii), the D-action on S^{2i-1} is the same as (ii) and the D-action on $S^{4j-3}\times Z/2$ is the restriction of the induced representation $\mathrm{Ind}_{\langle a^2, b\rangle}(\eta')$ from the representation η' of $\langle a^2\rangle$, that is $a(z, s)=(sz, -s)$ and $b(z, s)=(z, -s)$ with $s\in\{1, -1\}\cong Z/2$.

It is immediate that $X(i, j)$ is a D-U-manifold. Thus we get the map

(4.8) $$\xi: X(j, i)/D \longrightarrow BD.$$

First consider the case (ii) and the fibering

$$S^{4j-1}/\langle a\rangle \longrightarrow X(j, i)/D \longrightarrow S^{2i-1}/\langle b\rangle$$

which induces the spectral sequence

(4.9) $$H_*(S^{2i-1}/\langle b\rangle ; H_*(S^{4j-1}/\langle a\rangle)) \Longrightarrow H_*(X(j, i)/D).$$

The map ξ in (4.8) induces the map of spectral sequences (4.9) to (4.5). The E_2-term of the spectral sequence (4.9) is isomorphic to (4.5) for $E^2_{r,s}$ if $s\leqq 4j-1$ and $r<2i-1$. But $E^2_{2i-1,*}\cong \mathrm{Ker}\, T$ and $E^2_{r,s}=0$ if $t\geqq 2i$ or $s\geqq 4j$. Then the fundamental class of $X(j, i)$ is the largest dimensional Z-generator and is represented in E_∞ in (4.9) by the nonzero element of $E^2_{2i-1,4j-1}$. Hence we know $X(2j, i)=s^{-1}ec_2^{j-1}y_1y_2^{i-1}=s^{-1}ec_2^{j-1}y_1^i$.

Similarly but more easily we know that $X(2j, 0)=s^{-1}c^j$, $X(2j-1, 0)=s^{-1}y_1c_2^{j-1}$, and $X(0, i)=s^{-1}y_2^i$.

For the case (iii)

$$X(2j-1, i)/D=((S^{4j-3}\times Z/2)/\langle a\rangle\times S^{2i-1})/\langle b\rangle$$
$$=S^{4j-3}/\langle a^2\rangle\times S^{2i-1}/\langle b\rangle.$$

Thus we have trivial fibering

$$S^{4j-3}/\langle a^2\rangle \longrightarrow X(2j-1, i)/D \longrightarrow S^{2i-1}/\langle b\rangle$$

and ξ induces a map of spectral sequences from the above to (4.4). Let $H_*(S^{4j-3}/\langle a^2\rangle)=s^{-1}Z[w]/(2w, w^{2j-1})$. Then $\xi_* s^{-1}w^k=2s^{-1}u$. Hence $X(2j-1, i)=s^{-1}ec_2^{j-1}y_2^{i-1}$ because both elements correspond to $\{2s^{-1}u^{2j-1}\}\in E^2_{2i-1,4j-1}$ in (4.5).

The only element which is not presented by an $X(j, i)$ is $s^{-1}y_1^j$ for $j\geqq 2$. Note that there is an automorphism λ of D such that $\lambda: b\mapsto ab$, $\lambda: a\mapsto a^3$. Then $\lambda s^{-1}y_2=s^{-1}y_2+s^{-1}y_1$. Take $X'(0, i)=D\times_{\langle ab\rangle}S^{2i-1}$ and this manifold represents $s^{-1}y_1^i+s^{-1}y_2^i$. Thus we have known that $X(j, i)$ and $X'(0, i')$, $i'\geqq 2$ generates $BP_*(BD)$ as a BP_*-module from (4.3).

Next consider relations $\sum v_n Q_{n*} Q_{k*}{}^{-1}(x)=0$. First consider the case $x=X(0, i)$. Since $s^{-1}y_2=Q_{0*}y_2$, we see $Q_{0*}{}^{-1}s^{-1}y_2^i=y_2^i$. The Q_{n*}-operation acts on HZ/p_*.

$$
\begin{aligned}
Q_{n*}y_2^i &= \sum\langle y_2^i, Q_n x_2 y_2^k\rangle x_2 y_2^k\,, \quad \text{where we recall } x_2^2=y_2 \\
&= \sum\langle y_2^i, y_2^{2^n+k}\rangle x_2 y_2^k = x_2 y_2^{i-2^n}\,.
\end{aligned}
$$

Therefore we have

$$
\sum v_n X(0, i-2^n+1)=0, \quad \sum v_n X'(0, i-2^n+1)=0. \tag{4.10}
$$

This relation is well known and is also given by the relation in $BP_*(BZ/2)$ and [2] the product of the formal group law in BP_*-theory (for example see [**J-W**], [**K-M**]).

When $x=X(2j, 0)$, the fact $Q_{0*}{}^{-1}s^{-1}(c_2^j)=0$ induces only the trivial relation. As for $x=X(2j-1, 0)$, the formula

$$
Q_{n*}c_2^j y_1=\sum\langle c_2^j y_1, Q_n c_2^k x_1\rangle c_2^k x_1=0 \quad \text{for } n\geqq 1
$$

follows the relation

$$
2X(2j-1, 0)=0\,. \tag{4.11}
$$

At last we consider the case $ij>0$. Since $s^{-1}y_2^i c_2^j e=c_2^j y_2^i u$ (see (4.1)), we get

$$
\begin{aligned}
Q_{n*}c_2^j y_2^i e &= \sum\langle c_2^j y_2^i e, Q_n c_2^k y_2^l u\rangle c_2^k y_2^l u \\
&= \sum\langle c_2^j y_2^i e, c_2^k y_2^l Q_n u\rangle c_2^k y_2^l u \\
&= \sum\langle c_2^{j-k} y_2^{i-l} e, Q_n u\rangle c_2^k y_2^l u\,.
\end{aligned} \tag{4.12}
$$

LEMMA 4.13. *Let $f_0=1$, $f_1=u+y_2$ and $f_{n+1}=uf_n^2+y_2 f_n^2+f_{n-1}^4 y_2 u^2$. Then $Q_n u=x_2 u f_n$.*

PROOF. At first recall $Q_0 u=ux_2$. The Q_1-action is

$$
Q_1 u=Sq^2 Q_0 u+Q_0 Sq^2 u=Sq^2(ux_2)=u^2 x_2+ux_2^3=ux_2(u+x_2^2)\,.
$$

By the induction on $n\geqq 1$, we see

$$\begin{aligned} Q_{n+1}u &= (Sq^{2^{n+1}}Q_n+Q_nSq^{2^{n+1}})u \\ &= Sq^{2^{n+1}}Q_nu = Sq^{2^{n+1}}(x_2uf_n), \quad \text{where } |x_2uf_n|=2^{n+1}+1 \\ &= x_2u^2f_n{}^2+x_2{}^3uf_n{}^2+x_2{}^2u^2Sq^{|f_n|-1}f_n\,. \end{aligned}$$

If $f_n=\sum\lambda_iu^iy_2{}^j$, then

$$Sq^{|f_n|-1}f_n = \sum\lambda_i i(ux_2)u^{2(i-1)}y_2{}^{2j} = ux_2(\partial f_n/\partial u)^2\,.$$

Therefore $Q_{n+1}u=ux_2(uf_n{}^2+x_2{}^2f_n{}^2+x_2{}^2u^2(\partial f_n/\partial u)^2)$. q.e.d.

Let us write $f_n=\sum f_{n,i}u^iy^{2^n-i-1}$. Then we get

$$\begin{aligned} Q_{n*}c_2{}^jy_2{}^ie &= \sum\langle c_2{}^ky_2{}^le,\ \sum f_{n,t}u^ty_2{}^{2^n-1-t}e\rangle c_2{}^{j-k}y_2{}^{i-l}u \\ &= \sum f_{n,2t}c_2{}^{j-t}y_2{}^{i-(2^n-1-2t)}u\,. \end{aligned}$$

Hence we have the relation

$$\sum\nolimits_{n=0} v_n(\sum f_{n,2t}X(j-t,\ i+2t+1-2^n)) = 0\,. \tag{4.14}$$

Next consider the relation such that $v_1X(j,i)+\cdots=0$. If $Q_{1*}w=c_2{}^jy_2{}^iu$, then

$$\begin{aligned} c_2{}^jy_2{}^iu &= \sum\langle w,\ Q_1c_2{}^ky_2{}^lu\rangle c_2{}^ky_2{}^lu \\ &= \sum\langle w,\ c_2{}^ky_2{}^le(u+y_2)\rangle c_2{}^ky_2{}^lu \end{aligned}$$

shows $w=c_2{}^jy_2{}^{i+1}e$ or $w=c_2{}^jy_2{}^ieu$. Since $Q_{0*}c_2{}^jy_2{}^{i+1}e=c_2{}^jy_2{}^{i+1}u$, the case $w=c_2{}^jy_2{}^{i+1}e$ gives a relation such that $2X(j,i+1)+\cdots=0$, which is contained in (4.14). Hence we need only the case $w=c_2{}^jy_2{}^ieu$,

$$\begin{aligned} Q_{n*}w &= \sum\langle c_2{}^jy_2{}^ieu,\ Q_nc_2{}^ky_2{}^lu\rangle c_2{}^ky_2{}^lu \\ &= \sum\langle c_2{}^ky_2{}^leu,\ \sum f_{n,t}u^ty_2{}^{2^n-1-t}e\rangle c_2{}^{j-k}y_2{}^{i-l}u \\ &= \sum f_{n,2t+1}c_2{}^{j-t}y_2{}^{i-(2^n-1-2t-1)}u\,. \end{aligned}$$

Therefore we get

$$\sum\nolimits_{n=1} v_n(\sum\nolimits_{t=0} f_{n,2t+1}X(j-t,\ i-2^n+2t+2)) = 0 \tag{4.15}$$

THEOREM 4.16. *There is a BP_*-module isomorphism*

$$BP_*(BD)\cong BP_*\{X(j,i),\ X'(0,i')\,|\,j,\ i\geqq 0,\ i'\geqq 2\}/R$$

where $R=((4.10),\ (4.11),\ (4.14),\ (4.15))\bmod(2,\ v_1,\ \cdots)^2$.

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