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Extension of minimal immersions of spheres into spheres

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§ 1. Introduction.

The purpose of the present study is to get isometric minimal immersions of $S^{m+k}(1)$ into spheres which are extensions of isometric minimal immersions of $S^m(1)$ into spheres and to find some properties of such immersions.

Let $S^{n-1}(r)$ denote the sphere of radius r centered at the origin in $\boldsymbol{R}^n$. An isometric minimal immersion $f_{m,s}: S^m(1) \to S^{n-1}(r)$ is expressed by

$$f_{m,s}(u) = \sum_{A=1}^{n} f^A(u)\tilde{e}_A$$

where $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ is an orthonormal basis of $\boldsymbol{R}^n$ and $u \in S^m(1)$. By a theorem of Takahashi [7] f^A $(A=1, \cdots, n)$ are spherical harmonics of degree s,

$$\Delta f^A = \lambda_s f^A, \qquad \lambda_s = s(s+m-1).$$

Let $\{e_1, \cdots, e_{m+1}\}$ be an orthonormal basis of $\boldsymbol{R}^{m+1}$ and $S^m(1)$ be the unit sphere in $\boldsymbol{R}^{m+1}$ so that we can put $u = u^i e_i$ using summation convention. To an eigenfunction f of Δ with $\Delta f = \lambda_s f$, there corresponds a unique harmonic polynomial

$$F = F_{i_1 \cdots i_s} x^{i_1} \cdots x^{i_s}$$

of degree s such that

$$f(u) = F_{i_1 \cdots i_s} u^{i_1} \cdots u^{i_s}.$$

The harmonic polynomial F then is viewed as a symmetric harmonic tensor of degree s, satisfying

i) $F(v_1, \cdots, v_s)$ is symmetric in $v_1, \cdots, v_s$

ii) $\sum_i F(e_i, e_i, v_3, \cdots, v_s) = 0$

where $v_1, \cdots, v_s \in \boldsymbol{R}^{m+1}$.

Thus, to an isometric minimal immersion $f_{m,s}$ there corresponds a set of n symmetric harmonic tensors $\{F^1, \cdots, F^n\}$. Let $V(m, s)$ denote the vector space of symmetric harmonic tensors of degree s on $\boldsymbol{R}^{m+1}$. Then we know that $\dim V(m, s) = n(m, s)$ is given by

$$n(m, s) = (2s+m-1)(s+m-2)!/(s!(m-1)!),$$

and $n=n(m, s)$. If we take a basis $\{H^1, \cdots, H^n\}$ of $V(m, s)$ satisfying

$$\int_{S^m(1)} H^A(u)H^B(u)d\omega_m = c\delta^{AB},$$

where c is a certain number given later, then the corresponding isometric minimal immersion $h_{m,s}: S^m(1)\to S^{n-1}(r)$ such that $h_{m,s}(u)=\sum_A H^A(u)\tilde{e}_A$ is called a *standard minimal immersion* [**3**, §5] and the basis $\{H^1, \cdots, H^n\}$ is called the *standard basis.*

IMI(m, s) and SMI(m, s) denote respectively the *set of isometric minimal immersions* $f_{m,s}$ and the *set of standard minimal immersions* $h_{m,s}$, hence SMI$(m, s)\subset$IMI(m, s). $f_{m,s}$ and $\tilde{f}_{m,s}$ are called *equivalent* if there exists an orthogonal transformation $g\in O(n)$ on $\boldsymbol{R}^n$ such that $\tilde{f}_{m,s}=g\circ f_{m,s}$. Hence SMI$(m, s)$ is the unique equivalence class of standard minimal immersions in IMI(m, s).

To describe the equivalence we introduce the symmetric tensor product $B(m, s)=V(m, s)\otimes V(m, s)$. Any element of $B(m, s)$ is given by

$$\sum_{A,B} b_{AB}H^A\otimes H^B, \qquad b_{AB}=b_{BA}.$$

Let $(F^1, \cdots, F^n)$ and $(\tilde{F}^1, \cdots, \tilde{F}^n)$ correspond respectively to $f_{m,s}$ and $\tilde{f}_{m,s}$. Then these are equivalent if and only if $\sum_A F^A\otimes F^A=\sum_A \tilde{F}^A\otimes\tilde{F}^A$ as it is easy to see [**3**, §3]. Thus, to the set SMI(m, s) there corresponds the element $\sum_A H^A\otimes H^A$ of $B(m, s)$. To describe the relation between IMI(m, s) and SMI(m, s) we consider

$$C=\sum_A(F^A\otimes F^A-H^A\otimes H^A)\in B(m, s).$$

It is to be noticed that, if we take a set $\{F^1, \cdots, F^n\}\subset V(m, s)$ at haphazard, it may happen that there exist no $f_{m,s}\in$IMI(m, s) corresponding to this set. When there exists an $f_{m,s}\in$IMI(m, s) corresponding to the given set $\{F^1, \cdots, F^n\}$ C satisfies

$$(\alpha) \qquad C(w, w, v, \cdots, v; v, \cdots, v)=0 \qquad w, v\in\boldsymbol{R}^{m+1}.$$

The set of elements of $B(m, s)$ satisfying (α) is a subspace of $B(m, s)$ and is denoted by $W(m, s)$. It is known that when $C\in W(m, s)$ is given, $\sum_A H^A\otimes H^A+C$ can be put $\sum_A F^A\otimes F^A$ with the set $\{F^1, \cdots, F^n\}$ corresponding to an $f_{m,s}\in$ IMI(m, s) if and only if C belongs to a certain compact convex body $L(m, s)$ in $W(m, s)$ [**1, 3**]. Precisely, the equivalence classes of isometric minimal immersions are known to be parametrized by $L(m, s)$ [**1**]. It is explained in §7 as well.

The purpose of the present paper is to give an injective homomorphism $\Lambda: W(m, s)\to W(m+k, s)$ such that

$$(\beta) \qquad \Lambda L(m, s)=L(m+k, s)\cap\Lambda W(m, s)$$

which amounts to giving a method of obtaining extensions of $f_{m,s}$ belonging to $\mathrm{IMI}(m+k, s)$. These *extensions* are denoted by $\mathrm{Ext}_k f_{m,s}$. Extension is natural in the sense that it keeps equivalence classes by (β). Obviously a standard one is extended to a standard one. Furthermore, some properties of $f_{m,s}$ are inherited to its extension. Though not many examples of non-standard minimal immersions have been known, we can find many examples systematically from known ones.

§2 is given preparatorily to the essential part of the paper beginning with §3. In §2.1 we recall a relation between $S^m(1)$ and $S^{m+k}(1)$. The way of deduction used there gives in §2.2 a formula for the integral over $S^{m+k}(1)$ of some function on $S^m(1)$. Inner products $(\ ,\ )_m$ and $(\ ,\ )_{m,m+k}$ are defined. We begin to give the notion of extension in §3. Extension of a symmetric tensor T on $\boldsymbol{R}^{m+1}$ is defined. When a tensor $\tilde{T}$ on $\boldsymbol{R}^{m+k+1}$ is obtained by extension, $\tilde{T}$ is denoted by $\mathrm{ext}_k T$. It is proved that harmonic tensors are extended to harmonic tensors. An inner product $[\ ,\ ]$ is defined which is invariant by extension. Thus we can construct an orthonormal basis of $V(m+k, s)$ with particular relation to an orthonormal basis of $V(m, s)$. In §4 extension of a standard minimal immersion is treated. $h_{m+k,s}$ obtained by extension of $h_{m,s}$ is denoted by $\mathrm{Ext}_k h_{m,s}$. In §5 we construct from an immersion $f_{m,s}\in\mathrm{IMI}(m, s)$ an immersion $f_{m+k,s}\in\mathrm{IMI}(m+k, s)$ which we call an extension of $f_{m,s}$. Then $f_{m+k,s}$ is denoted by $\mathrm{Ext}_k f_{m,s}$. As a result we get an injective homomorphism $\Lambda: W(m, s)\to W(m+k, s)$ for which we prove the following theorem; Λ satisfies

$$\Lambda L(m, s) = L(m+k, s)\cap \Lambda W(m, s).$$

In §6 we consider the distance between the image of $S^m(1)$ by an isometric minimal immersion $f_{m,s}$ and the image of the same $S^m(1)$ by $\mathrm{Ext}_k f_{m,s}$. Some other result concerning the shape of $\mathrm{Ext}_k f_{m,s}(S^{m+k}(1))$ is also obtained. We prove in §7.1 that when an immersion $f_{m,s}\in\mathrm{IMI}(m, s)$ is given, there exist an orthonormal basis of $\boldsymbol{R}^n$ and a standard minimal immersion $h_{m,s}$ such that the tensors F^A and H^A associated with $f_{m,s}$ and $h_{m,s}$ satisfy $F^A=a^A H^A$ where a^A are non negative numbers. In §7.2 we consider the relation between $f_{m,s}(S^m(1))$ and $h_{m,s}(S^m(1))$ when $F^A=a^A H^A$ is satisfied, setting down some additional condition, and in §7.3 we consider the effect of extension. We introduce another notion of distance, denoted by $d(f_{m,s}, h_{m,s})$, between images of $S^m(1)$ by $f_{m,s}$ and by $h_{m,s}$ in §7.4. In §7.5 we define the distance $d(f_{m,s}, \mathrm{SMI}(m, s))$ and prove in §7.6

$$d(f_{m,s}, \mathrm{SMI}(m, s)) = d(\mathrm{Ext}_k f_{m,s}, \mathrm{SMI}(m+k, s)).$$

In §8 we prove that Λ leaves invariant the isotropic property.

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extent.

§ 2. Preliminaries.

2.1. Let c_d denote the volume of $S^d(1)$ and $d\omega_d$ the volume element of $S^d(1)$. We recall the relation among $d\omega_{m+k}$, $d\omega_m$, $d\omega_{k-1}$ and also that among c_{m+k}, c_m, c_{k-1} understanding $c_0=2$.

The unit ball b in $\boldsymbol{R}^{m+k+1}$ is given by

$$(x_1)^2+\cdots+(x_{m+1})^2+(y_1)^2+\cdots+(y_k)^2\leqq 1\,,$$

where we can put

$$x_\alpha = ru_\alpha \sin\theta\,, \qquad \alpha=1,\cdots, m+1\,,$$

$$y_\beta = rv_\beta \cos\theta\,, \qquad \beta=1,\cdots, k\,,$$

$u_1,\cdots, u_{m+1}, v_1,\cdots, v_k$ being considered to be such that

$$(u_1)^2+\cdots+(u_{m+1})^2 = (v_1)^2+\cdots+(v_k)^2 = 1$$

and $0\leqq r\leqq 1$. Taking local coordinates $\varphi_1,\cdots,\varphi_m$ for $S^m(1)$ and $\psi_1,\cdots,\psi_{k-1}$ for $S^{k-1}(1)$, we get

$$\frac{\partial(x_1,\cdots, x_{m+1}, y_1,\cdots, y_k)}{\partial(r,\theta,\varphi_1,\cdots,\varphi_m,\psi_1,\cdots,\psi_{k-1})} = r^{m+k}\begin{vmatrix} M_{1,1} & \cdots\cdots\cdots & M_{1,m+k+1} \\ \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots\cdots\cdots & \cdots \\ M_{m+k+1,1} & \cdots\cdots\cdots & M_{m+k+1,m+k+1}\end{vmatrix}$$

where

$$M_{1,\alpha} = u_\alpha \sin\theta\,, \qquad M_{1,m+1+\beta} = v_\beta\cos\theta\,,$$

$$M_{2,\alpha} = u_\alpha \cos\theta\,, \qquad M_{2,m+1+\beta} = -v_\beta\sin\theta\,,$$

$$M_{2+\lambda,\alpha} = (\partial u_\alpha/\partial\varphi_\lambda)\sin\theta\,, \qquad M_{2+\lambda,m+1+\beta} = 0\,,$$

$$M_{m+2+\mu,\alpha} = 0\,, \qquad M_{m+2+\mu,m+1+\beta} = (\partial v_\beta/\partial\psi_\mu)\cos\theta\,,$$

$$\lambda=1,\cdots, m, \quad \mu=1,\cdots, k-1\,.$$

Thus we have, for the volume element db of the unit ball,

$$db = r^{m+k}dr\,d\theta\sin^m\theta\cos^{k-1}\theta\,d\omega_m d\omega_{k-1}\,,$$

and get

$$d\omega_{m+k} = d\omega_m d\omega_{k-1}\sin^m\theta\cos^{k-1}\theta\,d\theta\,,$$

$$c_{m+k} = I_{m,k-1}c_m c_{k-1}\,,$$

where

$$I_{m,k-1} = \int_0^{\pi/2}\sin^m\theta\cos^{k-1}\theta\,d\theta\,.$$

2.2. As an application of the above formulas we get the following lemma for $S^{m+k}(1)$ expressed as

$$(x_1)^2+\cdots+(x_{m+1})^2+(y_1)^2+\cdots+(y_k)^2=1$$

and a homogeneous polynomial $P_{2s}(x)=P_{2s}(x_1, \cdots, x_{m+1})$ of degree $2s$.

LEMMA 2.2.1. *For the integral of $P_{2s}(x)$ we have*

$$\int P_{2s}(x)d\omega_{m+k}=I_{2s+m,\,k-1}c_{k-1}\int P_{2s}(u)d\omega_m\,, \tag{2.2.1}$$

where u is the unit vector of $\boldsymbol{R}^{m+1}$ as $d\omega_m$ indicates.

PROOF. Here and also in what follows the domain of integration is not explicitly shown when the volume element is written. The following calculation proves the lemma.

$$\begin{aligned}\int P_{2s}(x)d\omega_{m+k}&=\int_0^{\pi/2}\int P_{2s}(u\sin\theta)\sin^m\theta\, d\omega_m\cos^{k-1}\theta\, d\omega_{k-1}d\theta\\&=\int_0^{\pi/2}\int P_{2s}(u)d\omega_m d\omega_{k-1}\sin^{2s+m}\theta\cos^{k-1}\theta\, d\theta\\&=I_{2s+m,\,k-1}\int P_{2s}(u)d\omega_m\int d\omega_{k-1}.\end{aligned}$$

q. e. d.

Let $\{e_1, \cdots, e_{m+k+1}\}$ be an orthonormal basis of $\boldsymbol{R}^{m+k+1}$ and $\boldsymbol{R}^{m+1}$ be the subspace spanned by $e_1, \cdots, e_{m+1}$. Consider the projection $P: \boldsymbol{R}^{m+k+1}\to\boldsymbol{R}^{m+1}$ given by $Pe_1=e_1, \cdots, Pe_{m+1}=e_{m+1}, Pe_{m+2}=0, \cdots, Pe_{m+k+1}=0$. Then we get as an application of Lemma 2.2.1 the following corollary.

COROLLARY 2.2.2. *We have*

$$\begin{aligned}(T_1, T_2)_{m,\,m+k}&=\int T_1(P\tilde{u}, \cdots, P\tilde{u})T_2(P\tilde{u}, \cdots, P\tilde{u})d\omega_{m+k}\\&=I_{2s+m,\,k-1}c_{k-1}(T_1, T_2)_m\,,\end{aligned} \tag{2.2.2}$$

$\tilde{u}$ ranging over $S^{m+k}(1)$.

Here T_1 and T_2 are symmetric tensors of degree s on $\boldsymbol{R}^{m+1}$, $(T_1, T_2)_{m,\,m+k}$ is defined by this formula, while $(T_1, T_2)_m$ is defined by

$$(T_1, T_2)_m=\int T_1(u, \cdots, u)T_2(u, \cdots, u)d\omega_m\,.$$

§ 3. Extension of tensors on $\boldsymbol{R}^{m+1}$ to those on $\boldsymbol{R}^{m+k+1}$.

In § 2 we considered $P: \boldsymbol{R}^{m+k+1}\to\boldsymbol{R}^{m+1}$. Since $\boldsymbol{R}^{m+1}$ is a subspace of $\boldsymbol{R}^{m+k+1}$, when $P\tilde{v}$, $\tilde{v}\in\boldsymbol{R}^{m+k+1}$, is considered as a vector of $\boldsymbol{R}^{m+k+1}$, it is sometimes written as $\tilde{P}\tilde{v}$ if necessary. Naturally we have $P\tilde{P}\tilde{v}=P\tilde{v}$.

DEFINITION 3.1. Let T be a symmetric tensor of degree s on $\boldsymbol{R}^{m+1}$ and $\tilde{v}_i$ a vector in $\boldsymbol{R}^{m+k+1}$. Then the symmetric tensor $\tilde{T}$ on $\boldsymbol{R}^{m+k+1}$ defined by

(3.1) $$\tilde{T}(\tilde{v}_1, \cdots, \tilde{v}_s) = T(P\tilde{v}_1, \cdots, P\tilde{v}_s)$$

is called the *extension* of T and is denoted by $\mathrm{ext}_k T$.

It is easy to see that a tensor $\tilde{T}$ on $\boldsymbol{R}^{m+k+1}$ is the extension of a certain tensor T on $\boldsymbol{R}^{m+1}$ if and only if

(3.2) $$\tilde{T}(\tilde{P}\tilde{v}_1, \cdots, \tilde{P}\tilde{v}_s) = \tilde{T}(\tilde{v}_1, \cdots, \tilde{v}_s)$$

since $P\tilde{P}\tilde{v} = P\tilde{v}$.

It is also easy to see that we have

$$(\mathrm{ext}_k T_1, \mathrm{ext}_k T_2)_{m+k} = (T_1, T_2)_{m, m+k}$$

for symmetric tensors T_1 and T_2 on $\boldsymbol{R}^{m+1}$. Furthermore we have

$$[\mathrm{ext}_k T_1, \mathrm{ext}_k T_2] = [T_1, T_2]$$

where $[T_1, T_2]$ is defined by

$$[T_1, T_2] = \sum_i^* T_1(e_{i_1}, \cdots, e_{i_s}) T_2(e_{i_1}, \cdots, e_{i_s})$$

$\sum_i^*$ indicating summation where each of $i_1, \cdots, i_s$ ranges over $1, \cdots, m+1$.

LEMMA 3.2. *$\tilde{T} = \mathrm{ext}_k T$ is a symmetric harmonic tensor on $\boldsymbol{R}^{m+k+1}$ if and only if T is a symmetric harmonic tensor on $\boldsymbol{R}^{m+1}$.*

PROOF. We take an orthonormal basis of $\boldsymbol{R}^{m+k+1}$ as in § 2.2 and use indices $i=1, \cdots, m+1$ and $p=m+2, \cdots, m+k+1$. If T is harmonic, we have

$$\sum_i T(e_i, e_i, v, \cdots, v) = 0, \qquad v \in \boldsymbol{R}^{m+1}.$$

Then, since $Pe_i = e_i$ and $Pe_p = 0$, we get

$$\begin{aligned} &\sum_i \tilde{T}(e_i, e_i, \tilde{v}, \cdots, \tilde{v}) + \sum_p \tilde{T}(e_p, e_p, \tilde{v}, \cdots, \tilde{v}) \\ &= \sum_i \tilde{T}(e_i, e_i, \tilde{P}\tilde{v}, \cdots, \tilde{P}\tilde{v}) = \sum_i T(e_i, e_i, P\tilde{v}, \cdots, P\tilde{v}) = 0 \end{aligned}$$

which shows that $\tilde{T}$ is harmonic. We can easily see that the converse is also true. q. e. d.

Let us take a basis $\{T^1, \cdots, T^{n(m,s)}\}$ of $V(m, s)$ orthonormal in the sense of tensors, that is, in the sense of inner products [,]. Then the set $\{\tilde{T}^1, \cdots, \tilde{T}^{n(m,s)}\}$, where $\tilde{T}^P = \mathrm{ext}_k T^P$ ($P=1, \cdots, n(m, s)$), is orthonormal as well in $V(m+k, s)$. Hence, supplementing this with symmetric harmonic tensors $\tilde{T}^X$ ($X=n(m, s)+1, \cdots, n(m+k, s)$) suitably chosen, we get a basis $\{\tilde{T}^1, \cdots, \tilde{T}^{n(m+k,s)}\}$ of $V(m+k, s)$ orthonormal in the sense of tensors. This satisfies

(3.3) $$(\tilde{T}^A, \tilde{T}^B)_{m+k} = (s!(m+k-1)!!/(2s+m+k-1)!!)c_{m+k}\delta^{AB}$$

$$A, B = 1, \cdots, n(m+k, s).$$

To get this formula we can use the formulas $(H^A, H^B)_m=c\delta^{AB}$, $[H^A, H^B]=c'\delta^{AB}$ satisfied by the basis $\{H^1, \cdots, H^{n(m,s)}\}$ of $V(m, s)$ corresponding to a standard minimal immersion $h_{m,s}$ [**3**, § 5]. c and c' are computed according to the formulas given in [**3**, p. 322], thus

$$c=(r(m, s))^2 c_m/n(m, s), \qquad c'=m!!(2s+m-3)!!/s(s+m-1)!,$$

so that we get

$$(T^A, T^B)_m=(c/c')\delta^{AB}=(s!(m-1)!!/(2s+m-1)!!)c_m\delta^{AB}. \tag{3.4}$$

Replacing m with $m+k$, we get (3.3).

REMARK. We consider $f_{m,s}: S^m(1)\to S^{n(m,s)-1}(r(m, s))$.

§ 4. Extension of standard minimal immersions.

We take an orthonormal basis $\{T^1, \cdots, T^{n(m,s)}\}$ of the space $V(m, s)$ and the orthonormal basis $\{\tilde{T}^1, \cdots, \tilde{T}^{n(m+k,s)}\}$ of the space $V(m+k, s)$ as in § 3. Then the harmonic tensors $H^1, \cdots, H^{n(m,s)}$ and $\tilde{H}^1, \cdots, \tilde{H}^{n(m+k,s)}$ which are taken as

$$H^P=(c')^{1/2}T^P, \qquad \tilde{H}^P=(\tilde{c}')^{1/2}\tilde{T}^P, \qquad \tilde{H}^X=(\tilde{c}')^{1/2}\tilde{T}^X$$

are associated with standard minimal immersions $h_{m,s}$ and $h_{m+k,s}$ respectively since they satisfy

$$[H^P, H^Q]=c'\delta^{PQ}, \qquad [\tilde{H}^A, \tilde{H}^B]=\tilde{c}'\delta^{AB}$$

for $P, Q=1, \cdots, n(m, s)$ and $A, B=1, \cdots, n(m+k, s)$. The immersion $h_{m+k,s}$ obtained in this way is called the *extension of* $h_{m,s}$ and is denoted by $\mathrm{Ext}_k h_{m,s}$.

§ 5. $f_{m+k,s}$ obtained as an extension of $f_{m,s}$.

As was stated in § 4, we have a standard minimal immersion $h_{m+k,s}$ corresponding to a standard minimal immersion $h_{m,s}$ such that $H^P=(c')^{1/2}T^P$, $\tilde{H}^P=(\tilde{c}')^{1/2}\mathrm{ext}_k T^P$, hence

$$\tilde{H}^P=(\tilde{c}'/c')^{1/2}\mathrm{ext}_k H^P. \tag{5.1}$$

This suggests us a new correspondence between tensors on $\boldsymbol{R}^{m+1}$ and those on $\boldsymbol{R}^{m+k+1}$, $T\to\lambda\mathrm{ext}_k T$, where T is a tensor on $\boldsymbol{R}^{m+1}$ and λ is given by

$$\lambda^2=\tilde{c}'/c'=\frac{(m+k)!!}{m!!}\cdot\frac{(s+m-1)!}{(s+m+k-1)!}\cdot\frac{(2s+m+k-3)!!}{(2s+m-3)!!}. \tag{5.2}$$

Minding such a correspondence we get the following theorem.

THEOREM 5.1. *Let F^P $(P=1, \cdots, n(m, s))$ be tensors associated with an isometric minimal immersion $f_{m,s}$. Taking the number $\lambda>0$ given above, put* *tensors* $\tilde{F}^A$ $(A=1, \cdots, n(m+k, s))$ *on* $\boldsymbol{R}^{m+k+1}$ *as*

$$\tilde{F}^P(\tilde{v}) = \varepsilon_P \lambda F^P(P\tilde{v}),$$
$$\tilde{F}^X(\tilde{v}) = \varepsilon_X \tilde{H}^X(\tilde{v}) \quad (X=n(m, s)+1, \cdots, n(m+k, s)),$$

where $|\varepsilon_A|=1$. *Then there exists an isometric minimal immersion* $f_{m+k,s} \in \mathrm{IMI}(m+k, s)$ *such that* $\tilde{F}^A$ *are associated with* $f_{m+k,s}$.

PROOF. As we have

$$\sum_X \tilde{F}^X \otimes \tilde{F}^X - \sum_X \tilde{H}^X \otimes \tilde{H}^X = 0,$$

we get

$$\begin{aligned}\sum_A \tilde{F}^A(\tilde{v})\tilde{F}^A(\tilde{w}) - \sum_A \tilde{H}^A(\tilde{v})\tilde{H}^A(\tilde{w}) &= \sum_P \tilde{F}^P(\tilde{v})\tilde{F}^P(\tilde{w}) - \sum_P \tilde{H}^P(\tilde{v})\tilde{H}^P(\tilde{w}) \\ &= \lambda^2[\sum_P F^P(P\tilde{v})F^P(P\tilde{w}) - \sum_P H^P(P\tilde{v})H^P(P\tilde{w})] \\ &= \lambda^2 C(P\tilde{v}, \cdots, P\tilde{v}; P\tilde{w}, \cdots, P\tilde{w})\end{aligned}$$

where $C=\sum_P(F^P \otimes F^P - H^P \otimes H^P) \in W(m, s)$. Let us define a bi-symmetric tensor $\tilde{C}$ on $\boldsymbol{R}^{m+k+1}$ by

$$\tilde{C}(\tilde{v}, \cdots, \tilde{v}; \tilde{w}, \cdots, \tilde{w}) = \lambda^2 C(P\tilde{v}, \cdots, P\tilde{v}; P\tilde{w}, \cdots, P\tilde{w}). \tag{5.3}$$

Then it is easy to see that $\tilde{C}$ satisfies the conditions $\tilde{C} \in B(m+k, s)$ and (α) with m replaced by $m+k$. Thus $\tilde{C} \in L(m+k, s)$ and $\tilde{F}^A$ are tensors associated with an isometric minimal immersion. q.e.d.

Theorem 5.1 shows that, if C belongs to $L(m, s)$, then $\tilde{C}$ obtained by (5.3) belongs to $L(m+k, s)$.

DEFINITION 5.2. An element $\tilde{C}$ of $W(m+k, s)$ obtained from an element C of $W(m, s)$ by (5.3), with λ^2 given by (5.2), is called the *extension* of C and is denoted by ΛC. Λ induces a mapping $\Lambda: W(m, s) \to W(m+k, s)$.

THEOREM 5.3. *Λ is an injective homomorphism such that*

$$\Lambda L(m, s) = L(m+k, s) \cap \Lambda W(m, s).$$

PROOF. From Theorem 5.1 we get

$$\Lambda L(m, s) \subset L(m+k, s) \cap \Lambda W(m, s).$$

If the set $\{F^P; P=1, \cdots, n(m, s)\}$ is linearly dependent, then so are the set $\{\tilde{F}^P; P=1, \cdots, n(m, s)\}$, and hence the set $\{\tilde{F}^A; A=1, \cdots, n(m+k, s)\}$ as well. Thus we have

$$\Lambda \partial L(m, s) \subset \partial L(m+k, s) \cap \Lambda W(m, s),$$

which proves the theorem, since L is a convex body. q.e.d.

DEFINITION 5.4. In Theorem 5.1 we may put $\varepsilon_A = \pm 1$ for each of

$A=1, \cdots, n(m+k, s)$ arbitrarily. This is a natural result of the notion of equivalence. When $\varepsilon_A=1$ for every A, the resulting $f_{m+k,s}$ is called an *extension* of $f_{m,s}$ and is denoted by $\mathrm{Ext}_k f_{m,s}$.

When $f_{m,s}$ is given, there still exist many extensions $\mathrm{Ext}_k f_{m,s}$, since there exists some degree of freedom in the choice of $\tilde{H}^X$.

As an application of Theorem 5.1 and Theorem 5.3, we get a corollary of the following theorem due to Mashimo [**2**].

THEOREM A. *Let s be an integer $s\geqq 4$. Then there exists an isometric minimal immersion of $S^3(1)$ into $S^{2s+1}(r)$, $r^2=3/s(s+2)$. Let s be an even integer $s\geqq 6$. Then there exists an isometric minimal immersion of $S^3(1)$ into $S^s(r)$, $r^2=3/s(s+2)$.*

COROLLARY B. *If $s\geqq 4$, there exists an isometric minimal immersion of a $(3+k)$-sphere into a $(2s+1+n(3+k, s)-n(3, s))$-sphere. If s is even and $\geqq 6$, then there exists an isometric minimal immersion of a $(3+k)$-sphere into an $(s+n(3+k, s)-n(3, s))$-sphere.*

§ 6. Distance between an isometric minimal immersion and its extension.

6.1. Let $\mathrm{Ext}_k f_{m,s}$ be an extension of an isometric minimal immersion $f_{m,s}$. We define the *ground distance* $d_{m,m+k}$ between $f_{m,s}$ and $\mathrm{Ext}_k f_{m,s}$ in the following way. We can consider that $f_{m,s}(S^m(1))$ lies in $\boldsymbol{R}^{n(m,s)}$ and $\mathrm{Ext}_k f_{m,s}(S^{m+k}(1))$ lies in $\boldsymbol{R}^{n(m+k,s)}$, where $\boldsymbol{R}^{n(m,s)}$ is the subspace of $\boldsymbol{R}^{n(m+k,s)}$ generated by the first $n(m, s)$ vectors of a fixed orthonormal basis of $\boldsymbol{R}^{n(m+k,s)}$. We consider the distance $d_{m,m+k}(u)$ between $f_{m,s}(u)$ and $\mathrm{Ext}_k f_{m,s}(\tilde{u})$ where $u\in S^m(1)$, $\tilde{u}\in S^{m+k}(1)$ with $P\tilde{u}=u$. The relation $P\tilde{u}=u$ simply means that $\tilde{u}$ belongs to $\boldsymbol{R}^{m+1}\cap S^{m+k}(1)=S^m(1)$ and $\tilde{u}=u$. Under this circumstance $\boldsymbol{R}^{n(m,s)}$ and $S^m(1)$ are called the *ground space* and the *ground sphere* respectively.

Then, since we have put $\varepsilon_A=1$, we get

$$\mathrm{Ext}_k f_{m,s}(\tilde{u})=\tilde{F}^P(\tilde{u})\tilde{e}_P+\tilde{F}^X(\tilde{u})\tilde{e}_X=\lambda F^P(u)\tilde{e}_P+\tilde{H}^X(\tilde{u})\tilde{e}_X\,,$$

where $\tilde{e}_P$ lie in the ground space and $\tilde{e}_X$ are vertical to the ground space. On the other hand we have $f_{m,s}(u)=F^P(u)\tilde{e}_P$. Thus the distance $d_{m,m+k}(u)$ is given by

$$(d_{m,m+k}(u))^2=\sum_P(\lambda F^P(u)-F^P(u))^2+\sum_X(\tilde{H}^X(\tilde{u}))^2. \tag{6.1.1}$$

Since the right hand side becomes

$$\sum_P(\lambda F^P(u))^2+\sum_X(\tilde{H}^X(\tilde{u}))^2+(1-2\lambda)\sum_P(F^P(u))^2$$

which is equal to

$$(r(m+k, s))^2+(1-2\lambda)(r(m, s))^2,$$

where $r(m, s)$ (resp. $r(m+k, s)$) is the radius of the sphere on which $f_{m,s}(S^m(1))$ (resp. $\mathrm{Ext}_k f_{m,s}(S^{m+k}(1))$) lies, $d_{m,m+k}(u)$ does not depend on u and is denoted simply by $d_{m,m+k}$.

Thus we have the following definition and theorem.

DEFINITION 6.1.1. $d_{m,m+k}$ is called the *ground distance between* $f_{m,s}$ *and its extension* $\mathrm{Ext}_k f_{m,s}$.

THEOREM 6.1.2. *The ground distance* $d_{m,m+k}$ *between* $f_{m,s} \in \mathrm{IMI}(m, s)$ *and its extension* $\mathrm{Ext}_k f_{m,s}$ *is given by*

$$d_{m,m+k} = ((r(m+k, s))^2 + (1-2\lambda)(r(m, s))^2)^{1/2} \tag{6.1.2}$$

where λ *is the positive number given by* (5.2). *Furthermore the ground distance does not depend on the choice of the immersion* $f_{m,s}$ *and is determined only by* m, k *and* s.

Now let us consider the distance between the ground space and the image of the ground sphere by the extension $\mathrm{Ext}_k f_{m,s}$. Clearly the square of the distance is given by $\sum_X (\tilde{H}^X(\tilde{u}))^2$ for each point. On the other hand we have

$$(r(m+k, s))^2 = \lambda^2 \sum_P (F^P(u))^2 + \sum_X (\tilde{H}^X(\tilde{u}))^2 = \lambda^2 (r(m, s))^2 + \sum_X (\tilde{H}^X(\tilde{u}))^2, \tag{6.1.3}$$

hence

$$\sum_X (\tilde{H}^X(\tilde{u}))^2 = (r(m+k, s))^2 - \lambda^2 (r(m, s))^2 .$$

This admits us to give the following definition and theorem.

DEFINITION 6.1.3. Let $\mathrm{Ext}_k f_{m,s}$ be an extension of $f_{m,s} \in \mathrm{IMI}(m, s)$ and let $\tilde{u}$ be the position vector of a point of the ground sphere, hence $\tilde{u} = \tilde{P}\tilde{u}$. As the distance between the point $\mathrm{Ext}_k f_{m,s}(\tilde{u})$ and the ground space does not depend on $\tilde{u}$, it is called the *ground distance between the extension* $\mathrm{Ext}_k f_{m,s}$ *and the ground space.*

THEOREM 6.1.4. *Let* $f_{m,s}$ *be an immersion* $\in \mathrm{IMI}(m, s)$. *The ground distance between* $\mathrm{Ext}_k f_{m,s}$ *and the ground space is given by*

$$((r(m+k, s))^2 - \lambda^2 (r(m, s))^2)^{1/2} \tag{6.1.4}$$

hence depends only on m, k *and* s.

We give two examples where s is less than 4, hence all immersions are standard minimal immersions.

EXAMPLE 1. $s=2$, $m=2$, $k=1$. Then we have $n(2, 2)=5$, $n(3, 2)=9$, $(r(2, 2))^2=1/3$, $(r(3, 2))^2=3/8$. We take variables x, y, z in $\boldsymbol{R}^3$ and variables x, y, z, t in $\boldsymbol{R}^4$. Then we can choose

$$H^1 = xy, \qquad H^2 = xz, \qquad H^3 = yz, \qquad H^4 = (1/2)(x^2 - y^2),$$

$$H^5 = 12^{-1/2}(x^2+y^2-2z^2)$$

as a standard basis of $V(2, 2)$. As $\lambda=1$ in this case,

$$\tilde{H}^1 = H^1, \quad \tilde{H}^2 = H^2, \quad \tilde{H}^3 = H^3, \quad \tilde{H}^4 = H^4, \quad \tilde{H}^5 = H^5, \quad \tilde{H}^6 = xt,$$
$$\tilde{H}^7 = yt, \quad \tilde{H}^8 = zt, \quad \tilde{H}^9 = 24^{-1/2}(x^2+y^2+z^2-3t^2)$$

can be considered as a standard basis of $V(3, 2)$ extended from the basis of $V(2, 2)$ given above. The image of the ground sphere is obtained when we put $t=0$, hence $x^2+y^2+z^2=1$, hence $\tilde{H}^6=\tilde{H}^7=\tilde{H}^8=0$ and $\tilde{H}^9=24^{-1/2}$. Thus we get $d_{2,3}=24^{-1/2}$. The ground distance between the extension and the ground space is also $24^{-1/2}$.

EXAMPLE 2. $s=3, m=2, k=1$. Then we have $n(2, 3)=7$, $n(3, 3)=16$, $(r(2, 3))^2=1/6$, $(r(3, 3))^2=1/5$. We take the same variables as in Example 1 and can choose

$$H^1 = axyz, \quad H^2 = bx(y^2-z^2), \quad H^3 = by(x^2-z^2),$$
$$H^4 = bz(x^2-y^2), \quad H^5 = cx(2x^2-3y^2-3z^2),$$
$$H^6 = cy(2y^2-3x^2-3z^2), \quad H^7 = cz(2z^2-3x^2-3y^2),$$

with $a=(5/2)^{1/2}$, $b=a/2$, $c=24^{-1/2}$, as the standard basis of $V(2, 3)$. As we get $\lambda^2=24/25$ in this case,

$$\tilde{H}^1 = \lambda H^1, \quad \cdots, \quad \tilde{H}^7 = \lambda H^7, \quad \tilde{H}^8 = \lambda axyt, \quad \tilde{H}^9 = \lambda axzt,$$
$$\tilde{H}^{10} = \lambda ayzt, \quad \tilde{H}^{11} = \lambda bt(x^2-y^2), \quad \tilde{H}^{12} = 3^{-1/2}\lambda bt(x^2+y^2-2z^2),$$
$$\tilde{H}^{13} = 5^{-1/2}t(x^2+y^2+z^2-t^2), \quad \tilde{H}^{14} = 5^{-1}x(x^2+y^2+z^2-5t^2),$$
$$\tilde{H}^{15} = 5^{-1}y(x^2+y^2+z^2-5t^2), \quad \tilde{H}^{16} = 5^{-1}z(x^2+y^2+z^2-5t^2)$$

can be taken as a standard basis of $V(3, 3)$ extended from the basis of $V(2, 3)$ given above. Putting $t=0$ we get

$$\tilde{H}^1 = \lambda H^1, \ \cdots, \ \tilde{H}^7 = \lambda H^7, \quad \tilde{H}^8 = \cdots = \tilde{H}^{13} = 0,$$
$$\tilde{H}^{14} = 5^{-1}x, \quad \tilde{H}^{15} = 5^{-1}y, \quad \tilde{H}^{16} = 5^{-1}z.$$

After some computation we get $(d_{2,3})^2=11/30-(4/5)6^{-1/2}$. The ground distance between the extension and the ground space is $1/5$.

All these results coincide with (6.1.2) and (6.1.4).

6.2. The distance $d_{m,m+k}$ considered above is so to say the distance between $f_{m,s}(S^m(1))$ and $\mathrm{Ext}_k f_{m,s}(S^m(1))$. It is desirable to know more about $\mathrm{Ext}_k f_{m,s}(S^{m+k}(1))$. When $\tilde{u}$ is an arbitrary unit vector in $\boldsymbol{R}^{m+k+1}$, let us take $P\tilde{u}/\|P\tilde{u}\|$ as the unit vector in $\boldsymbol{R}^{m+1}$ corresponding to $\tilde{u}$. This allows us to consider the distance $d(\mathrm{Ext}_k f_{m,s}(\tilde{u}), f_{m,s}(P\tilde{u}/\|P\tilde{u}\|))=\tilde{d}_{m,m+k}(\tilde{u})$ for which we get

$$(\tilde{d}_{m,m+k}(\tilde{u}))^2 = (1-2\lambda\|Pu\|^s)(r(m, s))^2+(r(m+k, s))^2$$

since we have $F^P(P\tilde{u})=\|P\tilde{u}\|^s F^P(P\tilde{u}/\|P\tilde{u}\|)$.

The distance between the ground space and the image by the extension $\mathrm{Ext}_k f_{m,s}$ of $S^{m+k}(1)$ is also easy to compute. We get

$$((r(m+k,\ s))^2-(\lambda\|P\tilde{u}\|^s r(m,\ s))^2)^{1/2}.$$

Thus we get the following theorem.

THEOREM 6.2.1. *Let $f_{m+k,s}$ be an extension of $f_{m,s}\in \mathrm{IMI}(m,\ s)$. Then the distance between $\mathrm{Ext}_k f_{m,s}(\tilde{u})$, $\tilde{u}\in S^{m+k}(1)$, and the ground space $\boldsymbol{R}^{m+1}$ is not less than*

$$r(m+k,\ s)(1-(\lambda r(m,\ s)/r(m+k,\ s))^2)^{1/2}.$$

§ 7. Relation between an isometric minimal immersion and a standard minimal immersion.

7.1. As we consider for a while only immersions of $\mathrm{IMI}(m,\ s)$, we use in § 7.1 and in § 7.2 indices ranging as

$$A,\ B,\ C,\ \cdots=1,\ \cdots,\ n;\qquad P,\ Q,\ R,\ \cdots=1,\ \cdots,\ p;$$
$$X,\ Y,\ Z,\ \cdots=p+1,\ \cdots,\ n$$

where $n=n(m,\ s)$ and p is an integer $1\leqq p<n$.

Let us recall that tensors F^A or H^A associated with an $f_{m,s}\in\mathrm{IMI}(m,\ s)$ or an $h_{m,s}\in\mathrm{SMI}(m,\ s)$ depend on the choice of the orthonormal basis $\{\tilde{e}_1,\ \cdots,\ \tilde{e}_n\}$ of $\boldsymbol{R}^n$. As F^A belong to $V(m,\ s)$ and $\{H^1,\ \cdots,\ H^n\}$ is a standard basis of $V(m,\ s)$, we can put

$$F^A=\sum_B f^{AB}H^B, \tag{7.1.1}$$

where f^{AB} are numbers making an $n\times n$ matrix f. On the other hand, for $C\in W(m,\ s)$ there exists a symmetric matrix $[c^{AB}]$ such that

$$C=\sum_{A,B}c^{AB}H^A\otimes H^B.$$

We are going to study the behavior of the matrices $[f^{AB}]$ and $[c^{AB}]$ when we choose a suitable basis of $\boldsymbol{R}^n$, a suitable immersion from the given equivalence class $[f_{m,s}]$ and a suitable standard minimal immersion.

Taking a suitable basis $\{H^1,\ \cdots,\ H^n\}$ and hence the corresponding standard minimal immersion $h_{m,s}$, we have a diagonal form for C,

$$C=\sum_A c^A H^A\otimes H^A. \tag{7.1.2}$$

If moreover C satisfies

$$C=\sum_A (F^A\otimes F^A-H^A\otimes H^A),$$

where F^A are associated with $f_{m,s}$, we have

$$\sum_A F^A \otimes F^A = C + \sum_A H^A \otimes H^A = \sum_A (1+c^A) H^A \otimes H^A.$$

Thus we get

$$\sum_{A,B,C} f^{CA} f^{CB} H^A \otimes H^B = \sum_A (1+c^A) H^A \otimes H^A$$

and hence

$$\text{(7.1.3)} \qquad \sum_C f^{CA} f^{CB} = (1+c^A)\delta^{AB}, \qquad \sum_C (f^{CA})^2 = 1+c^A.$$

This shows that, whenever C belongs to $L(m, s)$, we have $1+c^A \geqq 0$ for each of $A=1, \cdots, n$. Conversely, if $1+c^A \geqq 0$, then there exists an isometric minimal immersion $g_{m,s}$ such that

$$G^A = (1+c^A)^{1/2} H^A$$

are the tensors associated with $g_{m,s}$. Thus the condition $1+c^A \geqq 0$ (all A) is the necessary and sufficient condition for C to belong to $L(m, s)$, and this condition is satisfied in our case. If $1+c^A=0$ for some A, then we have $C \in \partial L(m, s)$.

Let us consider the relation between $f_{m,s}$ and $g_{m,s}$. As we have

$$G^A = d^A H^A, \qquad F^A = \sum_B f^{AB} H^B,$$

where $d^A = (1+c^A)^{1/2}$, we get, if $d^A > 0$ for all A,

$$F^A = \sum_B (f^{AB}/d^B) G^B.$$

From this and (7.1.3) we get

$$\sum_A (f^{AB}/d^B)(f^{AC}/d^C) = \delta^{BC}$$

and this shows that the matrix $[f^{AB}/d^B]$ is an orthogonal matrix. Thus $g_{m,s}$ and $f_{m,s}$ belong to one and the same equivalence class.

If we have $d^P > 0$ but $d^X = 0$ for $P=1, \cdots, p$ and $X=p+1, \cdots, n$, then we get $f^{AX}=0$, $G^P = d^P H^P$, $G^X = 0$, hence

$$F^A = \sum_P f^{AP} H^P = \sum_P (f^{AP}/d^P) G^P + \sum_X g^{AX} G^X,$$

$$\sum_A (f^{AP}/d^P)(f^{AQ}/d^Q) = \delta^{PQ}$$

where we can choose g^{AX} freely. Thus, when g^{AX} are chosen such that $[f^{AP}/d^P, g^{AX}]$ is an orthogonal matrix, $g_{m,s}$ belongs to the equivalence class of $f_{m,s}$.

Thus we have proved the following theorem.

THEOREM 7.1.1. *Let an orthonormal basis of* $\boldsymbol{R}^n$ *be fixed,* $f_{m,s}$ *be an arbitrary isometric minimal immersion and* $h_{m,s}$ *be a suitable standard minimal immersion. Then there exists an isometric minimal immersion* $g_{m,s}$ *belonging to the equivalence class of* $f_{m,s}$ *such that tensors* G^A *and* H^A *associated with* $g_{m,s}$

and $h_{m,s}$ *respectively satisfy*

$$G^A = g^A H^A, \qquad g^A \geqq 0.$$

The relation between F^A and G^A can be written

$$F^A = \sum_B a^{AB} G^B, \qquad G^A = \sum_B a^{BA} F^B$$

where $a=[a^{AB}]$ is an orthogonal matrix. Let $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ be the orthonormal basis of $\boldsymbol{R}^n$ with respect to which F^A, G^A and H^A are the tensors considered above. Then taking another orthonormal basis $\{\tilde{e}'_1, \cdots, \tilde{e}'_n\}$ such that

$$\tilde{e}'_A = \sum_B a^{BA} \tilde{e}_B,$$

we get $F^A \tilde{e}_A = G^A \tilde{e}'_A$. This shows that G^A are the tensors associated with $f_{m,s}$ with respect to the new basis $\{\tilde{e}'_1, \cdots, \tilde{e}'_n\}$. Let $h'_{m,s}$ be another standard minimal immersion such that the tensors associated with $h'_{m,s}$ are H^A with respect to the new basis, namely the tensors associated are $\sum_B a^{AB} H^B$ with respect to the old basis. Then we can deduce from the equation $G^A = g^A H^A$, $g^A \geqq 0$ the following theorem.

THEOREM 7.1.2. *Let* $f_{m,s}$ *be an arbitrary isometric minimal immersion. If we choose a suitable orthonormal basis of* $\boldsymbol{R}^n$ *and a suitable standard minimal immersion* $h_{m,s}$, *then the tensors* F^A *and* H^A *associated with* $f_{m,s}$ *and* $h_{m,s}$ *respectively satisfy*

$$F^A = a^A H^A, \qquad a^A \geqq 0. \tag{7.1.4}$$

7.2. Assuming $f_{m,s}$ and $h_{m,s}$ satisfy the condition (7.1.4), we compute the pointwise distance between $f_{m,s}(S^m(1))$ and $h_{m,s}(S^m(1))$, namely $d(f_{m,s}(u), h_{m,s}(u))$ where u is a unit vector in $\boldsymbol{R}^{m+1}$. For this distance $\rho(u)$ we have

$$(\rho(u))^2 = \sum_A (F^A(u) - H^A(u))^2 = \sum_A (a^A - 1)^2 (H^A(u))^2$$

and also

$$(\rho(u))^2 = 2r^2 - 2\sum_A a^A (H^A(u))^2 = 2\sum_A (1 - a^A)(H^A(u))^2.$$

As it is easy to see, $\rho(u)$ coincides with $\min\{d(f_{m,s}(u), h_{m,s}(v))\,;\ \|v\|=1\}$ if $\rho(u)$ does not depend on u.

We can put

$$a^A = a_i \ (A = n_1 + \cdots + n_{i-1} + 1, \cdots, n_1 + \cdots + n_i)$$

where $i=1, \cdots, p$, $\sum_i n_i = n$ and $a_1 > a_2 > \cdots > a_p \geqq 0$. Let us define

$$\sigma_i(u) = \sum_{(i)} (H^A(u))^2 \tag{7.2.1}$$

where $\sum_{(i)}$ means the sum for $A = n_1 + \cdots + n_{i-1} + 1, \cdots, n_1 + \cdots + n_i$. Then we have

$$\sum_i \sigma_i(u) = \sum_A (H^A(u))^2 = r^2,$$

$$\sum_i (a_i)^2 \sigma_i(u) = \sum_A (a^A H^A(u))^2 = r^2,$$

$$(\rho(u))^2 = 2r^2 - 2\sum_i a_i \sigma_i(u).$$

In some following paragraphs we consider the case where $\sigma_i(u)$ do not depend on u. Then putting $\sigma_i(u)=\sigma_i$ we get from (7.2.1) $\sigma_i=(n_i/n)r^2$ because of $c=r^2c_m/n$ [**3**, §5], hence $\rho(u)=\rho$ does not depend on u.

$$\rho^2 = \Big(1-\sum_{i=1}^{p} n_i a_i/n\Big)2r^2. \tag{7.2.2}$$

Thus we have obtained the following lemma and corollary.

LEMMA 7.2.1. *Let $f_{m,s} \in \mathrm{IMI}(m, s)$ and $h_{m,s} \in \mathrm{SMI}(m, s)$ be such that (7.1.4) is satisfied and moreover the basis $\{H^1, \cdots, H^n\}$ corresponding to $h_{m,s}$ be such that $\sigma_i(u)$ defined by (7.2.1) do not depend on $u \in S^m(1)$. Then $d(f_{m,s}(u), h_{m,s}(u)) =\rho$ is given by (7.2.2).*

COROLLARY 7.2.2. *Let $h_{m,s}$ be a standard minimal immersion with the corresponding basis $\{H^1, \cdots, H^n\}$ satisfying $\sigma_i(u)=\sigma_i$. If $\alpha_1, \cdots, \alpha_p$ are numbers such that*

$$\sum_{i=1}^{p} \alpha_i \sum_{(i)} H^A \otimes H^A \in L(m, s),$$

then there exists an isometric minimal immersion $f_{m,s}$ satisfying (7.1.4) and such that $d(f_{m,s}(u), h_{m,s}(u))$ does not depend on the unit vector u.

EXAMPLE 1. From the result we have got in [**6**, §9], where some cases of $S^3(1) \to S^{24}(r)$, $r^2=1/8$, are treated, we can easily deduce that there exists a standard minimal immersion $h_{3,4}$ with $\{H^1, \cdots, H^{25}\}$ such that

$$\frac{1}{2}\Big(3\sum_{A=1}^{10} H^A \otimes H^A - 2\sum_{A=11}^{25} H^A \otimes H^A\Big) \in \partial L(3, 4).$$

Then we have an isometric minimal immersion $f_{3,4}$ such that

$$F^A = a^A H^A, \qquad a^1 = \cdots = a^{10} = \Big(\frac{5}{2}\Big)^{1/2}, \qquad a^{11} = \cdots = a^{25} = 0.$$

This satisfies

$$(d(f_{3,4}(u), h_{3,4}(u)))^2 = \Big(1-\Big(\frac{2}{5}\Big)^{1/2}\Big)/4.$$

EXAMPLE 2. We have the following $C \in \partial L(3, 4)$ as well,

$$C = -\frac{1}{3}\Big(3\sum_{A=1}^{10} H^A \otimes H^A - 2\sum_{A=11}^{25} H^A \otimes H^A\Big).$$

Then we can take $f_{3,4}$ satisfying (7.1.4) for which the associated F^A are given by

$$F^A = a^A H^A, \qquad a^1 = \cdots = a^{10} = 0, \qquad a^{11} = \cdots = a^{25} = \left(\frac{5}{3}\right)^{1/2},$$

hence

$$(d(f_{3,4}(u),\ h_{3,4}(u)))^2 = \left(1-\left(\frac{3}{5}\right)^{1/2}\right)/4.$$

7.3. Let us study about extensions of $f_{m,s}$ and $h_{m,s}$ when the conditions of Lemma 7.2.1 are satisfied. $f_{m+k,s}$ and $h_{m+k,s}$ which we take as extensions are such that the corresponding $\{\tilde{F}^1, \cdots, \tilde{F}^{n(m+k,s)}\}$ and $\{\tilde{H}^1, \cdots, \tilde{H}^{n(m+k,s)}\}$ satisfy

$$\begin{aligned}(7.3.1)\quad & \tilde{F}^P(\tilde{v}) = \lambda F^P(P\tilde{v}), \qquad \tilde{H}^P(\tilde{v}) = \lambda H^P(P\tilde{v}), \qquad P=1, \cdots, n(m, s),\\ & \tilde{F}^X(\tilde{v}) = \tilde{H}^X(\tilde{v}), \qquad X=n(m, s)+1, \cdots, n(m+k, s).\end{aligned}$$

Then we have

$$\begin{aligned}
&\tilde{F}^A = \tilde{a}^A \tilde{H}^A, \qquad A=1, \cdots, n(m+k, s),\\
&\tilde{a}^P = a^P, \qquad \tilde{a}^X = 1,\\
&\tilde{\sigma}_i(\tilde{u}) = \sum_{(i)} (\tilde{H}^P(\tilde{u}))^2 = \lambda^2 \sigma_i(P\tilde{u}) = \lambda^2 \|P\tilde{u}\|^{2s} \sigma_i,\\
&\sum_X (\tilde{H}^X(\tilde{u}))^2 = \tilde{r}^2 - \lambda^2 \|P\tilde{u}\|^{2s} \sum_i \sigma_i = (r(m+k, s))^2 - \lambda^2 \|P\tilde{u}\|^{2s} (r(m, s))^2,
\end{aligned}$$

hence $\tilde{\rho}(\tilde{u}) = d(f_{m+k,s}(\tilde{u}),\ h_{m+k,s}(\tilde{u}))$ is obtained from

$$\begin{aligned}
(\tilde{\rho}(\tilde{u}))^2 &= \sum_A (\tilde{F}^A(\tilde{u}) - \tilde{H}^A(\tilde{u}))^2 = \sum_P (\tilde{F}^P(\tilde{u}) - \tilde{H}^P(\tilde{u}))^2\\
&= \lambda^2 \sum_P (a^P - 1)^2 (H^P(P\tilde{u}))^2 = \lambda^2 \|P\tilde{u}\|^{2s} \rho^2
\end{aligned}$$

where $\rho = d(f_{m,s}(u),\ h_{m,s}(u))$.

Thus we have proved the following theorem.

THEOREM 7.3.1. *Suppose that $f_{m,s}$ and $h_{m,s}$ satisfy the conditions of Lemma 7.2.1, while $f_{m+k,s}$ and $h_{m+k,s}$ are extensions satisfying* (7.3.1). *Then the pointwise distance $\tilde{\rho}(\tilde{u}) = d(f_{m+k,s}(\tilde{u}),\ h_{m+k,s}(\tilde{u}))$ satisfies $\tilde{\rho}(\tilde{u}) = \lambda \|P\tilde{u}\|^s \rho$ where $\rho = d(f_{m,s}(u),\ h_{m,s}(u))$ does not depend on u.*

Then it is clear that, at the point $\tilde{u}$ where $P\tilde{u}$ vanishes, we have $\tilde{\rho}(\tilde{u}) = 0$. This shows that $f_{m+k,s}(S^{m+k}(1))$ and $h_{m+k,s}(S^{m+k}(1))$ come in contact there. On the other hand the pointwise distance $d(f_{m+k,s}(S^m(1)),\ h_{m+k,s}(S^m(1)))$, where $S^m(1)$ is the ground sphere, is equal to $\lambda\rho$.

7.4. Let M and N be submanifolds of a Euclidean space and $\varphi: M \rightarrow N$ be a suitable mapping. Then it is not unnatural to define the distance $d(M, N)$ from M to N by

$$(d(M, N))^2 = \int (d(x, \varphi(x)))^2 d\omega \Big/ \int d\omega$$

where $x \in M$ and $d\omega$ is the volume element of M. As a variation of such an idea we define the distance $d(f_{m,s}, h_{m,s})$ as follows.

DEFINITION 7.4.1 The number $d(f_{m,s}, h_{m,s}) \geqq 0$ is defined by

$$2\frac{r^2}{n} c_m (d(f_{m,s}, h_{m,s}))^2 = \int (d(f_{m,s}(u), h_{m,s}(u)))^2 d\omega_m$$

and is called the *relative distance* or, simply, the *distance* between $f_{m,s}$ and $h_{m,s}$.

From (7.1.1) we get

$$\begin{aligned}(d(f_{m,s}(u), h_{m,s}(u)))^2 &= \sum_A (F^A(u) - H^A(u))^2 = 2r^2 - 2\sum_A F^A(u) H^A(u) \\ &= 2r^2 - 2\sum_{A,B} f^{AB} H^A(u) H^B(u),\end{aligned}$$

where $A, B = 1, \cdots, n$. Then we get, by some computations [3, (5.1) and (5.12)],

$$\int (d(f_{m,s}(u), h_{m,s}(u)))^2 d\omega_m = 2r^2 c_m - 2(\mathrm{Tr}(f)/n) r^2 c_m .$$

This proves the following lemma.

LEMMA 7.4.2. *The distance $d(f_{m,s}, h_{m,s})$ is given by*

$$(d(f_{m,s}, h_{m,s}))^2 = \mathrm{Tr}(1 - f) \tag{7.4.1}$$

where 1 *is the unit matrix of order n.*

7.5. $d(f_{m,s}, h_{m,s})$ depends on $h_{m,s}$. We now define and consider $d(f_{m,s}, \mathrm{SMI}(m, s))$ where $\mathrm{SMI}(m, s)$ is the equivalence class of $h_{m,s}$.

DEFINITION 7.5.1. $d(f_{m,s}, \mathrm{SMI}(m, s))$ is the least value of $d(f_{m,s}, h_{m,s})$, $h_{m,s} \in \mathrm{SMI}(m, s)$.

We get $d(f_{m,s}, \mathrm{SMI}(m, s))$ when $\mathrm{Tr}(f)$ takes the largest value. From Theorem 7.1.2, for any $f_{m,s}$, there exist an orthonormal basis of $\boldsymbol{R}^n$ and a standard minimal immersion $h_{m,s}$ such that

$$F^A = a^A H^A, \qquad a^A \geqq 0. \tag{7.5.1}$$

If an arbitrary orthonormal basis of $\boldsymbol{R}^n$ and an arbitrary standard minimal immersion are taken, we have an equation of the form

$$\sum_B S^{AB} F^B = a^A \sum_B T^{AB} H^B$$

instead of (7.5.1) where $S = [S^{AB}]$ and $T = [T^{AB}]$ are orthogonal matrices. Then we have

$$F^A = \sum_B f^{AB} H^B, \qquad f^{AB} = \sum_C S^{CA} a^C T^{CB}$$

and consequently

$$\mathrm{Tr}(f) = \sum_C (TS^{-1})^{CC} a^C.$$

TS^{-1} being an orthogonal matrix, no diagonal element is larger than 1. As a^C are nonnegative, we have

$$\mathrm{Tr}(f) \leqq \sum_A a^A.$$

This proves the following theorem.

THEOREM 7.5.2. *The distance* $d(f_{m,s}, \mathrm{SMI}(m, s))$ *is given by* $d(f_{m,s}, h_{m,s})$ *where* $f_{m,s}$ *and* $h_{m,s}$ *satisfy* (7.1.4), *hence*

$$(d(f_{m,s}, \mathrm{SMI}(m, s)))^2 = n - \sum_A a^A.$$

7.6. As an application we consider $d(\mathrm{Ext}_k f_{m,s}, \mathrm{SMI}(m+k, s))$. We now use indices ranging as follows,

$$P = 1, \cdots, n(m, s); \qquad X = n(m, s)+1, \cdots, n(m+k, s);$$
$$A = 1, \cdots, n(m+k, s).$$

The orthonormal basis of $\boldsymbol{R}^{n(m,s)}$ and $h_{m,s}$ are supposed to be such that we have $F^P = a^P H^P$, $a^P \geqq 0$. Then, as we have shown in § 5, we can take $\mathrm{Ext}_k f_{m,s}$, $\mathrm{Ext}_k h_{m,s}$ and the orthonormal basis of $\boldsymbol{R}^{n(m+k,s)}$ such that $\tilde{F}^P = a^P \tilde{H}^P$, $\tilde{F}^X = \tilde{H}^X$. Thus we get

$$\tilde{F}^A = \tilde{a}^A \tilde{H}^A$$

where $\tilde{a}^P = a^P$, $\tilde{a}^X = 1$. This proves the following theorem.

THEOREM 7.6.1. *Let* $\mathrm{Ext}_k f_{m,s}$ *be an extension of* $f_{m,s}$. *Then we have*

$$d(f_{m,s}, \mathrm{SMI}(m, s)) = d(\mathrm{Ext}_k f_{m,s}, \mathrm{SMI}(m+k, s)),$$

namely, extension leaves invariant the distance between an isometric minimal immersion and the equivalence class of standard minimal immersions.

§ 8. Isotropic property.

Isotropic property of isometric minimal immersions of spheres into spheres was studied in [4] and [8]. Isotropic property considered in the present paper is defined as follows.

DEFINITION 8.1. Let C be an element of $W(m, s)$. If $C(v, \cdots, v, w, \cdots, w; v, \cdots, v, w, \cdots, w)$, which is r-linear in v, identically vanishes when $r \leqq 2j+1$, then C is said to be *j-isotropic*.

Let $C \in W(m, s)$ and $\tilde{C} \in W(m+k, s)$ be such that $\tilde{C} = \Lambda C$. Then we have

$$\tilde{C}(\tilde{v}, \cdots, \tilde{v}, \tilde{w}, \cdots, \tilde{w}; \tilde{v}, \cdots, \tilde{v}, \tilde{w}, \cdots, \tilde{w})$$
$$= \lambda^2 C(v, \cdots, v, w, \cdots, w; v, \cdots, v, w, \cdots, w)$$

where $v=P\tilde{v}$, $w=P\tilde{w}$. This proves the following theorem.

THEOREM 8.2. *The mapping Λ leaves invariant the isotropic property of elements of $W(m, s)$.*

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