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Notes on the mean value property for certain degenerate elliptic operators

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Introduction.

The purpose of this paper is to study the mean value property and its applications to a certain class of degenerate elliptic operators. We shall treat the operators L_α defined by

$$L_\alpha(x, \partial_x) = -x_n \Delta - \alpha \partial_{x_n} \quad \text{for} \quad x=(x', x_n) \in \boldsymbol{R}_+^n, \tag{0.1}$$

where α is a real parameter and $\boldsymbol{R}_+^n$ is the Euclidian halfspace defined by $\{x=(x', x_n) \mid x' \in \boldsymbol{R}^{n-1}, x_n>0\}$.

Let Ω be a domain of $\boldsymbol{R}_+^n$ and we set

$$\begin{aligned} \underline{\Omega} &= \Omega \cup (\partial\Omega \cap \partial \boldsymbol{R}_+^n), \\ \partial\Omega^i &= \partial\Omega \setminus \partial\boldsymbol{R}_+^n. \end{aligned} \tag{0.2}$$

By $C^0(\Omega)$ and $C^0(\underline{\Omega})$ we denote the sets of all continuous functions on Ω and $\underline{\Omega}$ respectively.

With the operators L_α, we shall associate the modified mean value functions $M_{\alpha,\rho}u(a)$ of $u \in C^0(\Omega)$ (resp. $u \in C^0(\underline{\Omega})$) at a point $a \in \Omega$ (resp. $a \in \underline{\Omega}$). More precisely

DEFINITION 0.1. Let $a=(a', a_n)$ be an arbitrary point in Ω (resp. $\underline{\Omega}$), and let α and ρ be arbitrary positive numbers satisfying $\rho < \operatorname{dist}(a, \partial\Omega)$ (resp. $\rho < \operatorname{dist}(a, \partial\Omega^i)$). For $u \in C^0(\Omega)$ (resp. $u \in C^0(\underline{\Omega})$) we set

$$\begin{aligned} &M_{\alpha,\rho}u(a) \\ &\quad = C(\alpha)\rho^{1-n-\alpha}\int_0^1 \{s(1-s)\}^{\alpha/2-1} ds \int_{\partial B_\rho^+} x_n^\alpha u(x'+a', \gamma(x_n, a_n, s)) dS_x \end{aligned} \tag{0.3}$$

where

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$$\gamma(x_n, a_n, s) = \sqrt{(x_n - a_n)^2 s + (x_n + a_n)^2(1-s)},$$
$$\partial B_\rho^+ = \{x = (x', x_n) \in \boldsymbol{R}^n \mid |x| = \rho, x_n > 0\},$$
$$C(\alpha) = \frac{2^{\alpha-1}\pi^{-n/2}\Gamma((n+\alpha)/2)}{\Gamma(\alpha/2)},$$

and dS_x is the $(n-1)$-dimensional Lebesgue measure.

Since it holds that

$$|x_n - a_n| \leqq \gamma(x_n, a_n, s) \leqq x_n + a_n \tag{0.4}$$

and

$$\cup_{0\leqq s\leqq 1} \cup_{x\in\partial B_\rho^+} (a' + x', \gamma(x_n, a_n, s)) \subseteqq B_\rho^+(a) = \{x \in \boldsymbol{R}_+^n : |x - a| \leqq \rho\},$$

it is easy to see that the modified mean value functions $M_{\alpha,\rho}u(a)$ are well defined. We introduce the following notions as a substantial extension of the classical theory for the Laplacian in $\boldsymbol{R}^n$:

DEFINITION 0.2. Let Ω be a domain of $\boldsymbol{R}_+^n$ and u be of class $C^0(\Omega)$ (resp. $C^0(\underline{\Omega})$). Then u is said to be α-superharmonic in Ω (resp. α-superharmonic in $\underline{\Omega}$) if

$$M_{\alpha,\rho}u(a) \leqq u(a) \tag{0.5}$$

for any $a \in \Omega$ (resp. $a \in \underline{\Omega}$) and any $\rho \in (0, \operatorname{dist}(a, \partial\Omega))$ (resp. $\rho \in (0, \operatorname{dist}(a, \partial\Omega^i))$).

In case that $-u$ is α-superharmonic, u is said to be α-subharmonic.

DEFINITION 0.3. Let u be of class $C^0(\Omega)$ (resp. $C^0(\underline{\Omega})$). Then u is said to be α-harmonic in Ω (resp. α-harmonic in $\underline{\Omega}$) if

$$M_{\alpha,\rho}u(a) = u(a) \tag{0.6}$$

for any $a \in \Omega$ (resp. $\alpha \in \underline{\Omega}$) and any $\rho \in (0, \operatorname{dist}(a, \partial\Omega))$ (resp. $\rho \in (0, \operatorname{dist}(a, \partial\Omega^i))$).

Then we shall establish the mean value theorems for the operator L_α and give a necessary and sufficient condition for the α-harmonicity. As corollaries we shall make clear the structure of the α-harmonic functions which is deeply connected to the degeneracy of the operators L_α on the hyperplane $\{x \in \boldsymbol{R}^n \mid x_n = 0\}$.

In order to see in advance the role of $M_{\alpha,\rho}u(x)$ as well as what it means, we shall trace the definition of $M_{\alpha,\rho}u(x)$ to its origin assuming that $\Omega = \boldsymbol{R}_+^n$ and α is a positive integer in the rest of this subsection.

First we show:

LEMMA 0.1. *Let α be an arbitrary positive integer, and let u be of class $C^0(\boldsymbol{R}^1)$. Then it holds that*

$$M_{\alpha,\rho}u(|a|)=\frac{1}{\rho^{\alpha}|S^{\alpha}|}\int_{|z-a|=\rho}u(|z|)dS_{z}, \tag{0.7}$$

where

$$a=(a_1, a_2, \cdots, a_{\alpha+1})\in \boldsymbol{R}^{\alpha+1},\quad z=(z_1, z_2, \cdots, z_{\alpha+1})\in \boldsymbol{R}^{\alpha+1}.$$

PROOF. We shall make use of the polar coordinate system defined by

$$\begin{gathered} z=a+\rho\omega,\qquad a=|a|\omega',\qquad \text{with}\ \ \omega, \omega'\in S^{\alpha}, \\ |z|^2=|a|^2+\rho^2+2|a|\rho\cos\phi,\qquad \cos\phi=\omega\cdot\omega', \end{gathered} \tag{0.8}$$

$$\rho^{\alpha}dS_{\omega}^{\alpha}=(\rho\sin\phi)^{\alpha-1}dS_{\omega''}^{\alpha-1}\rho d\phi\qquad \text{with}\ \ 0<\phi<\pi,\ \ \omega''\in S^{\alpha-1}.$$

So we get

$$\begin{aligned} &\int_{|z-a|=\rho}u(|z|)dS_{z} \\ &\quad=|S^{\alpha-1}|\int_0^{\pi}\rho^{\alpha}(\sin\phi)^{\alpha-1}u(\sqrt{|a|^2+\rho^2+2\rho|a|\cos\phi})d\phi. \end{aligned} \tag{0.9}$$

Carrying out the change of variables defined by

$$\sin\phi=2\sqrt{s(1-s)},\qquad \text{that is,}\quad \cos\phi=2s-1, \tag{0.10}$$

the desired estimate follows. Q. E. D.

In a similar way we can show the following (the proof is omitted):

LEMMA 0.2. *Let u be of class $C^0(\boldsymbol{R}_{x'}^{n-1}, \boldsymbol{R}_{x_n}^{1})$, and let (a', b) be an arbitrary point in $\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{\alpha+1}$. Then we have*

$$M_{\alpha,\rho}u(a)=\frac{1}{\rho^{n+\alpha-1}|S^{n+\alpha-1}|}\int_{|(x',z)-(a',b)|=\rho}u(x', |z|)dS_{(x',z)}. \tag{0.11}$$

Here

$$\begin{gathered} a=(a', a_n)\ \ \text{with}\ \ a_n=|b|,\quad x=(x', x_n)\ \ \text{with}\ \ x_n=|z|, \\ (a', b), (x', z)\in \boldsymbol{R}^{n-1}\times\boldsymbol{R}^{\alpha+1}. \end{gathered}$$

Now it is clear that the definition of $M_{\alpha,\rho}u(a)$ comes from the equality (0.11) provided α is a positive integer.

Here we remark that the integral on the righthand side of (0.3) is not convergent if $\alpha\in(-1, 0]$, nevertheless we see that $M_{\alpha,\rho}u(a)$ can be continued analytically with respect to α in that case. In particular if $\alpha=0$, we have

$$\begin{aligned} &M_{0,\rho}u(a) \\ &\quad=2^{-1}\pi^{-n/2}\Gamma\Big(\frac{n}{2}\Big)\rho^{1-n}\int_{\partial B_{\rho}^{+}}\{u(x'+a', x_n+a_n)+u(x'+a', |x_n-a_n|)\}dS_x. \end{aligned} \tag{0.12}$$

The potential theory has been developed together with the study of uniformly elliptic differential operators represented by the Laplacian Δ, and as a result it has become an extensive field of research in both mathematics and mathematical physics (see [**5**], [**6**], [**7**] and [**12**]). Suitably extended versions are also applicable to nonlinear elliptic equations to some extent (see [**1**], [**2**], [**10**], [**13**], [**14**], [**15**] and [**16**]). But the development of the potential theory seems to be rather limitted in the study of genuinely degenerate operators. Therefore we also indicate in this paper that the classical potential theory for the uniformly elliptic operators can be extended on the same lines to the degenerate operator L_α as a typical example.

This paper is organized in the following way. In § 1 we shall describe the modified mean value property and give a necessary and sufficient condition for the α-harmonicity in terms of the mean value theorem. In § 2 we prove Theorems 1.1 and 1.2 stated in the previous section. The proof of Theorem 1.1 will be parallel to that for the Laplacian. In Appendix 1 we collect mostly without proofs the basic properties of the potential kernel K_α and we also give the proof of a lemma which is needed in § 1. In Appendix 2 we shall prove a theorem on representation of superharmonic function.

§ 1. Main results.

In this section we shall describe the fundamental properties of the operator L_α defined by (0.1) in terms of the modified mean value $M_{\alpha,\rho}u(a)$ prepared in the previous section. Let Ω be a domain of $\boldsymbol{R}^n_+$. First we assume that $\alpha>0$.

THEOREM 1.1. *Suppose that $\alpha>0$ and that u is of class $C^0(\Omega)$ (resp. $C^0(\underline{\Omega})$). Moreover we suppose that u is α-harmonic, that is:*

$$M_{\alpha,\rho}u(a)=u(a) \tag{1.1}$$

holds for any $a\in\Omega$ (resp. $a\in\underline{\Omega}$) and any $\rho\in(0, \operatorname{dist}(a, \partial\Omega))$ (resp. $\rho\in(0, \operatorname{dist}(a, \partial\Omega^i))$). Then u is of class $C^\infty(\Omega)$ (resp. $C^\infty(\underline{\Omega})$) and satisfies $L_\alpha u=0$ in Ω (resp. $\underline{\Omega}$).

THEOREM 1.2. *Suppose that $\alpha>0$ and that u is of class $C^2(\Omega)$ (resp. $C^2(\underline{\Omega})$). Then u is α-superharmonic in Ω (resp. $\underline{\Omega}$) if and only if $L_\alpha u\geqq 0$ holds in Ω (resp. $\underline{\Omega}$).*

From these theorems we immediately have:

COROLLARY 1.3. *Suppose that $\alpha>0$ and that u is of class $C^2(\Omega)$ (resp. $C^2(\underline{\Omega})$). Moreover we suppose that u satisfies $L_\alpha u=0$ in Ω (resp. $\underline{\Omega}$). Then u is of class $C^\infty(\Omega)$ (resp. $C^\infty(\underline{\Omega})$).*

Now we proceed to the case $\alpha<1$. In this case we shall treat α-harmonic functions with the Dirichlet boundary condition, and characterize them using the previous results. For simplicity we assume that $\Omega=\boldsymbol{R}_{+}^{n}$. Let $u\in C^{0}(\overline{\boldsymbol{R}_{+}^{n}})$ be a solution of the equation $L_{\alpha}u=0$ with the Dirichlet boundary condition $u|_{x_{n}=0}=0$. By u^{+} we denote the extension of u to $\boldsymbol{R}^{n}$ obtained by setting $u=0$ on $\boldsymbol{R}^{n}\setminus\overline{\boldsymbol{R}_{+}^{n}}$. Then it follows from Lemma A.3 in Appendix 1 that $L_{\alpha}u^{+}=0$ in $\boldsymbol{R}^{n}$, where the differentiations are taken in the distribution sense. Then by virtue of the fundamental solutions (A.2) for $L_{\alpha}(\alpha<2)$, (A.5) and the approximation arguments (cf. see [**9**] for example) it follows that:

For any point $x_{0}\in\boldsymbol{R}_{+}^{n}$, there is a real analytic function v on $\overline{\boldsymbol{R}_{+}^{n}}$ such that

$$u(x)=x_{n}^{1-\alpha}v(x),\qquad \text{in some neighborhood of } x_{0}. \tag{1.2}$$

In fact one can show the equality that $\varphi(x)u(x)=\mathcal{E}_{\alpha}(L_{\alpha}(\varphi u^{+}))(x)$ in some neighborhood of each point $x_{0}\in\boldsymbol{R}_{+}^{n}$, where $\varphi\in C_{0}^{\infty}(\overline{\boldsymbol{R}_{+}^{n}})$, $\varphi=1$ near x_{0}. Here we used the fact $L_{\alpha}(\varphi u^{+})$ can be approximated by C^{2} functions.

Noting that $L_{\alpha}(x_{n}^{1-\alpha})=0$, we have

$$L_{\alpha}u=x_{n}^{1-\alpha}L_{\alpha}v+[L_{\alpha},\ x_{n}^{1-\alpha}]v=x_{n}^{1-\alpha}L_{2-\alpha}v\,. \tag{1.3}$$

Since $2-\alpha>0$, it follows at once from the theorems that

THEOREM 1.4. *Suppose that $\alpha<1$ and that $u\in C^{0}(\overline{\boldsymbol{R}_{+}^{n}})$ satisfies the equation*

$$\begin{aligned} L_{\alpha}u&=0,\qquad \textit{in}\ \ \boldsymbol{R}_{+}^{n},\\ u|_{x_{n}=0}&=0\,. \end{aligned} \tag{1.4}$$

Then

$$u(x)=x_{n}^{1-\alpha}M_{2-\alpha,\rho}v(x)\qquad \textit{with}\ \ v(x)=x_{n}^{\alpha-1}u(x)\,, \tag{1.5}$$

where v is a real analytic function on $\overline{\boldsymbol{R}_{+}^{n}}$ and ρ is an arbitrary positive number.

§2. The proofs of Theorem 1.1 and 1.2.

We prepare further notations. Take a nonnegative smooth function Φ such that $\Phi(\rho)=1$ in some neighborhood of the origin and such that

$$\int_{0}^{\infty}\rho^{n+\alpha-1}\Phi(\rho)d\rho=1,\qquad \Phi(|x|)\in C_{0}^{\infty}(\boldsymbol{R}^{n})\,. \tag{2.1}$$

Now we define a mollification of u as follows:

$$\begin{aligned} u_{\varepsilon,\alpha}(x)=C(\alpha)\varepsilon^{-n-\alpha}&\int_{0}^{1}\{s(1-s)\}^{\alpha/2-1}ds\\ &\times\int_{\boldsymbol{R}^{n-1}}\int_{0}^{\infty}r^{\alpha}u(x'-y',\ r)\Phi\Big(\frac{\sqrt{|y'|^{2}+\gamma(x_{n},\ r,\ s)^{2}}}{\varepsilon}\Big)dy'dr\,, \end{aligned} \tag{2.2}$$

for $u\in C^0(\boldsymbol{R}_+^n)$ (resp. $u\in C^0(\overline{\boldsymbol{R}_+^n})$) and $\varepsilon>0$. Then obviously $u_{\varepsilon,\alpha}\in C^\infty(\boldsymbol{R}_+^n)$ (resp. $u_{\varepsilon,\alpha}\in C^\infty(\overline{\boldsymbol{R}_+^n})$) for any $\alpha>0$ and $\varepsilon>0$. Moreover we show:

LEMMA 2.1. *It holds that*

$$u_{\varepsilon,\alpha}(x)=C(\alpha)\varepsilon^{-n-\alpha}\int_0^1\{s(1-s)\}^{\alpha/2-1}ds \times\int_{R^{n-1}}\int_0^\infty u(x'-y',\gamma(x_n,r,s))r^\alpha\Phi\Big(\frac{\sqrt{|y'|^2+r^2}}{\varepsilon}\Big)dy'dr \tag{2.3}$$

for any $\alpha>0$ *and* $\varepsilon>0$.

PROOF OF THEOREM 1.1.

Admitting this for a moment, we show that $u_{\varepsilon,\alpha}(x)=u(x)$ for a sufficiently small $\varepsilon>0$ if u is α-harmonic in Ω (resp. $\underline{\Omega}$). In fact we immediately have

$$\begin{aligned} u_{\varepsilon,\alpha}(x) &= C(\alpha)\int_0^\infty\Phi(\rho/\varepsilon)\varepsilon^{-n-\alpha}d\rho\int_0^1\{s(1-s)\}^{\alpha/2-1}ds \\ &\quad\times\int_{\partial B_\rho^+}r^\alpha u(x'-y',\gamma(x_n,r,s))dS_{(y',r)} \\ &= u(x)\int_0^\infty\rho^{n+\alpha-1}\Phi(\rho)d\rho=u(x), \end{aligned}\tag{2.4}$$

where ε and ρ are sufficiently small positive numbers and $dS_{(y',r)}$ is the $(n-1)$-dimensional Lebesgue measure. Therefore we have proved that u is smooth if u is α-harmonic. The equality $L_\alpha u=0$ easily follows from (2.12) if u is smooth and α-harmonic. Q. E. D.

PROOF OF LEMMA 2.1. Let us set for $t>0$

$$\begin{aligned} \varphi\#\phi(t) &= 2^{\alpha-1}\int_0^\infty r^\alpha\varphi(r)dr\int_0^1\{s(1-s)\}^{\alpha/2-1}\phi(\gamma(t,r,s))ds \\ &= \int_0^\infty r^\alpha\varphi(r)dr\int_0^\pi\phi(\sqrt{t^2+r^2-2tr\cos\theta})\sin^{\alpha-1}\theta d\theta, \end{aligned}\tag{2.5}$$

for $\varphi,\phi\in C_0^0(\boldsymbol{R}_+^n)$. Here we used the change of variable defined by $\cos\theta=2s-1$.

To prove Lemma 2.1 we have only to show the equality $\varphi\#\phi=\phi\#\varphi$, which implies the validity of the commutativity law. First we put $r=t\tau$ and then divide the integral into two terms. So that we obtain

$$\varphi\#\phi(t)=I_1+I_2,$$

$$\begin{aligned} I_1 &= t^{\alpha+1}\int_0^1\varphi(t\tau)\tau^\alpha d\tau\int_0^\pi\phi(t\sqrt{1+\tau^2-2\tau\cos\theta})\sin^{\alpha-1}\theta d\theta, \\ I_2 &= t^{\alpha+1}\int_1^\infty\varphi(t\tau)\tau^\alpha d\tau\int_0^\pi\phi(t\sqrt{1+\tau^2-2\tau\cos\theta})\sin^{\alpha-1}\theta d\theta. \end{aligned}\tag{2.6}$$

Secondly we carry out a change of variables in the inner integral defined by $\sigma^2=1+\tau^2-2\tau\cos\theta$, so we get

(2.7)
$$I_1=2^{1-\alpha}t^{\alpha+1}\int_0^1\varphi(t\tau)\tau d\tau\int_{1-\tau}^{1+\tau}g(\sigma,\tau)^{\alpha-2}\psi(t\sigma)\sigma d\sigma\,,$$
$$I_2=2^{1-\alpha}t^{\alpha+1}\int_1^\infty\varphi(t\tau)\tau d\tau\int_{\tau-1}^{\tau+1}g(\sigma,\tau)^{\alpha-2}\psi(t\sigma)\sigma d\sigma\,,$$

where

$$g(\sigma,\tau)=\sqrt{|2(\sigma^2+\tau^2)-(\tau^2-\sigma^2)^2-1|}\,.$$

Therefore we get

(2.8)
$$\varphi\#\psi(t)=2^{1-\alpha}t^{\alpha+1}\int_{\Lambda^{\sigma,\tau}}\sigma\tau g(\sigma,\tau)^{\alpha-2}\varphi(t\tau)\psi(t\sigma)d\sigma d\tau\,,$$

where

$$\Lambda^{\sigma,\tau}=\{(\sigma,\tau)\in \boldsymbol{R}_+\times\boldsymbol{R}_+\,|\,|1-\tau|\leqq\sigma\leqq 1+\tau\}\,.$$

After all we have proved the commutativity law for the #-product, because $g(\sigma,\tau)$ and $\Lambda^{\sigma,\tau}$ are symmetric with respect to σ and τ.

PROOF OF THEOREM 1.2. We shall deal with the case that $\Omega=\overline{\boldsymbol{R}^n_+}$. Since the operator L_α is elliptic in $\boldsymbol{R}^n_+$, the proof in the general case follows in a similar way.

Hence we assume that u is of class $C^2(\overline{\boldsymbol{R}^n_+})$. First we recall that

$$\gamma(x_n,a_n,s)=[x_n^2+a_n^2+2a_nx_n(1-2s)]^{1/2}\quad\text{and}\quad\gamma(0,a_n,s)=a_n\,.$$

Assume that $a_n>0$, then by Taylar's expansion formula we get

(2.9)
$$\begin{aligned}u(x'+a',\ &\gamma(x_n,a_n,s))-u(a)\\ &=\sum_{j=1}^{n-1}x_j\partial_ju(a)+(1-2s)x_n\partial_nu(a)\\ &\quad+2^{-1}\sum_{i,j\leqq n-1}x_ix_j\partial_i\partial_ju(a)+(1-2s)\sum_{j=1}^{n-1}x_nx_j\partial_n\partial_ju(a)\\ &\quad+2^{-1}(1-2s)^2x_n^2\partial_n^2u(a)+2s(1-s)a_n^{-1}x_n^2\partial_nu(a)+o(|x|^2)\,.\end{aligned}$$

If $a_n=0$, simply we have

(2.9*)
$$u(x'+a',\gamma(x_n,a_n,s))-u(a)=\sum_{j=1}^{n}x_j\partial_ju(a)+o(|x|)\,.$$

We prepare the following lemmas. (The proof is omitted.)

LEMMA 2.2. *Suppose that $k(j)\geqq 0$ for $j=1,\cdots,n$. Then we have*

(2.10)
$$\int_{S^{n-1}}|x_1|^{k(1)}\cdot|x_2|^{k(2)}\cdots|x_n|^{k(n)}dS_x$$
$$=2\prod_{i=1}^{n}\Gamma\Big(\frac{k(i)+1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big(\sum_{j=1}^{n}k(j)+n\Big)\Big)^{-1}.$$

LEMMA 2.3. *Let* $\mathrm{Re}\,\beta>0$. *Then we have*

(2.11)
$$\pi^{1/2}\Gamma(\beta)=2^{\beta-1}\Gamma\Big(\frac{\beta}{2}\Big)\Gamma\Big(\frac{\beta+1}{2}\Big).$$

Using these lemmas we have

(2.12)
$$M_{\alpha,\rho}u(a)=u(a)-\rho^2\{2a_n(n+\alpha)\}^{-1}L_\alpha u(a)+o(\rho^2)$$
$$M_{\alpha,\rho}u(a)=u(a)+c(\alpha,\,n)\alpha\rho\partial_n u(a)+o(\rho),\qquad \text{if}\quad a_n=0,$$

where

$$c(\alpha,\,n)=\frac{\alpha 2^{\alpha-2}\Gamma(\alpha/2)^2\Gamma((n+\alpha)/2)}{\pi^{1/2}\Gamma(\alpha)\Gamma((n+\alpha+1)/2)}.$$

Since ρ and a are arbitrary, we have $L_\alpha u\geqq 0$ if u is α-superharmonic. We note that if u is α-harmonic in $\boldsymbol{R}_+^n$, then the equality $L_\alpha u=0$ follows from (2.12). Moreover if u is α-harmonic in $\overline{\boldsymbol{R}_+^n}$, then $\partial_n u(x)$ also vanishes on the boundary.

We proceed to the proof of the converse. Assume that $L_\alpha u\geqq 0$. Then it suffices to show that $\partial_\rho M_{\alpha,\rho}u(y)\leqq 0$ for any $y\in\overline{\boldsymbol{R}_+^n}$. In fact the α-superharmonicity follows from this inequality and the property

(2.13)
$$\lim_{\rho\to 0}M_{\alpha,\rho}u(y)=u(y).$$

Let us set $v=u(x'+y',\,\gamma(x_n,\,y_n,\,s))$. By $\partial v/\partial s$ and $\partial v/\partial\gamma$ we simply denote the derivatives of v with respect to s and γ respectively. Then we have

(2.14)
$$\frac{\partial}{\partial s}v=-\frac{2x_ny_n}{\gamma}\frac{\partial}{\partial\gamma}v,$$

(2.15)
$$\frac{\partial^2}{\partial s^2}v=\frac{4x_n^2y_n^2}{\gamma^2}\Big(\frac{\partial^2}{\partial\gamma^2}-\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\Big)v,$$

and hence

(2.16)
$$\Big(\Delta_x+\frac{\alpha}{x_n}\frac{\partial}{\partial x_n}\Big)v$$
$$=\Big(\Delta_{x'}+\frac{\partial^2}{\partial\gamma^2}+\frac{\alpha}{\gamma}\frac{\partial}{\partial\gamma}\Big)v-\frac{4y_n^2s(1-s)}{\gamma^2}\Big(\frac{\partial^2}{\partial\gamma^2}v-\frac{1}{\gamma}\frac{\partial}{\partial\gamma}v\Big)+\frac{\alpha y_n(1-2s)}{\gamma x_n}\frac{\partial}{\partial\gamma}v$$
$$=\Big(\Delta_{x'}+\frac{\partial^2}{\partial\gamma^2}+\frac{\alpha}{\gamma}\frac{\partial}{\partial\gamma}\Big)v-\frac{1}{x_n^2}\Big(s(1-s)\frac{\partial^2}{\partial s^2}v+\frac{\alpha}{2}(1-2s)\frac{\partial}{\partial s}v\Big),$$

where $\Delta_x=\sum_{j=1}^n\partial_{x_j}^2$ and $\Delta_{x'}=\sum_{j=1}^{n-1}\partial_{x_j}^2$.

By the homogeneity with respect to ρ we have

$$(2.17)\quad C(\alpha)^{-1}\partial_\rho M_{\alpha,\rho}u(y)$$
$$=\rho^{1-\alpha-n}\int_0^1\{s(1-s)\}^{\alpha/2-1}ds\int_{\partial B_\rho^+}x_n^\alpha\partial_\nu u(x'+y',\ \gamma(x_n,\ y_n,\ s))dS_x$$
$$=\lim_{\varepsilon\to 0}\rho^{1-\alpha-n}\int_0^1\{s(1-s)\}^{\alpha/2-1}ds\int_{\partial B_{\rho,\varepsilon}^+}x_n^\alpha\partial_\nu u(x'+y',\ \gamma(x_n,\ y_n,\ s))dS_x,$$

where ν is the outer normal and $B_{\rho,\varepsilon}^+=\{x:\ |x|\leqq\rho,\ \varepsilon\leqq x_n\leqq\rho\}$ for $0<\varepsilon<\rho$.

We also note that $\partial_{x_n}\gamma=\gamma^{-1}[(x_n-y_n)s+(x_n+y_n)(1-s)]$ is bounded. Then by Green's formula we get

$$(2.18)\quad C(\alpha)^{-1}\partial_\rho M_{\alpha,\rho}u(y)$$
$$=\lim_{\varepsilon\to 0}\rho^{1-\alpha-n}\int_0^1\{s(1-s)\}^{\alpha/2-1}ds$$
$$\times\int_{B_{\rho,\varepsilon}^+}x_n^\alpha\Big(\Delta_x+\alpha x_n^{-1}\frac{\partial}{\partial x_n}\Big)u(x'+z',\ \gamma(x_n,\ y_n,\ s))dx$$
$$=\lim_{\varepsilon\to 0}\rho^{1-\alpha-n}\int_0^1\{s(1-s)\}^{\alpha/2-1}ds$$
$$\times\int_{B_{\rho,\varepsilon}^+}x_n^\alpha\Big(\Delta_{x'}+\frac{\partial^2}{\partial\gamma^2}+\alpha\gamma^{-1}\frac{\partial}{\partial\gamma}\Big)u(x'+z',\ \gamma(x_n,\ y_n,\ s))dx$$
$$-\lim_{\varepsilon\to 0}\rho^{1-\alpha-n}\int_0^1\{s(1-s)\}^{\alpha/2-1}ds$$
$$\times\int_{B_{\rho,\varepsilon}^+}x_n^\alpha\Big[\frac{4y_n^2s(1-s)}{\gamma^2}\Big(\frac{\partial^2}{\partial\gamma^2}-\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\Big)-\frac{\alpha y_n(1-2s)}{\gamma x_n}\frac{\partial}{\partial\gamma}\Big]u(x'+y',\ \gamma)dx$$
$$=\lim_{\varepsilon\to 0}\rho^{1-\alpha-n}\int_0^1\{s(1-s)\}^{\alpha/2-1}ds$$
$$\times\int_{B_{\rho,\varepsilon}^+}x_n^\alpha\Big(\Delta_{x'}+\frac{\partial^2}{\partial\gamma^2}+\alpha\gamma^{-1}\frac{\partial}{\partial\gamma}\Big)u(x'+z',\ \gamma(x_n,\ y_n,\ s))dx$$
$$-\lim_{\varepsilon\to 0}\rho^{1-\alpha-n}\int_0^1\{s(1-s)\}^{\alpha/2-1}ds$$
$$\times\int_{B_{\rho,\varepsilon}^+}x_n^{\alpha-2}\Big(s(1-s)\frac{\partial^2}{\partial s^2}+\frac{\alpha}{2}(1-2s)\frac{\partial}{\partial s}\Big)u(x'+y',\ \gamma)dx$$

(using integration by parts with respect to s)

$$=\lim_{\varepsilon\to 0}\rho^{1-\alpha-n}\int_0^1\{s(1-s)\}^{\alpha/2-1}ds$$
$$\times\int_{B_{\rho,\varepsilon}^+}x_n^\alpha\Big[\Big(\Delta_z+\alpha z_n^{-1}\frac{\partial}{\partial z_n}\Big)u(x'+z',\ z_n)\Big]_{\substack{z'=y'\\ z_n=\gamma(x_n,\ y_n,\ s)}}dx$$

(using (2.19) in Lemma 2.4)

$$= \rho^{1-\alpha-n}\int_0^1 \{s(1-s)\}^{\alpha/2-1}ds$$

$$\times\int_{B_\rho^+} x_n^\alpha \left[\left(\Delta_z+\alpha z_n^{-1}\frac{\partial}{\partial z_n}\right)u(x'+z',\, z_n)\right]_{\substack{z'=y'\\ z_n=\gamma(x_n,\, y_n,\, t)}} dx \leqq 0,$$

(since u is α-superharmonic)

where $\Delta_z=\sum_{j=1}^n \partial^2/\partial z_j^2$. Q. E. D.

LEMMA 2.4. *Let* $\alpha>0$. *Then*

$$I(y,\, \alpha) = \int_0^1 x^\alpha dx \int_0^1 \frac{1}{\gamma(x,\, y,\, s)}[s(1-s)]^{\alpha/2-1}ds \tag{2.19}$$

is finite if $y\geqq 0$.

PROOF OF LEMMA 2.4. Let us set

$$\begin{aligned} J(x,\, y,\, \alpha) &= \int_0^{1/2} \frac{x^\alpha s^{\alpha/2-1}}{(|x-y|^2+4xys)^{1/2}}ds, \\ K(A,\, \alpha) &= \int_0^A t^{\alpha/2-1}(1+t)^{1/2}dt, \qquad \text{for} \quad A=\frac{2xy}{|x-y|^2}. \end{aligned} \tag{2.20}$$

Then we have

$$I(y,\, \alpha) \leqq 2\max(1,\, 2^{1-\alpha/2})\int_0^1 J(x,\, y,\, \alpha)x^\alpha dx, \tag{2.21}$$

$$J(x,\, y,\, \alpha) = (4xy)^{-\alpha/2}x^\alpha|x-y|^{\alpha-1}K(A,\, \alpha), \tag{2.22}$$

and

$$K(A,\, \alpha) \leqq Const. \begin{cases} \left(\dfrac{A}{1+A}\right)^{\alpha/2}, & \text{for} \quad 0<\alpha<1, \\ \left(\dfrac{A}{1+A}\right)^{1/2}\cdot\log(2+A), & \text{for} \quad \alpha=1, \\ A^{\alpha/2}(1+A)^{-1/2}, & \text{for} \quad 1<\alpha. \end{cases} \tag{2.23}$$

Hence $J(x,\, y,\, \alpha)$ satisfies

$$J(x,\, y,\, \alpha) \leqq Const. \begin{cases} \left(\dfrac{x}{|x+y|}\right)^\alpha |x-y|^{\alpha-1}, & \text{for} \quad 0<\alpha<1, \\ \dfrac{x}{|x+y|}\left|\log\dfrac{|x+y|}{|x-y|}\right|, & \text{for} \quad \alpha=1, \\ \dfrac{x^\alpha}{|x+y|}, & \text{for} \quad 1<\alpha. \end{cases} \tag{2.24}$$

So that the desired estimates follow. Q. E. D.

Appendix 1. Potential kernels and the proof of Lemma A.3.

In this section we shall introduce potential kernels closely related to the operators L_α and A_α defined by

$$A_\alpha = -\nabla\cdot(x_n^\alpha \nabla\cdot) = x_n^{\alpha-1} L_\alpha. \tag{A.1}$$

Let us set

$$\begin{aligned} &E_\alpha(x, y) = D_\alpha y_n^{\alpha-1}|x-y^*|^{-\alpha}|x-y|^{2-n}F(\alpha, \omega), \qquad \alpha > 0, \\ &\mathcal{E}_\alpha(x, y) = D_{2-\alpha} x_n^{1-\alpha}|x-y^*|^{\alpha-2}|x-y|^{2-n}F(2-\alpha, \omega), \qquad \alpha < 2, \\ &K_\alpha(x, y) = D_\alpha |x-y^*|^{-\alpha}|x-y|^{2-n}F(\alpha, \omega), \qquad \alpha > 0, \end{aligned} \tag{A.2}$$

where

$$y^* = (y', -y_n),$$

$$\omega = 1 - \frac{|x-y|^2}{|x-y^*|^2} = \frac{4x_n y_n}{|x-y^*|^2},$$

$$D_\alpha = \frac{2^{\alpha-2}\pi^{-n/2}\Gamma((n+\alpha-2)/2)\Gamma(\alpha/2)}{\Gamma(\alpha)}$$

and $F(\alpha, \omega)$ is a hypergeometric function defined by

$$\begin{aligned} F(\alpha, \omega) &= \frac{\Gamma(\alpha)}{\Gamma((\alpha+2-n)/2)\Gamma(\alpha/2)} \sum_{j=0}^{\infty} \frac{\Gamma(j+(\alpha+2-n)/2)\Gamma(j+\alpha/2)}{\Gamma(j+\alpha)} \frac{\omega^j}{j!}, \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)^2} |x-y^*|^\alpha |x-y|^{n-2} \\ &\quad \times \int_0^1 (|x-y|^2(1-\theta) + |x-y^*|^2\theta)^{-(n-2+\alpha)/2} [\theta(1-\theta)]^{\alpha/2-1} d\theta. \end{aligned} \tag{A.3}$$

Moreover $F(\alpha, \omega)$ satisfies the following estimates (the proof is omitted):

PROPOSITION A.1. *There exist positive numbers c_1 and c_2 such that*

$$\begin{aligned} &c_1^{-1} \leqq F(\alpha, \omega) \leqq c_1, \qquad n \geqq 3, \qquad \alpha > 0, \\ &c_2^{-1} \leqq \frac{F(\alpha, \omega)}{\log[2+|x-y^*|/|x-y|]} \leqq c_2, \qquad n = 2, \qquad \alpha > 0. \end{aligned} \tag{A.4}$$

Here c_1 and c_2 are independent of x and y.

Then $E_\alpha(x, y)$, $\mathcal{E}_\alpha(x, y)$ and $K_\alpha(x, y)$ are Green functions for the operators L_α and A_α in the following sense (cf. [8] and [11]):

PROPOSITION A.2. *Suppose that $f \in C^0(\overline{\boldsymbol{R}_+^n}) \cap \mathcal{E}'$ and $g \in C^2(\overline{\boldsymbol{R}_+^n}) \cap \mathcal{E}'$. Let us set*

$$\text{(A.5)}\qquad \begin{cases} u(x)=E_\alpha f(x)=\int_{R_+^n} E_\alpha(x,\ y)f(y)dy, \\ v(x)=\mathcal{E}_\alpha f(x)=\int_{R_+^n} \mathcal{E}_\alpha(x,\ y)f(y)dy, \\ w(x)=K_\alpha f(x)=\int_{R_+^n} K_\alpha(x,\ y)f(y)dy. \end{cases}$$

Then it holds that

$$\text{(A.6)}\qquad \begin{cases} L_\alpha u(x)=f(x), & \operatorname{Re}\alpha>0, \\ L_\alpha v(x)=f(x) \ \text{ and } \ v|_{x_n=0}=0, & \operatorname{Re}\alpha<1, \\ A_\alpha w(x)=f(x), & \operatorname{Re}\alpha>0. \end{cases}$$

Moreover it holds that

$$\text{(A.7)}\qquad \begin{cases} g(x)=E_\alpha(L_\alpha g)(x), & \operatorname{Re}\alpha>0, \\ g(x)=K_\alpha(A_\alpha g)(x), & \operatorname{Re}\alpha>0, \\ g(x)=\mathcal{E}_\alpha(L_\alpha g)(x), & \operatorname{Re}\alpha<1,\ g(x)|_{x_n=0}=0. \end{cases}$$

Here, by $\mathcal{E}'$ we mean the set of distributions having compact support.

Here we note that kernels are connected one another in the following way: For $\alpha>0$,

$$\text{(A.8)}\qquad \begin{aligned} &K_\alpha(x,\ y)=K_\alpha(y,\ x)=y_n^{1-\alpha}E_\alpha(x,\ y)=x_n^{1-\alpha}\mathcal{E}_{2-\alpha}(x,\ y), \\ &E_\alpha(x,\ y)=\mathcal{E}_{2-\alpha}(y,\ x) \ \text{ and } \ A_\alpha=x_n^{\alpha-1}L_\alpha={}^tL_\alpha(x_n^{\alpha-1}\cdot). \end{aligned}$$

Here ${}^tL_\alpha$ is the formal transpose of L_α. In particular if $n\geqq 3$ and $\alpha=n-2$, then we can compute $F(\alpha,\ \omega)$ to obtain

$$\text{(A.9)}\qquad K_{n-2}(x,\ y)=2^{n-4}\pi^{-n/2}\Gamma\Big(\frac{n-2}{2}\Big)\cdot(|x-y^*|\,|x-y|)^{2-n}.$$

By virtue of these kernels, it is not difficult to develop the so-called potential theory for the operator L_α (or A_α) in the same lines of the classical theory.

Lastly we show the following lemma which was needed in § 2.

Lemma A.3. *Let $\alpha<1$ and $u\in C^0(\overline{R_+^n})$. Assume that u satisfies*

$$\text{(A.10)}\qquad L_\alpha u=0, \qquad u|_{x_n=0}=0, \qquad \textit{then}$$

$$\text{(A.11)}\qquad \lim_{x_n\to 0} x_n\partial_{x_n}u(x)=0.$$

Proof. First we remark that since the operator L_α is elliptic in R_+^n, $x_n\partial_{x_n}u(x)$ is continuous there. Let $x_0\in R_+^n$ and choose a smooth function φ so that

$$\varphi(x)=\begin{cases}1, & x\in B_{d/4}(x_0),\\ 0, & x\notin B_{d/2}(x_0).\end{cases} \quad \text{and} \quad |\partial^k\varphi|\leqq c_k d^{-k}, \tag{A.12}$$

where $d=\mathrm{dist}(x_0, \{x_n=0\})$ and c_k is a positive number independent of each (x_0, x). Then we have from (A.7)

$$\begin{aligned} L_\alpha(u\varphi) &= uL_\alpha\varphi-2x_n\nabla\varphi\cdot\nabla u, \\ (u\varphi)(x) &= \mathcal{E}_\alpha(uL_\alpha\varphi-2x_n\nabla\varphi\cdot\nabla u)(x). \end{aligned} \tag{A.13}$$

Integration by parts gives

$$x_n\partial_{x_n}(u\varphi)(x)=\int_{B_{d/2}(x_0)\setminus B_{d/4}(x_0)}\mathcal{P}(x, y)u(y)dy, \tag{A.14}$$

for any $x\in B_{d/2}(x_0)$, where

$$\mathcal{P}(x, y)=x_n\partial_{x_n}[\mathcal{E}_\alpha(x, y)(L_\alpha\varphi)(y)+2\sum_{j=1}^n\partial_{y_j}(y_n\partial_{y_j}\varphi(y)\mathcal{E}_\alpha(x, y))].$$

By virtue of (A.12) and simple calculations, one can show

$$|\mathcal{P}(x, y)|\leqq C[|x-y|^{-n}+d^{-1}|x-y|^{1-n}], \tag{A.15}$$

for $(x, y)\in B_{d/2}(x_0)\times B_{d/2}(x_0)$. Therefore we have

$$\begin{aligned} |x_n\partial_{x_n}u(x)| &\leqq \int_{B_{d/2}(x_0)\setminus B_{d/4}(x_0)}|\mathcal{P}(x, y)||u(y)|dy \\ &\leqq C\,\mathrm{Sup}_{B_{d/2}(x_0)}|u(x)|, \quad \text{for any} \quad x\in B_{d/4}(x_0). \end{aligned} \tag{A.16}$$

This proves the assertion. Q. E. D.

Appendix 2. Representation of α-superharmonic functions by potentials.

In this section we shall prove a theorem on representation of α-superharmonic function.

THEOREM A.4. *Let $\alpha>0$. Suppose that f is an α-superharmonic function in $\overline{\boldsymbol{R}^n_+}$. Then for any bounded domain $\Omega\subset\boldsymbol{R}^n_+$ there exist a measure μ_Ω concentrated on Ω, and a function h_Ω α-harmonic in Ω such that*

$$f(x)=K_\alpha\mu_\Omega(x)+h_\Omega(x), \qquad x\in\Omega.$$

Moreover this expression is unique.

Here

$$K_\alpha\mu_\Omega(x)=\int_\Omega K_\alpha(x, y)d\mu_\Omega(y),$$

$K_\alpha(x, y)$ is defined by (A.2).

PROOF. If f is smooth, then the assertion holds with $\mu_\Omega=-\nabla\cdot(x_n^\alpha\nabla f)=x_n^{\alpha-1}L_\alpha f$. So we will show in the general case that $\nabla\cdot(x_n^\alpha\nabla f)$ in the distribution sense turns out to be a measure in $\boldsymbol{R}_+^n$. To this end we consider the following measure ν_ρ^a with $a\in\boldsymbol{R}_+^n$ and $\rho>0$:

$$\nu_\rho^a(f)=\rho^{-2}[f(a)-M_{\alpha,\rho}f(a)] \qquad \text{with} \quad f\in C^0(\boldsymbol{R}_+^n). \tag{A.17}$$

Now we assume that f is α-superharmonic. Then from (2.12) it follows that $\nu_\rho^a(f)>0$, $\nu_\rho^a(f)$ is locally integrable with respect to a, and $\lim_{\rho\to 0}\nu_\rho^a(f)=[2(n+\alpha)a_n]^{-1}L_\alpha f(a)$ weakly. So that $x_n^{-1}L_\alpha f(x)$ is a measure as a limit of non-negative, locally integrable measure in $\boldsymbol{R}_+^n$, and this proves the assertion because $-\nabla\cdot(x_n^\alpha\nabla f)=x_n^\alpha\{x_n^{-1}L_\alpha f\}$ holds.

Secondly we show that $h_\Omega\equiv f-K_\alpha\mu_\Omega$ is α-harmonic. Indeed, according to the method of mollification in Lemma 2.1 we can easily verify the smoothness of h_Ω. Lastly we prove the uniqueness. Assume that $K_\alpha\mu_1+h_1=K_\alpha\mu_2+h_2$ holds with h_1, h_2 being α-harmonic. Then the signed measure $\mu=\mu_1-\mu_2$ has the potential $K_\alpha\mu=h_2-h_1=h$. Then the assertion follows from the next uniqueness lemma.

LEMMA A.5. *Let $\alpha>0$. Suppose that in the region Ω the potential of a signed measure $K_\alpha\mu(x)$ equals an α-harmonic function $h(x)$ almost everywhere. Then $\mu\equiv 0$ in Ω.*

PROOF. It suffices to show that $\mu(g)=0$ for any function g with continuous derivatives of the second order and with compact support in Ω. Then, we have by Green's formula and the mollification lemma 2.1,

$$\begin{aligned}\mu(g)&=\iint K_\alpha(x,\ y)x_n^{\alpha-1}L_\alpha g(x)dx\,d\mu(y)\\&=-\iint K_\alpha(x,\ y)\nabla\cdot(x_n^\alpha\nabla g(x))dx\,d\mu(y)\\&=-\int h(x)\nabla\cdot(x_n^\alpha\nabla g(x))dx=-\int\nabla\cdot(x_n^\alpha\nabla h(x))g(x)dx\\&=\int x_n^{\alpha-1}L_\alpha h(x)\cdot g(x)dx=0,\end{aligned} \tag{A.18}$$

as required. Q. E. D.

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