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Some properties for the measure-valued branching diffusion processes

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1. Introduction.

The purpose of this paper is to investigate some fundamental properties for an occupation time of a measure-valued branching diffusion process $X(t)$. The process $X(t)$ arises as a high density limit of a critical branching Brownian motion on $\boldsymbol{R}^d$ (see Dawson [1] and Watanabe [7]), hence $X(t)$ may be considered as a model describing an evolution of population with spatial migration.

One of the most important problems is concerned with the limiting distribution of the process $X(t)$ as $t\to\infty$. It is well-known that if the initial state $X(0)$ is a finite measure, then the total mass process of $X(t)$ is equivalent to a one-dimensional continuous state critical branching process and hence extinction occurs almost surely. But if $X(0)$ has an infinite total mass, then interesting phenomena arise. Namely, assuming that $X(0)$ is the Lebesgue measure on $\boldsymbol{R}^d$, Dawson [1] proved the following:

(i) If $d\leqq 2$, then $X(t)$ converges vaguely to the zero measure as $t\to\infty$ in probability.

(ii) If $d\geqq 3$, then the distribution of $X(t)$ converges weakly to a non-trivial stationary distribution as $t\to\infty$.

Furthermore, under the same initial condition, Iscoe [3] obtained the following limit theorems for the occupation time process $Y(t)=\int_0^t X(s)ds$.

(iii) If $d=1$, then $P(\lim_{t\to\infty} Y(t, K)<\infty)=1$ for every compact set K.

(iv) If $d=2$, then $P(\lim_{t\to\infty} Y(t, G)=\infty)=1$ for every non-empty open set G.

(v) If $d\geqq 3$, then $P(\lim_{t\to\infty} Y(t)/t=\lambda(\text{vaguely}))=1$, where λ denotes the Lebesgue measure on $\boldsymbol{R}^d$.

However, since the above results (iii) and (iv) seem rather crude, we would like to investigate more detailed properties for the occupation time process $Y(t)$.

It is well known that the Brownian local time is often used to characterize the limiting process concerning an occupation time of a one-dimensional Brownian

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motion (cf. [2], p. 137). For the measure-valued branching diffusion process $X(t)$, if $Y(t)$ has a density with respect to the Lebesgue measure, the density process $Y(t, x)$ would play a role of the Brownian local time, therefore, the limiting process for a suitably rescaled process of $Y(t)$ could be characterized by means of $Y(t, x)$. Indeed, it will be justified in the case $d \leqq 3$.

The main purpose of this paper is to show the existence of a density process $Y(t, x)$ for the occupation time process $Y(t)$, to investigate some smoothness properties of $Y(t, x)$, and to prove two limit theorems for rescaled processes of $Y(t)$. In Section 2 the results will be summarized, and the following sections 3 to 6 will be devoted to the proofs. The first four theorems are concerned with the existence of the density process and its smoothness. The last two theorems are concerned with scaling limit theorems for $Y(t)$. Theorem 1 states that $X(t, dx)$ has a jointly continuous density $X(t, x)$ with respect to the Lebesgue measure, which was proved in [5]. In Theorem 2 we show that the occupation time process $Y(t, dx)$ has a jointly continuous density $Y(t, x)$ with respect to the Lebesgue measure when $d \leqq 3$, for which we need a smoothness condition for the initial state in the case $d=2$ or 3. Theorem 2 will be proved in Section 3. Theorem 3 states that $Y(t, x)$ is lower semi-continuous in general when $d=2$ or 3, which will be proved in Section 4. Furthermore, we can discuss the differentiability of $Y(t, x)$ when $d=1$. By Theorem 1, $Y(t, x)$ clearly is continuously differentiable in t. In Theorem 4 we show that $Y(t, x)$ is differentiable in x also, which will be proved in Section 5. Theorems 5 and 6 are limit theorems for the rescaled process of $Y(t)$, which will be proved in Section 6 by applying the preceding theorems together with a scaling property of $Y(t)$.

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2. Summary of results.

Let $C(\boldsymbol{R}^d)$ be the Banach space of bounded continuous functions on $\boldsymbol{R}^d$ with the usual sup norm $\|\cdot\|$ and $C_K(\boldsymbol{R}^d)$ be the subspace of $C(\boldsymbol{R}^d)$ whose members have compact support. For any Radon measures μ and ν on $\boldsymbol{R}^d$, $\mu \leqq \nu$ means that $\mu(A) \leqq \nu(A)$ holds for any Borel set A of $\boldsymbol{R}^d$. A Radon measure μ on $\boldsymbol{R}^d$ is said to be atomless if $\mu(\{x\})=0$ for any $x \in \boldsymbol{R}^d$. The Lebesgue measure is denoted by λ. We denote, by $|x|$, the Euclidean norm of $x \in \boldsymbol{R}^d$. Let $M_p(\boldsymbol{R}^d)$, $p \geqq 0$, be the space of all Radon measures on $\boldsymbol{R}^d$ such that $\int(1+|x|)^{-p}\mu(dx)<\infty$. We set $M(\boldsymbol{R}^d)=\bigcup_{p \geqq 0} M_p(\boldsymbol{R}^d)$.

For a function ϕ and a measure μ we use the notations

$$(2.1)\qquad \langle\mu, \phi\rangle = \int\phi(x)\mu(dx),$$

$$(2.2)\qquad (\phi\mu)(dx) = \phi(x)\mu(dx),$$

$$(2.3)\qquad (\mu\phi)(x) = \int\phi(x-y)\mu(dy).$$

The topology of $M_p(\boldsymbol{R}^d)$ is defined as follows; $\{\mu_n\}_{n\geq 1}$ converges to μ as $n\to\infty$ if $\lim_{n\to\infty}\langle\mu_n, \phi\rangle=\langle\mu, \phi\rangle$ holds for any $\phi\in C_K(\boldsymbol{R}^d)$ and $\phi(x)=(1+|x|)^{-p}$.

Let $p\geqq 0$ be fixed. Then for any $\mu\in M_p(\boldsymbol{R}^d)$ there exists a unique diffusion process, or strong Markov process with continuous sample paths $(P_\mu, X(t))$ on the state space $M_p(\boldsymbol{R}^d)$ such that $X(0)=\mu$ and the transition function is characterized by the following Laplace functional

$$(2.4)\qquad E_\nu[\exp(-\langle X(t), \phi\rangle)] = \exp(-\langle\nu, u(t)\rangle),$$
$$\nu\in M(\boldsymbol{R}^d),\quad t\geqq 0,\quad \phi\in C_K(\boldsymbol{R}^d),\quad \phi\geqq 0,$$

where $u(t)=u(t, x)$ is the solution of

$$(2.5)\qquad \frac{\partial u}{\partial t} = \frac{\Delta}{2}u - \frac{1}{2}u^2,\qquad u(0)=\phi.$$

See Iscoe [3] and Appendix in Konno and Shiga [5].

First we mention a result of Konno and Shiga [5].

THEOREM 1([5]). *If $d=1$ and $\mu\in M(\boldsymbol{R})$ then there exists a family of nonnegative random variables $\{X(t, x), t>0, x\in\boldsymbol{R}\}$ such that the following* (i) *and* (ii) *hold* (P_μ-a.s.);

(i) *$X(t, x)$ is jointly continuous in $t>0$ and $x\in\boldsymbol{R}$,*

(ii) *for every $\phi\in C_K(\boldsymbol{R})$ and $t>0$,*

$$\langle X(t), \phi\rangle = \int X(t, x)\phi(x)dx.$$

Set

$$(2.6)\qquad Y(t) = \int_0^t X(s)ds,$$

$$(2.7)\qquad p_t(x) = p(t, x) = (2\pi t)^{-d/2}\exp(-|x|^2/(2t)),\qquad t>0,\quad x\in\boldsymbol{R}^d,$$

$$(2.8)\qquad q_t(x) = q(t, x) = \int_0^t p_s(x)ds,\qquad t\geqq 0,\quad x\in\boldsymbol{R}^d.$$

Our first result is

THEOREM 2. *Let $d\leqq 3$ and $\mu\in M(\boldsymbol{R}^d)$. When $d=2$ or 3 we assume that*

$$(2.9)\qquad (\mu q_t)(x) \textit{ is jointly continuous in } t\geqq 0 \textit{ and } x\in\boldsymbol{R}^d.$$

Then there exists a family of nonnegative random variables $\{Y(t, x), t\geqq 0, x\in\boldsymbol{R}^d\}$

such that the following (i) *and* (ii) *hold* (P_μ-*a.s.*);

(i) $Y(t, x)$ *is jointly continuous in* $t \geqq 0$ *and* $x \in \boldsymbol{R}^d$,

(ii) *for every* $\phi \in C_K(\boldsymbol{R}^d)$ *and* $t \geqq 0$,

$$\langle Y(t), \phi\rangle = \int Y(t, x)\phi(x)dx.$$

Moreover, for any $R>0$ *there exists* $c(R)>0$ *such that*

$$E_\mu[\exp(|\theta Y(t, x)|)] < \infty \tag{2.10}$$

holds for every $t \leqq R$, $x \in \boldsymbol{R}^d$ *and* $|\theta| < c(R)$.

We shall prove this theorem in Section 3.

REMARK 1. If $d=1$ and $\mu \in M(\boldsymbol{R})$ then (2.9) is always satisfied by Proposition 3.1 in Section 3.

PROPOSITION 1. *Let* $d=2$ *or* 3 *and* $\mu(dx)=g(x)dx$. *Then* (2.9) *is satisfied if one of the following conditions holds;*

(i) $g(x) \leqq C(1+|x|)^p$ *holds for some* $C>0$ *and* $p \geqq 0$,

(ii) $g(x)=|x-a|^{-p}$ *for some* $a \in \boldsymbol{R}^d$ *and* $0<p<2$.

This will be proved at the end of Section 3.

If $d=2$ or 3, for a general $\mu \in M(\boldsymbol{R}^d)$ we obtain a weaker result than Theorem 2. To this end we prepare a lemma which asserts that $Y(t)-Y(\varepsilon)$, $t \geqq \varepsilon$, has a continuous density for every fixed $\varepsilon>0$.

LEMMA 1. *Let* $d=2$ *or* 3 *and* $\mu \in M(\boldsymbol{R}^d)$. *Then for any fixed* $\varepsilon>0$ *there exists a family of nonnegative random variables* $\{Y_\varepsilon(t, x),\ t \geqq \varepsilon,\ x \in \boldsymbol{R}^d\}$ *such that the following* (i) *and* (ii) *hold* (P_μ-*a.s.*);

(i) $Y_\varepsilon(t, x)$ *is jointly continuous in* $t \geqq \varepsilon$ *and* $x \in \boldsymbol{R}^d$,

(ii) *for any* $\phi \in C_K(\boldsymbol{R}^d)$ *and* $t \geqq \varepsilon$,

$$\int_\varepsilon^t \langle X(s), \phi\rangle ds = \int Y_\varepsilon(t, x)\phi(x)dx.$$

This will be proved in Section 4.

We extend $Y_\varepsilon(t, x)$ by defining

$$Y_\varepsilon(t, x) = 0, \qquad t<\varepsilon, \quad x \in \boldsymbol{R}^d. \tag{2.11}$$

Then $Y_\varepsilon(t, x)$ increases as ε decreases. Hence we can define

$$Y(t, x) = \lim_{\varepsilon \downarrow 0} Y_\varepsilon(t, x), \qquad t \geqq 0, \quad x \in \boldsymbol{R}^d. \tag{2.12}$$

THEOREM 3. *Let* $d=2$ *or* 3, $\mu \in M(\boldsymbol{R}^d)$ *and* $Y(t, x)$ *be given by* (2.12). *Then the following* (i) *and* (ii) *hold* (P_μ-*a.s.*);

(i) *$Y(t, x)$ is jointly lower semicontinuous in $t \geqq 0$ and $x \in \boldsymbol{R}^d$,*

(ii) *for every $\phi \in C_K(\boldsymbol{R}^d)$ and $t \geqq 0$,*

$$\langle Y(t), \phi \rangle = \int Y(t, x)\phi(x)dx.$$

Moreover, $Y(t, x)$ can be modified to be continuous on the set of continuity points for $(\mu q_t)(x)$.

The above theorem will be proved in Section 4.

REMARK 2. Let μ be given by one of the following (i) and (ii);

(i) $d=2$ or 3 and $\mu=\delta_a$,

(ii) $d=3$ and $\mu(dx)=|x-a|^{-p}dx$ for some $2 \leqq p < 3$.

Then it is easy to see that $(\mu q_t)(a)=\infty$, $t>0$, but $(\mu q_t)(x)$ is continuous on $[0, \infty)\times(\boldsymbol{R}^d-\{a\})$.

We can discuss the differentiability of $Y(t, x)$ in the case $d=1$. We denote, by $D_x f(x)$ (resp. $D_x^+ f(x)$, $D_x^- f(x)$), the derivative (resp. right derivative, left derivative) of $f(x)$. If $d=1$ then $D_t Y(t, x)=X(t, x)$ is continuous on $(0, \infty)\times \boldsymbol{R}$ by Theorem 1.

THEOREM 4. *If $d=1$ and $\mu \in M(\boldsymbol{R})$ then the following* (i) *and* (ii) *hold* (P_μ-*a.s.*);

(i) *$Z(t, x) = Y(t, x) - E_\mu[Y(t, x)]$ is differentiable with respect to x,*

(ii) *$D_x Z(t, x)$ is jointly continuous in $t \geqq 0$ and $x \in \boldsymbol{R}$,*

$$D_x^+ E_\mu[Y(t, x)] - D_x^- E_\mu[Y(t, x)] = -2\mu(\{x\}), \qquad t>0, \quad x \in \boldsymbol{R}. \tag{2.13}$$

In particular, if μ is atomless, then $D_x E_\mu[Y(t, x)]$ is continuous and so $Y(t, x)$ is differentiable with respect to x and $D_x Y(t, x)$ is continuous (P_μ-a.s.). Moreover, for any $R>0$ there exists $c(R)>0$ such that

$$E_\mu[\exp(|\theta D_x Z(t, x)|)] < \infty \tag{2.14}$$

holds for every $t \leqq R$, $x \in \boldsymbol{R}$ and $|\theta| < c(R)$.

We shall prove this theorem in Section 5.

Finally we shall study the continuous process $(\langle Y(t), \phi \rangle, Y(t, a))$ for some fixed $\phi \in C_K(\boldsymbol{R}^d)$ and $a \in \boldsymbol{R}^d$ in the case $d \leqq 3$. Let W^n be the space of all continuous functions from $[0, \infty)$ to $\boldsymbol{R}^n$ endowed with the topology of uniform convergence on each finite interval. For any continuous process $(P, X(t))$, P^X denotes the probability measure on W^n induced from P by $X=(X(t))$. Two continuous processes $(P_1, X(t))$ and $(P_2, Y(t))$ are said to be equivalent if P_1^X and P_2^Y coincide. A family of continuous processes $\{(P_\rho, X_\rho(t)), \rho>0\}$ is said to be convergent to a continuous process $(P, X(t))$ as $\rho \to \infty$ if $P_\rho^{X_\rho}$ converges to P^X weakly as $\rho \to \infty$.

For any $\mu\in M(\boldsymbol{R}^{d})$ and $\rho>0$ we define $\mu(\rho\cdot)\in M(\boldsymbol{R}^{d})$ by

$$\langle\mu(\rho\cdot), \phi\rangle=\langle\mu, \phi(\rho^{-1}\cdot)\rangle. \tag{2.15}$$

Hence if $\mu=g\lambda$ then we have

$$\mu(\rho\cdot)=\rho^{d}g(\rho\cdot)\lambda. \tag{2.16}$$

THEOREM 5. *Let $d\leqq 3$ and $\mu\in M(\boldsymbol{R}^{d})$. When $d=2$ or 3 we assume that* (2.9) *holds. Then for every fixed $\phi\in C_{K}(\boldsymbol{R}^{d})$ and $a\in\boldsymbol{R}^{d}$, the family of continuous processes*

$$(P_{\rho^{2}\mu(\rho^{-1}\cdot)}, \rho^{d-4}(\langle Y(\rho^{2}t), \phi\rangle, Y(\rho^{2}t, a))), \qquad \rho>0,$$

converges to the continuous process $(P_{\mu}, (\langle\lambda, \phi\rangle, 1)Y(t, 0))$ as $\rho\rightarrow\infty$.

THEOREM 6. *If $d=1$ and $\mu\in M(\boldsymbol{R})$ is atomless, then for any fixed $\phi\in C_{K}(\boldsymbol{R})$ and $a\in\boldsymbol{R}$ the family of continuous processes*

$$\left(P_{\rho^{2}\mu(\rho^{-1}\cdot)}, \rho^{-2}\left(\int(Y(\rho^{2}t, x)-Y(\rho^{2}t, 0))\phi(x)dx, D_{x}Y(\rho^{2}t, a)\right)\right), \qquad \rho>0,$$

conrerges to the continuous process $\left(P_{\mu}, \left(\int x\phi(x)dx, 1\right)D_{x}Y(t, 0)\right)$ as $\rho\rightarrow\infty$.

REMARK 3. If $\mu(dx)=|x|^{2-d}\chi(x)dx$, where χ is an indicator function of a cone in $\boldsymbol{R}^{d}$, then by (2.16) we have $\rho^{2}\mu(\rho^{-1}\cdot)=\mu$. Hence, in Theorems 5 and 6, the probability laws of the rescaled processes coincide with the original ones.

REMARK 4. Theorem 5 implies that the rescaled process $\rho^{d-4}Y(\rho^{2}t)$ under $P_{\rho^{2}\mu(\rho^{-1}\cdot)}$ converges as $\rho\rightarrow\infty$ to the measure valued process $Y(t, 0)\lambda$ under P_{μ}, and an analogous statement holds for Theorem 6 also.

We shall prove the last two theorems in Section 6.

3. Proof of Theorem 2.

We shall prove Theorem 2 by showing the following three propositions. Set

$$Y_{h}(t, a)=\int p_{h}(a-x)Y(t, dx). \tag{3.1}$$

PROPOSITION 3.1. *If $\mu\in M(\boldsymbol{R}^{d})$ then for any $t\geqq 0$ and $a\in\boldsymbol{R}^{d}$ we have*

$$\lim_{h\rightarrow 0}E_{\mu}[Y_{h}(t, a)]=(\mu q_{t})(a). \tag{3.2}$$

If $d=1$ then $(\mu q_{t})(x)$ is jointly continuous in $t\geqq 0$ and $x\in\boldsymbol{R}^{d}$.

Set

$$Z_h(t, x) = Y_h(t, x) - E_\mu[Y_h(t, x)]. \tag{3.3}$$

PROPOSITION 3.2. *If $d \leqq 3$, $\mu \in M(\boldsymbol{R}^d)$ and (2.9) holds for $d=2, 3$, then for any $t \geqq 0$ and $a \in \boldsymbol{R}^d$ there exists*

$$\underset{h\to 0}{\text{l.i.m.}}\, Z_h(t, a) = Z_0(t, a) \quad \textit{with respect to } P_\mu. \tag{3.4}$$

For the existence of a continuous version of $Z_0(t, x)$, by Totoki's theorem it suffices to get the following estimates.

PROPOSITION 3.3. *If $d \leqq 3$, $\mu \in M(\boldsymbol{R}^d)$ and (2.9) holds for $d=2, 3$, then for each $n \geqq 1$ there exist positive constants α, β, c_n such that*

$$E_\mu[|Z_0(t, a) - Z_0(t, b)|^\alpha] \leqq c_n |a-b|^{d+1+\beta}, \tag{3.5}$$

$$E_\mu[|Z_0(t, a) - Z_0(s, a)|^\alpha] \leqq c_n |t-s|^{d+1+\beta}, \tag{3.6}$$

hold for every $0 \leqq s, t \leqq n$ and $|a|, |b| \leqq n$.

We shall show (2.10) after the proof of Proposition 3.3.

PROOF OF PROPOSITION 3.1. It is needed to estimate several moments of $Y(t)$. We use the following expression of Laplace functionals, which is found in Iscoe [3]

$$E_\mu[\exp(-\langle Y(t), \phi\rangle)] = \exp(-\langle \mu, v(t)\rangle), \qquad t \geqq 0,\ \phi \in C_K(\boldsymbol{R}^d),\ \phi \geqq 0, \tag{3.7}$$

in which $v(t) = v(t, x)$ is the solution of

$$\frac{\partial v}{\partial t} = \frac{\Delta}{2} v - \frac{1}{2} v^2 + \phi, \qquad v(0) = 0. \tag{3.8}$$

We define

$$P_t\phi(x) = \int p_t(x-y)\phi(y)dy, \qquad Q_t\phi(x) = \int q_t(x-y)\phi(y)dy, \tag{3.9}$$

$$(\mu P_t)(dx) = (\mu p_t)(x)dx, \qquad (\mu Q_t)(dx) = (\mu q_t)(x)dx. \tag{3.10}$$

Then we have

$$\begin{cases} \langle \mu P_t, \phi\rangle = \langle \mu, P_t\phi\rangle = \int (\mu p_t)(x)\phi(x)dx, \\ \langle \mu Q_t, \phi\rangle = \langle \mu, Q_t\phi\rangle = \int (\mu q_t)(x)\phi(x)dx. \end{cases} \tag{3.11}$$

By (3.7) and (3.8) we obtain

$$E_\mu[\langle Y(t), \phi\rangle] = \langle \mu, Q_t\phi\rangle, \tag{3.12}$$

$$E_\mu[\langle Y(t) - \mu Q_t, \phi\rangle^2] = \int_0^t \langle \mu, P_{t-s}(Q_s\phi)^2\rangle ds. \tag{3.13}$$

Hence we have

$$E_\mu[Y_h(t, a)] = \int_h^{t+h} ds \int p_s(a-x)\mu(dx), \tag{3.14}$$

and (3.2) follows easily.

We remark that for any $p \geqq 0$ there exists $C = C(p) > 0$ such that

$$|x|^p p_t(x) \leqq Ct^{(p-d)/2}, \tag{3.15}$$

$$(1+|x|)^p p_t(x) \leqq Ct^{-d/2}(1+t)^p, \tag{3.16}$$

hold for every $t>0$ and $x \in \boldsymbol{R}^d$. Then it is easy to see that if $\mu \in M(\boldsymbol{R}^d)$ then there exist $p \geqq 0$ and $C>0$ such that

$$(\mu p_t)(x) \leqq Ct^{-d/2}(1+t)^p(1+|x|)^p, \quad t>0, \quad x \in \boldsymbol{R}^d. \tag{3.17}$$

Hence if $d=1$ and $\mu \in M(\boldsymbol{R})$ then we have

$$\lim_{t \to 0} (\mu q_t)(x) = 0 \tag{3.18}$$

uniformly on each compact set of R. Thus the latter part of Proposition 3.1 follows.

PROOF OF PROPOSITION 3.2. Set

$$q_{t,h}(x) = P_h q_t(x) = \int_h^{t+h} p_s(x) ds. \tag{3.19}$$

Then by (3.13) we have

$$E_\mu[(Z_h(t, a) - Z_k(t, a))^2] = \int_0^t \langle \mu, P_{t-s}(q_{s,h}(a-\cdot) - q_{s,k}(a-\cdot))^2 \rangle ds. \tag{3.20}$$

Since $\lim_{h \to 0} q_{t,h}(x) = q_t(x)$ it suffices to show that

$$\lim_{h \to 0} \int_0^t \langle \mu, P_{t-s}(q_{s,h}(a-\cdot) - q_s(a-\cdot))^2 \rangle ds = 0. \tag{3.21}$$

We remark that

$$q_{t,h}(x) - q_t(x) = (P_t q_h)(x) - q_h(x), \tag{3.22}$$

$$P_t(P_s q_h)^2(x) \leqq P_t P_s q_h(x) \|P_s q_h\| = P_{t+s} q_h(x) P_s q_h(0). \tag{3.23}$$

Hence it suffices to show that

$$\int_0^t (\mu P_t q_h)(a) P_s q_h(0) ds, \tag{3.24}$$

$$\int_0^t ds \int (\mu p_s)(x) q_h(a-x)^2 dx \tag{3.25}$$

converge to 0 as $h \to 0$. We remark that

$$q_t(x)^2=\int_0^t ds\int_0^t dr(2\pi(r+s))^{-d/2}p\Big(\frac{rs}{r+s},\ x\Big). \tag{3.26}$$

Then it suffices to show that (3.24) and

$$\int_0^h dr\int_0^h dq(r+q)^{-d/2}\int_0^t\Big(\mu p\Big(s+\frac{rq}{r+q},\ \cdot\Big)\Big)(a)ds \tag{3.27}$$

converge to 0 as $h\to 0$. But by the continuity of $(\mu q_t)(x)$ it suffices to show that

$$\int_0^t ds\int_s^{s+h} r^{-d/2}dr, \tag{3.28}$$

$$\int_0^h dr\int_0^h (r+q)^{-d/2}dq \tag{3.29}$$

converge to 0 as $h\to 0$. But this follows from the assumption $d\leqq 3$. Thus we have shown Proposition 3.2.

Before proceeding to the proof of Proposition 3.3 we shall give some remarks. By (3.7), (3.8) and a routine work we have

$$E_\mu[\exp(\theta\langle Y(t)-\mu Q_t,\ \phi\rangle)]=\exp\Big(2\sum_{n=2}^{\infty}\Big(\frac{\theta}{2}\Big)^n\langle\mu,\ v_n(t)\rangle\Big), \tag{3.30}$$

where $v_n(t)$, $n\geqq 2$, are determined by

$$\begin{cases} v_1(t)=Q_t\phi, \\ v_n(t)=\sum_{k=1}^{n-1}\int_0^t dsP_{t-s}(v_k(s)v_{n-k}(s)), \qquad n\geqq 2. \end{cases} \tag{3.31}$$

Then by (2.4), (2.5) and the Markov property we have

$$E_\mu[\exp(\theta\langle Y(t+s)-Y(s)-\mu Q_{t+s}+\mu Q_s,\ \phi\rangle)]=\exp\Big(2\sum_{n=2}^{\infty}\Big(\frac{\theta}{2}\Big)^n\langle\mu,\ v_n(s,\ t)\rangle\Big), \tag{3.32}$$

where $v_n(s, t)$, $n\geqq 2$, are determined by

$$\begin{cases} v_1(s,\ t)=P_s v_1(t), \\ v_n(s,\ t)=P_s v_n(t)+\sum_{k=1}^{n-1}\int_0^s drP_{s-r}(v_k(r,\ t)v_{n-k}(r,\ t)), \qquad n\geqq 2. \end{cases} \tag{3.33}$$

It is not difficult to justify (3.30) and (3.32) for each fixed $\mu\in M(\boldsymbol{R}^d)$, $\phi\in C_K(\boldsymbol{R}^d)$ and $s,\ t\geqq 0$ if $|\theta|$ is sufficiently small. We shall apply (3.30) and (3.32) to prove Proposition 3.3. We notice that if the continuous density $Y(t,\ x)$ exists then $Y(t,\ a)$ is written as $\int\delta_a(x)Y(t,\ x)dx$ where δ_a is the delta function at a. But (3.30) and (3.32) may not have a meaning for $\phi=\delta_a$. However (3.30) and (3.32) hold even for a $\phi=\delta_a$ in a weak sense. To state this precisely we introduce the following notion. For a random variable X, we say

$$E[\exp(\theta X)]=\exp\Big(\sum_{n=1}^{\infty}a_n\theta^n\Big) \tag{3.34}$$

holds formally or we have formally (3.34), if $E[|X|^n]<\infty$ and

$$E[X^n]=D_\theta^n\Big(\exp\Big(\sum_{k=1}^{n}a_k\theta^k\Big)\Big)\Big|_{\theta=0}$$

holds for every $n\geqq 1$, where $\{a_n\}$ is a sequence of real numbers. Then it is easy to see that we have formally (3.30) and (3.32) for $\phi=\delta_a$. The following lemma is often useful.

LEMMA 3.1. *Let X be a random variable such that* (3.34) *holds formally.*

(i) *If for some integer N there exist $r, b>0$ such that*

$$|a_n|\leqq br^n, \qquad \text{for} \quad 1\leqq n\leqq 2N, \tag{3.35}$$

then there exists $C=C(b, N)>0$ such that

$$E[X^{2N}]\leqq Cr^{2N}. \tag{3.36}$$

(ii) *If $\sum_{n=1}^{\infty}a_n\theta_0^n$ converges for some $\theta_0>0$ then*

$$E[\exp(|\theta X|)]<\infty \qquad \text{for} \quad |\theta|<\theta_0. \tag{3.37}$$

The proof of this lemma is easy and so we omit it.

Before proving (3.5) we need two more lemmas. By (3.30), (3.31) and the definition of $Z_0(t, x)$ we have formally that

$$E_\mu[\exp(\theta(Z_0(t, a)-Z_0(t, b)))]=\exp\Big(2\sum_{n=2}^{\infty}\Big(\frac{\theta}{2}\Big)^n\langle\mu, v_n(t)\rangle\Big), \tag{3.38}$$

where $v_n(t)$, $n\geqq 2$, are given by (3.31) with

$$v_1(t, x)=q_t(a-x)-q_t(b-x). \tag{3.39}$$

LEMMA 3.2. *Let $d\leqq 3$, choose α such that*

$$0<\alpha<\min\{1, 2-d/2\}, \tag{3.40}$$

and set

$$\bar{v}(t, x)=\int_0^{2t}s^{-\alpha/2}(p_s(a-x)+p_s(b-x))ds. \tag{3.41}$$

Then for each $R>0$ there exist positive constants $a_n=a_n(R)$, $n=1, 2, \cdots$, such that

$$|v_n(t, x)|\leqq a_n|a-b|^{n\alpha}\bar{v}(t, x) \tag{3.42}$$

holds for every $t\leqq R$, $a, b, x\in\boldsymbol{R}^d$ and $n\geqq 1$.

Before proceeding to the proof of this lemma we shall give some remarks. It is easy to see that

$$(3.43) \qquad (p_t(x)+p_t(y))(p_s(x)+p_s(y)) \leqq 3(p_t(x)p_s(x)+p_t(y)p_s(y))$$
$$\leqq 3(st)^{-d/4}\Big(p\Big(\frac{ts}{t+s},\ x\Big)+p\Big(\frac{ts}{t+s},\ y\Big)\Big),$$
$$s,\ t>0,\quad x,\ y\in \boldsymbol{R}^d.$$

For any $0<\alpha\leqq 1$ there exists $C=C(\alpha)>0$ such that

$$(3.44) \qquad |p_t(x)-p_t(y)| \leqq Ct^{-\alpha/2}|x-y|^{\alpha}(p_{2t}(x)+p_{2t}(y)), \qquad t>0,\quad x,\ y\in \boldsymbol{R}^d.$$

This is seen as follows. Set $D(\alpha)=\max\{2r^{2-\alpha}\exp(-r^2);\ r\geqq 0\}$. Then for any $t\geqq s\geqq 0$ we have

$$\exp(-s^2)-\exp(-t^2)=\int_s^t 2r\exp(-r^2)dr \leqq \alpha^{-1}D(\alpha)(t^{\alpha}-s^{\alpha}) \leqq \alpha^{-1}D(\alpha)(t-s)^{\alpha}.$$

Then (3.44) follows from this and

$$p_t(x)-p_t(y)=(8\pi t)^{d/2}(p_{2t}(x)-p_{2t}(y))(p_{2t}(x)+p_{2t}(y)).$$

PROOF OF LEMMA 3.2. Since the case $n=1$ follows from (3.44) it suffices to show (3.42) in the case $n\geqq 2$ assuming that it holds for every $1\leqq k\leqq n-1$. Then we have

$$(3.45) \qquad |v_n(t)| \leqq \sum_{k=1}^{n-1} a_k a_{n-k}|a-b|^{n\alpha}\int_0^t dsP_{t-s}(\bar{v}(s)^2), \qquad t\leqq R.$$

We set $z(t,\ x)=p_t(a-x)+p_t(b-x)$ for a while. By (3.43) we obtain

$$(3.46) \qquad \bar{v}(t,\ x)^2 \leqq 3\int_0^{2t}ds\int_0^{2t}dr(rs)^{-(d+2\alpha)/4}z\Big(\frac{rs}{r+s},\ x\Big).$$

Then it follows that

$$(3.47) \qquad \int_0^t dsP_{t-s}(\bar{v}(s)^2)(x) \leqq 3\int_0^t ds\int_0^{2s}dr\int_0^{2s}dq(rq)^{-(d+2\alpha)/4}z\Big(t-s+\frac{rq}{r+q},\ x\Big)$$
$$\leqq 3\int_0^{2t}dr\int_0^{2t}dq(rq)^{-(d+2\alpha)/4}\int_0^{2t}z(s,\ x)ds$$
$$=C_1(\alpha)t^{2-(d+2\alpha)/2}\int_0^{2t}(p_s(a-x)+p_s(b-x))ds.$$

From this we have

$$(3.48) \qquad \int_0^t dsP_{t-s}(\bar{v}(s)^2)(x) \leqq C_2(\alpha)t^{2-(d+\alpha)/2}\bar{v}(t,\ x)$$

and (3.42) follows from (3.45).

Combining Lemma 3.2 with (3.47) we obtain

LEMMA 3.3. *If $d\leqq 3$ and α satisfies* (3.40) *then for any $R>0$ there exist positive constants $b_n=b_n(R)$, $n=2, 3, \cdots$, such that*

$$(3.49) \qquad |v_n(t,\ x)| \leqq b_n|a-b|^{n\alpha}(q_{2t}(a-x)+q_{2t}(b-x))$$

holds for every $t\leqq R$, a, b, $x\in\boldsymbol{R}^d$ *and* $n\geqq 2$.

(3.5) in Proposition 3.3 now follows from (3.38), Lemma 3.3, Lemma 3.1, (i) and the continuity of $(\mu q_t)(x)$.

Before proving (3.6) we prepare two more lemmas. By (3.32) we have formally that

$$E_\mu[\exp(\theta(Z_0(t+s,\ a)-Z_0(s,\ a)))]=\exp\Big(2\sum_{n=2}^{\infty}\Big(\frac{\theta}{2}\Big)^n\langle\mu,\ v_n(s,\ t)\rangle\Big), \tag{3.50}$$

where $v_n(s,\ t)$, $n\geqq 2$, are given by (3.31) and (3.33) with

$$v_1(t,\ x)=q_t(a-x). \tag{3.51}$$

LEMMA 3.4. *If* $d\leqq 3$ *then there exist positive constants* a_n, $n=1, 2, \cdots$, *such that*

$$v_n(t,\ x)\leqq a_n t^{(2-d/2)(n-1)}q_{2t}(a-x) \tag{3.52}$$

holds for every $t\geqq 0$, a, $x\in\boldsymbol{R}^d$ *and* $n\geqq 1$, *and*

$$f(\theta)=\sum_{n=1}^{\infty}a_n\theta^n \tag{3.53}$$

has a positive radius of convergence.

PROOF. Since (3.52) is obvious in the case $n=1$, we show (3.52) in the case $n\geqq 2$ assuming that it holds for every $1\leqq k\leqq n-1$. By (3.31) we have

$$v_n(t)\leqq\sum_{k=1}^{n-1}a_k a_{n-k}t^{(2-d/2)(n-2)}w(t) \tag{3.54}$$

where

$$w(t,\ x)=\int_0^t ds\int_0^{2s}dr\int_0^{2s}dq(2\pi(r+q))^{-d/2}p\Big(t-s+\frac{rq}{r+q},\ a-x\Big). \tag{3.55}$$

By $q+r\geqq(qr)^{1/2}$, w is dominated by

$$\int_0^{2t}dr\int_0^{2t}dq(qr)^{-d/4}\int_0^{2t}p_s(a-x)ds=Kt^{2-d/2}q_{2t}(a-x)$$

where $K=(1-d/4)^{-2}2^{2-d/2}$. Hence we have shown (3.52). By the above argument, a_n, $n\geqq 1$, may be determined by

$$a_1=1,\qquad a_n=K\sum_{k=1}^{n-1}a_k a_{n-k},\qquad n\geqq 2. \tag{3.56}$$

Then $f(\theta)$ satisfies $f(\theta)-\theta=Kf(\theta)^2$, i.e., $f(\theta)=(1-(1-4K\theta)^{1/2})/(2K)$. Then (3.53) follows.

LEMMA 3.5. *If* $d\leqq 3$ *then for any* $R>0$ *there exist positive constants* $b_n=b_n(R)$, $n=1, 2, \cdots$, *such that*

(3.57) $$v_n(s, t, x) \leqq b_n t^{(n-1)/2}(P_s v_1(t))(x)$$

holds for every $s, t \leqq R$, $a, x \in \boldsymbol{R}^d$ *and* $n \geqq 1$, *where* $v_1(t, x) = q_{2t}(a-x)$.

PROOF. Let $R>0$ be fixed and assume that $s, t \leqq R$. Since the case $n=1$ is clear we show (3.57) in the case $n \geqq 2$ assuming that it holds for every $1 \leqq k \leqq n-1$. Set

(3.58) $$\begin{cases} w_0(s, t) = P_s v_n(t), \\ w_k(s, t) = \int_0^s dr P_{s-r}(v_k(r, t) v_{n-k}(r, t)), \qquad 1 \leqq k \leqq n-1. \end{cases}$$

It suffices to show that there exists $c(R)>0$ such that

(3.59) $$w_k(s, t) \leqq c(R) t^{(n-1)/2} P_s v_1(t)$$

holds for every $0 \leqq k \leqq n-1$. If $k=0$ then this is obvious by (3.52). If $1 \leqq k \leqq n-1$ then we have

(3.60) $$w_k(s, t) \leqq c(R) t^{(n-2)/2} \int_0^s dr P_{s-r}(P_r v_1(t))^2.$$

But by (3.23) we have

$$P_{s-r}(P_r v_1(t))^2 \leqq P_s v_1(t) \int_r^{2t+r} (2\pi q)^{-d/2} dq,$$

and it follows that

(3.61) $$w_k(s, t) \leqq c(R) t^{(n-2)/2} P_s v_1(t) \int_0^s dr \int_r^{2t+r} q^{-d/2} dq \leqq c_1(R) t^{(n-1)/2} P_s v_1(t).$$

Hence we have completed the proof.

PROOF OF (3.6) OF PROPOSITION 3.3. We assume that $s, t, |a| \leqq N$ holds for some fixed $N \geqq 1$. We remark that

(3.62) $$(P_s v_1(t))(x) = \int_s^{s+2t} p_r(a-x) dr.$$

Then by the continuity of $(\mu q_t)(x)$ there exists $C(N)>0$ such that

(3.63) $$\langle \mu, P_s v_1(t) \rangle \leqq C(N).$$

By $n-1 \geqq n/2$ for $n \geqq 2$ and (3.57) there exist positive constants $c_n = c_n(N)$, $n=2, 3, \cdots$, such that

(3.64) $$\langle \mu, v_n(s, t) \rangle \leqq c_n t^{n/4}$$

holds for $n \geqq 2$. Hence (3.6) follows from (3.50) and Lemma 3.1, (i).

PROOF OF (2.10). By Lemma 3.4 we have

(3.65) $$\langle\mu, v_n(t)\rangle \leqq a_n t^{(2-d/2)(n-1)}(\mu q_{2t})(a), \qquad n\geqq 1.$$

Then (2.10) follows from Lemma 3.4, Lemma 3.1, (ii) and the continuity of $(\mu q_t)(x)$ if we set $s=0$ in (3.50).

PROOF OF PROPOSITION 1. It suffices to show (2.9). To this end we have only to show that

(3.66) $$\lim_{t\to 0}(\mu q_t)(x) = 0$$

uniformly on each compact set of $\boldsymbol{R}^d$. If $p\geqq 0$ then there exists $C>0$ such that

(3.67) $$\begin{aligned}\int(1+|y|)^p p_t(x-y)dy &= \int(1+|x+t^{1/2}y|)^p p_1(y)dy\\ &\leqq C(1+t)^p(1+|x|)^p\int(1+|y|)^p p_1(y)dy.\end{aligned}$$

Hence (3.66) holds in the case (i). We shall show (3.66) in the case (ii). To this end we need a lemma, see Lemma 3.2 in [4].

LEMMA 3.6. *If ϕ is a nonnegative and decreasing function on $[0, \infty)$ then*

(3.68) $$\int p_t(x-z)\phi(|z|)dz \leqq \int p_t(y-z)\phi(|z|)dz$$

holds for every $t>0$ and $|x|\geqq|y|$.

By this lemma we have

(3.69) $$\int|y-a|^{-p}p_t(x-y)dy \leqq \int|y|^{-p}p_t(y)dy = t^{-p/2}\int|y|^{-p}p_1(y)dy.$$

Then (3.66) follows easily in the case (ii) by $p<2$.

4. Proof of Theorem 3.

We shall first show Lemma 1 by following the argument used in Section 3. By (3.11) and (3.17) we have

LEMMA 4.1. *If $d\leqq 3$ then for each fixed $t>0$ and $\mu\in M(\boldsymbol{R}^d)$*

(4.1) $$(\mu P_t)(dx) = (\mu p_t)(x)dx$$

satisfies condition (i) *in Proposition* 1.

We set

(4.2) $$Y_\varepsilon(t+\varepsilon) = \int_\varepsilon^{t+\varepsilon} X(s)ds, \qquad t\geqq 0,$$

(4.3) $$Y_{\varepsilon,h}(t+\varepsilon, a) = \int p_h(a-x)Y_\varepsilon(t+\varepsilon, dx), \qquad h>0,\ t\geqq 0,\ a\in\boldsymbol{R}^d.$$

Then it is easy to see that

$$\lim_{h\to 0} E_\mu[Y_{\varepsilon,h}(t+\varepsilon, a)] = (\mu P_\varepsilon q_t)(a) \tag{4.4}$$

and this is continuous on $[0, \infty)\times \boldsymbol{R}^d$ by Lemma 4.1 and Proposition 1. Set

$$Z_{\varepsilon,h}(t+\varepsilon, a) = Y_{\varepsilon,h}(t+\varepsilon, a) - E_\mu[Y_{\varepsilon,h}(t+\varepsilon, a)]. \tag{4.5}$$

Then by (3.13) and the Markov property we have

$$\begin{aligned} E_\mu[(Z_{\varepsilon,h}(t+\varepsilon, a) - Z_{\varepsilon,k}(t+\varepsilon, a))^2] &= E_\mu[E_{X(\varepsilon)}[(Y_h(t, a) - Y_k(t, a))^2]] \\ &= \int_0^t ds \langle \mu P_\varepsilon, P_{t-s}(q_{s,h}(a-\cdot) - q_{s,k}(a-\cdot))^2\rangle. \end{aligned} \tag{4.6}$$

Then by Lemma 4.1 and Proposition 1 we can follow the proof of Proposition 3.2. Hence there exists

$$\underset{h\to 0}{\text{l.i.m.}}\, Z_{\varepsilon,h}(t+\varepsilon, a) = Z_\varepsilon(t+\varepsilon, a), \qquad t \geqq 0, \quad a \in \boldsymbol{R}^d, \tag{4.7}$$

with respect to P_μ. Hence it suffices to show (3.5) and (3.6) for $Z_\varepsilon(t+\varepsilon, a)$. We shall show (3.5) in our case. By (2.4), (3.30) and the Markov property we have formally that

$$\begin{aligned} E_\mu[\exp(\theta(Z_\varepsilon(t+\varepsilon, a) - Z_\varepsilon(t+\varepsilon, b)))] &= E_\mu[E_{X(\varepsilon)}[\exp(\theta(Z(t, a) - Z(t, b)))]] \\ &= \exp\Big(2\sum_{n=2}^\infty \Big(\frac{\theta}{2}\Big)^n \langle \mu, v_n(\varepsilon, t)\rangle\Big), \end{aligned} \tag{4.8}$$

where $v_n(s, t)$, $n \geqq 2$, are determined by (3.31) and (3.33) with

$$v_1(t, x) = q_t(a-x) - q_t(b-x). \tag{4.9}$$

The next lemma follows from Lemmas 3.2 and 3.3 as we have shown Lemma 3.5 from Lemma 3.4.

LEMMA 4.2. *Let $d \leqq 3$ and α be chosen to satisfy* (3.40). *Then for any fixed $R>0$ there exist positive constants $a_n = a_n(R)$, $n=2, 3, \cdots$, such that*

$$|v_n(s, t, x)| \leqq a_n |a-b|^{n\alpha} (P_s \bar{v}(t))(x) \tag{4.10}$$

holds for any $s, t \leqq R$, $a, b, x \in \boldsymbol{R}^d$ and $n \geqq 2$, where $\bar{v}(t, x) = q_{2t}(a-x) + q_{2t}(b-x)$.

Then (3.5) in our case follows from Lemmas 4.1 and 4.2 as we have shown (3.5) by using Lemma 3.3 in Section 3.

We shall show (3.6) in our case. By (3.50) and the Markov property we have formally that

$$\begin{aligned} &E_\mu[\exp(\theta(Z_\varepsilon(t+s+\varepsilon, a) - Z_\varepsilon(s+\varepsilon, a)))] \\ &\quad = E_\mu\Big[\exp\Big(2\sum_{n=2}^\infty \Big(\frac{\theta}{2}\Big)^n \langle X(\varepsilon), v_n(s, t)\rangle\Big)\Big], \end{aligned} \tag{4.11}$$

where $v_n(s, t)$, $n \geqq 2$, are those given by (3.50). Then by (2.4) and (2.5) we have formally that

$$E_\mu[\exp(\theta(Z_\varepsilon(t+s+\varepsilon, a)-Z_\varepsilon(s+\varepsilon, a)))] = \exp\Big(2\sum_{n=2}^{\infty}\Big(\frac{\theta}{2}\Big)^n\langle\mu, u_n(\varepsilon, s, t)\rangle\Big), \tag{4.12}$$

where $u_n(r, s, t)$, $n \geqq 2$, are given by

$$\begin{cases} u_1(r, s, t) = P_r v_1(s, t), \\ u_n(r, s, t) = P_r v_n(s, t) + \sum_{k=1}^{n-1}\int_0^r P_{r-r_1}(u_k(r_1, s, t)u_{n-k}(r_1, s, t))dr_1, \quad n \geqq 2. \end{cases} \tag{4.13}$$

Then by the same method of the proof of Lemma 3.5 we obtain

Lemma 4.3. *If $d \leqq 3$ then for any fixed $R>0$ there exist positive constants $a_n = a_n(R)$, $n=2, 3, \cdots$, such that*

$$u_n(r, s, t, x) \leqq a_n t^{(n-1)/2}(P_{r+s}v_1(t))(x), \tag{4.14}$$

holds for every $r, s, t \leqq R$, $a, x \in \boldsymbol{R}^d$ and $n \geqq 2$, where $v_1(t, x) = q_{2t}(a-x)$.

Then (3.6) in our case follows from Lemmas 4.1 and 4.3 as we have shown (3.6) by using Lemma 3.5 in Section 3. Thus we have shown Lemma 1.

We shall show Theorem 3. Since the first part is obvious by the definition of $Y(t, x)$ it suffices to show the second part. Let A be the set of continuity points for $(\mu q_t)(x)$ and set

$$A_n = \{(t, x) \in A : t, |x|, (\mu q_t)(x) \leqq n\}, \qquad n=1, 2, \cdots.$$

Then by Lemmas 3.3 and 3.5 we can show Proposition 3.3 for each A_n. Then $Y(t, x)$ can be modified to be continuous on each A_n, see [**6**, p. 186, Remark 2]. Hence it suffices to prove

Lemma 4.4. *If $f(t, x)$ is continuous on each A_n then f is continuous on A.*

Proof. It suffices to show that

$$\lim_{n\to\infty} f(t_n, x_n) = f(t_0, x_0) \tag{4.15}$$

holds for any $\{(t_n, x_n);\ n \geqq 0\} \subset A$ assuming that $\lim_{n\to\infty}(t_n, x_n) = (t_0, x_0)$. Then there exists m such that $(t_0, x_0) \in A_m$. Since $(\mu q_t)(x)$ is continuous at (t_0, x_0) there exists n_0 such that $(t_n, x_n) \in A_{m+1}$ holds for all $n \geqq n_0$. Then (4.15) follows from the continuity of f on A_{m+1}.

5. Proof of Theorem 4.

The method of the proof is the same as that used in Section 3. Set

$$\bar{Y}_h(t, x) = h^{-1}(Y(t, x+h) - Y(t, x)), \tag{5.1}$$

$$W_h(t, x) = \bar{Y}_h(t, x) - E_\mu[\bar{Y}_h(t, x)]. \tag{5.2}$$

We shall show

PROPOSITION 5.1. *Let $\mu \in M(\boldsymbol{R})$. For any $t \geqq 0$ and $x \in \boldsymbol{R}$ there exist*

$$\lim_{h \downarrow 0} E_\mu[\bar{Y}_h(t, x)] = f_+(t, x), \qquad \lim_{h \uparrow 0} E_\mu[\bar{Y}_h(t, x)] = f_-(t, x), \tag{5.3}$$

$$\operatorname*{l.i.m.}_{h \to 0} W_h(t, x) = W_0(t, x) \quad \textit{with respect to } P_\mu. \tag{5.4}$$

In particular, if μ is atomless, then $f_+ = f_-$ and these are jointly continuous in $t \geqq 0$ and $x \in \boldsymbol{R}$. Finally, for each $n \geqq 1$ there exist positive constants α, β, c_n such that

$$E_\mu[|W_0(t, a) - W_0(t, b)|^\alpha] \leqq c_n |a-b|^{2+\beta}, \tag{5.5}$$

$$E_\mu[|W_0(t, a) - W_0(s, a)|^\alpha] \leqq c_n |t-s|^{2+\beta}, \tag{5.6}$$

hold for every $0 \leqq s, t \leqq n$ and $|a|, |b| \leqq n$.

The precise forms of f_+ and f_- are given by (5.15) below. Let $W(t, x)$ be the continuous version of $W_0(t, x)$. Then it is easy to see that $Z(t, x) = Z(t, 0) + \int_0^x W(t, y)dy$ holds with a.s. P_μ. Then (2.13) follows from the differentiability of $Z(t, x)$ and (5.15). We shall show (2.14) at the end of this section.

We shall first show (5.3). Set

$$q_{t,h}(x) = h^{-1}(q_t(x+h) - q_t(x)). \tag{5.7}$$

By (3.2) we have

$$E_\mu[\bar{Y}_h(t, a)] = \int q_{t,h}(a-x)\mu(dx). \tag{5.8}$$

Set

$$u_t(x) = u(t, x) = -\int_0^t \frac{x}{s} p_s(x) ds. \tag{5.9}$$

Then we obtain

$$\begin{cases} (D_x q_t)(x) = u_t(x), & \text{if } \ x \neq 0, \\ (D_x^+ q_t)(0) = -1, \quad (D_x^- q_t)(0) = 1, & \text{if } \ t > 0, \end{cases} \tag{5.10}$$

$$(P_s u_t)(x) = -\int_s^{s+t} \frac{x}{r} p_r(x) dr. \tag{5.11}$$

It is easy to see that

$$u_t(x) = -2\,\mathrm{sgn}(x)\int_{|x|}^{\infty} p_t(y)dy\,, \tag{5.12}$$

where $\mathrm{sgn}(x)=x/|x|$ if $x\neq 0$ and $\mathrm{sgn}(0)=0$. Then we have

$$|u(t, x)| \leqq 1, \qquad t\geqq 0, \quad x\in \boldsymbol{R}. \tag{5.13}$$

By (5.12) and the fact that p_1 is rapidly decreasing, for any $p\geqq 0$ there exists $C=C(p)>0$ such that

$$|u(t, x)| \leqq C(1+t^{-1/2}|x|)^{-p}, \qquad t>0, \quad x\in \boldsymbol{R}. \tag{5.14}$$

Then from (5.10) and (5.14) it follows that

$$\begin{cases} D_x^+(\mu q_t)(x) = (\mu u_t)(x) - \mu(\{x\}), \\ D_x^-(\mu q_t)(x) = (\mu u_t)(x) + \mu(\{x\}), \qquad t>0, \quad x\in \boldsymbol{R}, \end{cases} \tag{5.15}$$

and especially if μ is atomless then

$$D_x(\mu q_t)(x) = (\mu u_t)(x), \qquad t\geqq 0, \quad x\in \boldsymbol{R}. \tag{5.16}$$

Thus we have shown (5.3). The continuity of $(\mu u_t)(x)$ follows from

LEMMA 5.1. *If $\mu\in M(\boldsymbol{R})$ then there exist $p\geqq 0$ and $C>0$ such that*

$$(\mu P_s|u_t|)(x) \leqq C(s^{-1}t)^{1/2}(1+s)^p(1+t)^p(1+|x|)^p \tag{5.17}$$

holds for every $s, t>0$ and $x\in \boldsymbol{R}$. If μ is atomless then $(\mu u_t)(x)$ is continuous.

PROOF. By (3.9) and (5.9) we have

$$(\mu P_s|u_t|)(x) = \int_0^t dr\int(\mu p_s)(x-y)\frac{|y|}{r}p_r(y)dy. \tag{5.18}$$

By (3.17) this is dominated by

$$\begin{aligned} &Cs^{-1/2}(1+s)^p\int_0^t dr\int(1+|x-y|)^p\frac{|y|}{r}p_r(y)dy \\ &\leqq Cs^{-1/2}(1+s)^p(1+t)^p(1+|x|)^p\int_0^t r^{-1/2}dr\int(1+|y|)^p|y|p_1(y)dy, \end{aligned}$$

and (5.17) follows. To see the latter part it suffices to show that

$$\lim_{t\to 0}(\mu u_t)(x) = 0 \tag{5.19}$$

uniformly on each finite interval. Choose $p>0$ to satisfy $\int(1+|x|)^{-p}\mu(dx)<\infty$. Then by (5.14) we have

$$\begin{aligned} |(\mu u_t)(x)| &\leqq C\int(1+t^{-1/2}|x-y|)^{-p}\mu(dy) \\ &\leqq C(1+|x|)^p\int f(t, x-y)(1+|y|)^{-p}\mu(dy), \end{aligned} \tag{5.20}$$

where $f(t, x)=(1+t^{-1/2}|x|)^{-p}(1+|x|)^{p}$. Remark that $\lim_{t\to 0}\sup_{|x|\geq\varepsilon}f(t, x)=0$ for any $\varepsilon>0$ and $f(t, x)\leqq 1$ if $t\leqq 1$. Then it suffices to notice that if μ is a finite atomless measure on $\boldsymbol{R}$ then for any $\varepsilon>0$ and $K>0$ there exists $\delta>0$ such that $\mu(I)<\varepsilon$ holds for any interval I contained in $[-K, K]$ with $\lambda(I)<\delta$.

Next we shall show (5.4). By (3.13) we have

$$E_{\mu}[(W_{h}(t, a)-W_{k}(t, a))^{2}]=\int_{0}^{t}ds\langle\mu, P_{t-s}(q_{s,h}(a-\cdot)-q_{s,k}(a-\cdot))^{2}\rangle. \tag{5.21}$$

By (5.10) and (5.13) it suffices to show

$$\lim_{h\to 0}\int_{0}^{t}ds\langle\mu, P_{t-s}(|q_{s,h}(a-\cdot)-u_{s}(a-\cdot)|)\rangle=0.$$

But this follows from (3.11), (3.17), (5.10) and (5.14).

We shall show (5.5). By (3.30) and the definition of $W_0(t, x)$ we have formally that

$$E_{\mu}[\exp(\theta(W_{0}(t, a)-W_{0}(t, b)))]=\exp\Big(2\sum_{n=2}^{\infty}\Big(\frac{\theta}{2}\Big)^{n}\langle\mu, v_{n}(t)\rangle\Big), \tag{5.22}$$

where $v_n(t)$, $n\geqq 2$, are given by (3.31) with

$$v_{1}(t, x)=u_{t}(a-x)-u_{t}(b-x). \tag{5.23}$$

Set

$$\bar{v}(t)=\int_{0}^{t}P_{t-s}(|v_{1}(s)|)ds. \tag{5.24}$$

LEMMA 5.2. *If $d=1$ and $R>0$ is fixed arbitrarily then there exist positive constants $a_n=a_n(R)$, $n=1, 2, \cdots$, such that*

$$\langle\lambda, v_{1}(t)^{2}\rangle\leqq a_{1}|a-b|, \tag{5.25}$$

$$\langle\mu, v_{2}(t)\rangle\leqq a_{2}|a-b|, \tag{5.26}$$

$$|v_{n}(t, x)|\leqq a_{n}|a-b|^{n/4}\bar{v}(t, x),\qquad x\in\boldsymbol{R},\quad n\geqq 3, \tag{5.27}$$

hold for every t, $|a|$, $|b|\leqq R$.

If this lemma is shown then we can prove (5.5) as follows. By (5.17) and (5.27), for any fixed $R>0$, there exist positive constants $c_n=c_n(R)$, $n=3, 4, \cdots$, such that

$$\langle\mu, |v_{n}(t)|\rangle\leqq c_{n}|a-b|^{n/4} \tag{5.28}$$

holds for every $n\geqq 3$ and t, $|a|$, $|b|\leqq R$. Then (5.5) follows from (5.26), (5.28) and Lemma 3.2, (i).

PROOF OF LEMMA 5.2. In the course of the proof we assume that s, t, $|a|$, $|b| \leqq R$ holds for some fixed $R>0$. By an elementary calculation we have

$$\int v_1(t, x)^2 dx = 2\int_0^t ds\int_0^t \frac{dr}{r+s}\Big(p_{r+s}(0)-p_{r+s}(a-b)+\frac{(a-b)^2}{r+s}p_{r+s}(a-b)\Big). \tag{5.29}$$

Since $1-e^{-x}+2xe^{-x} \leqq 4x/(1+x)$ holds for $x \geqq 0$ we obtain

$$\begin{aligned}\langle \lambda, v_1(t)^2\rangle &\leqq 8\int_0^t ds\int_0^t dr \frac{(r+s)^{-3/2}|a-b|^2}{r+s+|a-b|^2} \\ &\leqq 8|a-b|\int_0^\infty ds\int_0^\infty dr(r+s)^{-3/2}(r+s+1)^{-1}\end{aligned} \tag{5.30}$$

and (5.25) follows.

Next we shall show (5.26). We may assume that $a<b$. If $x>b$ or $x<a$ then by (5.12) we have

$$|v_1(t, x)| \leqq 2\Big|\int_{|x-a|}^{|x-b|} p_t(y)dy\Big| \leqq 2|a-b|(p_t(a-x)+p_t(b-x)), \tag{5.31}$$

and if $a<x<b$ then by (5.13) we have $|v_1(t, x)| \leqq 2$. Hence we obtain

$$v_2(t) \leqq 4|a-b|^2 w(t)+4z(t), \tag{5.32}$$

where

$$\begin{cases} w(t, x) = \int_0^t ds\int p_{t-s}(x-y)(p_s(a-y)+p_s(b-y))^2 dy, \\ z(t, x) = \int_a^b q_t(x-y)dy. \end{cases} \tag{5.33}$$

By (3.17) there exists $c(R)>0$ such that

$$\langle \mu, z(t)\rangle \leqq c(R)|a-b|. \tag{5.34}$$

By $(x+y)^2 \leqq 2(x^2+y^2)$ and $t/2 \leqq t-s/2 \leqq t$ for $0 \leqq s \leqq t$ we have

$$\begin{aligned} w(t, x) &\leqq \int_0^t dss^{-1/2}(p(t-s/2, a-x)+p(t-s/2, b-x)) \\ &\leqq 4t^{1/2}(p_t(a-x)+p_t(b-x)).\end{aligned} \tag{5.35}$$

Then by (3.17) there exists $c(R)>0$ such that

$$\langle \mu, w(t)\rangle \leqq c(R). \tag{5.36}$$

Hence (5.26) follows from (5.32), (5.34) and (5.36).

Before proceeding to the proof of (5.27) we shall show the following estimates. There exist positive constants $b_n=b_n(R)$, $n=2, 3, \cdots$, such that

$$|v_n(t)| \leqq b_n|a-b|^{n/2}, \tag{5.37}$$

$$|v_n(t)| \leqq b_n\bar{v}(t), \tag{5.38}$$

hold for every $n \geqq 2$.

By (5.25) we have

$$v_2(t, x) = \int_0^t ds \int p_{t-s}(x-y) v_1(s, y)^2 dy \tag{5.39}$$
$$\leqq \int_0^t ds (t-s)^{-1/2} \langle \lambda, v_1(s)^2 \rangle \leqq 2a_1 t^{1/2} |a-b| ,$$

and by (5.13) we have

$$v_2(t) \leqq 2\int_0^t ds P_{t-s}(|v_1(s)|) = 2\bar{v}(t). \tag{5.40}$$

Hence it suffices to show (5.37) and (5.38) in the case $n \geqq 3$ assuming that they hold for every $2 \leqq k \leqq n-1$. Set

$$w_{n,k}(t) = \int_0^t ds P_{t-s}(|v_k(s) v_{n-k}(s)|), \qquad 1 \leqq k \leqq n-1. \tag{5.41}$$

Then it is sufficient to show that there exists $c(R)>0$ such that

$$w_{n,k}(t) \leqq c(R) \min\{|a-b|^{n/2}, \bar{v}(t)\} \tag{5.42}$$

holds for any $1 \leqq k \leqq n-1$. By (5.37) and $n-1 \geqq 2$ we have

$$w_{n,1}(t) = w_{n,n-1}(t) \leqq b_{n-1} |a-b|^{(n-1)/2} \int_0^t ds P_{t-s}(|v_1(s)|) \tag{5.43}$$
$$= b_{n-1} |a-b|^{(n-1)/2} \bar{v}(t).$$

By (5.25) and $P_t(\phi^2) \geqq (P_t \phi)^2$ we obtain

$$w_{n,1}(t) = w_{n,n-1}(t) \leqq \int_0^t ds (P_{t-s}(v_1(s)^2))^{1/2} \cdot b_{n-1} |a-b|^{(n-1)/2} \tag{5.44}$$
$$\leqq c_1(R) \int_0^t ds (t-s)^{-1/4} \langle \lambda, v_1(s)^2 \rangle^{1/2} |a-b|^{(n-1)/2} \leqq c_2(R) |a-b|^{n/2}.$$

If $2 \leqq k \leqq n-2$ then by our assumption we have

$$w_{n,k}(t) \leqq t b_k b_{n-k} |a-b|^{n/2}, \tag{5.45}$$

and by $|v_1(t)| \leqq 2$ it follows that

$$w_{n,k}(t) \leqq b_k b_{n-k} \int_0^t ds P_{t-s}(\bar{v}(s)^2) \leqq c_1(R) \int_0^t P_{t-s}(\bar{v}(s)) ds \tag{5.46}$$
$$= 2c_1(R) \int_0^t ds \int_0^s dr P_{t-r}(|v_1(r)|) \leqq c_2(R) \bar{v}(t).$$

Thus we have shown (5.37) and (5.38).

Then (5.27) is shown as follows. It suffices to show that there exist positive constants $c_{n,k} = c_{n,k}(R)$, $1 \leqq k \leqq n-1$, $n=3, 4, \cdots$, such that

$$w_{n,k}(t) \leqq c_{n,k} |a-b|^{n/4} \bar{v}(t) \tag{5.47}$$

holds for every $n \geqq 3$ and $1 \leqq k \leqq n-1$. By (5.41) we have only to show this for $1 \leqq k \leqq n/2$. If $k=1$ then by $n-1 \geqq 2$ we have

$$w_{n,1}(t) \leqq b_{n-1}|a-b|^{(n-1)/2}\int_0^t ds P_{t-s}(|v_1(s)|) \leqq c(R)|a-b|^{n/4}\bar{v}(t).$$

If $2 \leqq k \leqq n/2$ then by $n-k \geqq n/2$ we obtain

$$w_{n,k}(t) \leqq b_k b_{n-k}\int_0^t ds P_{t-s}(\bar{v}(s)|a-b|^{(n-k)/2}) \leqq c(R)|a-b|^{n/4}\bar{v}(t).$$

Thus we have completed the proof of Lemma 5.2.

We shall show (5.6). By (3.32) and the definition of $W_0(t, x)$ we have formally that

$$E_\mu[\exp(\theta(W_0(s+t, a)-W_0(s, a)))] = \exp\Big(2\sum_{n=2}^{\infty}\Big(\frac{\theta}{2}\Big)^n\langle\mu, v_n(s, t)\rangle\Big), \tag{5.48}$$

where $v_n(s, t)$, $n \geqq 2$, are given by (3.31) and (3.33) with

$$v_1(t, x) = u_t(a-x). \tag{5.49}$$

LEMMA 5.3. *If $d=1$ then there exist positive constants a_n, $n=1, 2, \cdots$, and b_n, $n=2, 3, \cdots$, such that*

$$|v_n(t, x)| \leqq a_n t^{n-1}, \qquad t \geqq 0, \quad a, x \in \boldsymbol{R}, \quad n \geqq 1, \tag{5.50}$$

$$|v_n(s, x)| \leqq b_n s^{n-2} Q_s(|v_1(t)|)(x), \qquad s \leqq t, \quad a, x \in \boldsymbol{R}, \quad n \geqq 2, \tag{5.51}$$

and $f(\theta)=\sum_{n=2}^{\infty} b_n\theta^n$ has a positive radius of convergence.

PROOF. By (3.31) and (5.13) if we define a_n, $n \geqq 1$, by

$$a_1 = 1, \qquad a_n = \sum_{k=1}^{n-1} a_k a_{n-k}, \qquad n \geqq 2, \tag{5.52}$$

then (5.50) holds. We shall show (5.51) by defining b_n, $n \geqq 2$, as follows

$$b_2 = 1, \qquad b_n = 2a_{n-1} + \sum_{k=2}^{n-1} b_k a_{n-k}, \qquad n \geqq 3. \tag{5.53}$$

By (5.9) we have

$$t \to |v_1(t, x)| = |u_t(a-x)| \text{ is nondecreasing.} \tag{5.54}$$

Then by (5.13) we have

$$v_2(s) = \int_0^s dr P_{s-r}(v_1(r)^2) \leqq \int_0^s dr P_{s-r}(|v_1(t)|) = Q_s(|v_1(t)|). \tag{5.55}$$

Hence it suffices to prove (5.51) in the case $n \geqq 3$ assuming that it holds for any $2 \leqq k \leqq n-1$. Set

$$w_k(s) = \int_0^s dr P_{s-r}(|v_k(r) v_{n-k}(r)|), \qquad 1 \leqq k \leqq n-1. \tag{5.56}$$

By (5.54) and $n \geqq 3$ we have

$$w_1(s) = w_{n-1}(s) \leqq a_{n-1}\int_0^s dr r^{n-3} P_{s-r}(|v_1(t)|) \leqq a_{n-1}s^{n-2}Q_s(|v_1(t)|). \tag{5.57}$$

If $2 \leqq k \leqq n-2$ then by (5.50) and (5.51) we have

$$w_k(s) \leqq b_k a_{n-k}\int_0^s dr r^{n-3} P_{s-r}(Q_r(|v_1(t)|)). \tag{5.58}$$

Since $P_{s-r}Q_r(|v_1(t)|) \leqq Q_s(|v_1(t)|)$ we obtain

$$w_k(s) \leqq b_k a_{n-k} s^{n-2} Q_s(|v_1(t)|). \tag{5.59}$$

Hence (5.51) holds for b_n, $n \geqq 2$, given by (5.53).

Set $g(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$. Then $g(\theta) = (1-(1-4\theta)^{1/2})/2$. By (5.53) $f(\theta)$ satisfies $f(\theta) = 2\theta g(\theta) + f(\theta)(g(\theta)-\theta)$, i.e.,

$$f(\theta) = 2\theta g(\theta)(1-g(\theta)+\theta)^{-1}.$$

Hence f has a positive radius of convergence.

LEMMA 5.4. *If $d=1$ then for any $R>0$ there exist positive constants $c_n = c_n(R)$, $n=1, 2, \cdots$, and $d_n = d_n(R)$, $n=2, 3, \cdots$, such that*

$$|v_1(s, t, x)| \leqq c_1 \min\{1, (s^{-1}t)^{1/2}\}, \tag{5.60}$$

$$|v_n(s, t, x)| \leqq c_n t^{n/4}, \qquad n \geqq 2, \tag{5.61}$$

$$|v_n(s, t, x)| \leqq d_n t^{(n-2)/4}((st)^{1/2} P_s(|v_1(t)|) + P_s Q_t(|v_1(t)|))(x), \qquad n \geqq 2, \tag{5.62}$$

hold for every a, $x \in \boldsymbol{R}$ and s, $t \leqq R$.

PROOF. In the course of the proof we assume that s, $t \leqq R$ holds for some fixed $R>0$. By (5.11) and (5.13) it is easy to see that (5.60) holds. Then (5.61) is easily seen by (5.50), (3.33) and the induction argument. Since

$$\begin{aligned} v_2(s, t) &\leqq P_s v_2(t) + \int_0^s dr P_{s-r}(P_r v_1(t))^2 \\ &\leqq b_2 P_s Q_t(|v_1(t)|) + \int_0^s dr P_s(|v_1(t)|) \cdot \|P_r v_1(t)\|, \end{aligned} \tag{5.63}$$

(5.62) in the case $n=2$ follows from (5.60). Then it suffices to show (5.62) in the case $n \geqq 3$ assuming that it holds for every $2 \leqq k \leqq n-1$. Set

$$\begin{cases} w_0(s, t) = P_s(|v_n(t)|), \\ w_k(s, t) = \int_0^s dr P_{s-r}(|v_k(r, t) v_{n-k}(r, t)|), \qquad 1 \leqq k \leqq n-1. \end{cases} \tag{5.64}$$

By (5.51) we have

$$w_0(s, t) \leqq b_n t^{n-2} P_s Q_t(|v_1(t)|). \tag{5.65}$$

Since $n-1 \geqq 2$, by (5.60) and (5.62), we obtain

$$(5.66)\qquad \begin{aligned} w_1(s,t) &= w_{n-1}(s,t) \\ &\leqq c_1 d_{n-1} t^{(n-3)/4}\int_0^s dr(r^{-1}t)^{1/2}((rt)^{1/2}P_s(|v_1(t)|)+P_sQ_t(|v_1(t)|)) \\ &\leqq 2c_1 d_{n-1} t^{(n-2)/4}(st^{3/4}P_s(|v_1(t)|)+s^{1/2}t^{1/4}P_sQ_t(|v_1(t)|)). \end{aligned}$$

If $2\leqq k\leqq n-2$ then by (5.61) and (5.62) we have

$$(5.67)\qquad \begin{aligned} w_k(s,t) &\leqq c_k d_{n-k} t^{(n-2)/4}\int_0^s dr((rt)^{1/2}P_s(|v_1(t)|)+P_sQ_t(|v_1(t)|)) \\ &\leqq c_k d_{n-k} t^{(n-2)/4}(s^{3/2}t^{1/2}P_s(|v_1(t)|)+sP_sQ_t(|v_1(t)|)). \end{aligned}$$

Then (5.62) follows from these estimates and $|v_n(s,t)|\leqq\sum_{k=1}^{n-1}w_k(s,t)$.

Then we can prove (5.6) as follows. Let $R>0$ be fixed arbitrarily. By (5.17) there exists $c(R)>0$ such that

$$(5.68)\qquad \langle\mu, P_s(|v_1(t)|)\rangle \leqq c(R)s^{-1/2}t^{1/2},$$

$$(5.69)\qquad \langle\mu, P_sQ_t(|v_1(t)|)\rangle \leqq c(R)t^{1/2},$$

hold for s, t, $|a|\leqq R$. Then by (5.62) there exist positive constants $e_n=e_n(R)$, $n=2, 3, \cdots$, such that

$$(5.70)\qquad \langle\mu, |v_n(s,t)|\rangle \leqq e_n t^{n/4}$$

holds for every s, t, $|a|\leqq R$ and $n\geqq 2$. Then (5.6) follows from (5.48) and Lemma 3.1, (i).

Finally, (2.14) follows from (5.17), (5.51) and Lemma 3.1, (ii) if we set $s=0$ in (5.48).

6. Proofs of Theorems 5 and 6.

The following lemma is fundamental in this section.

LEMMA 6.1. *If $\mu\in M(\boldsymbol{R}^d)$ then, for any $\rho>0$, the process*

$$(P_{\rho^2\mu(\rho^{-1}\cdot)}, \rho^{-2}X(\rho^2 t, \rho\cdot))$$

is equivalent to the process $(P_\mu, X(t))$.

PROOF. Since each process has the same initial point μ it suffices to show that the transition functions coincide. Let $u(t, x, \phi)$ be the solution of (2.5). Then $\rho^2 u(\rho^2 t, \rho x, \rho^{-2}\phi(\rho^{-1}\cdot))$ also satisfies (2.5). Hence by (2.4) we have

$$\begin{aligned} E_{\rho^2\nu(\rho^{-1}\cdot)}\Big[\exp\Big(-\rho^{-2}\int\phi(x)X(\rho^2 t, \rho dx)\Big)\Big] &= \exp\Big(-\int u(t,x,\phi)\nu(dx)\Big) \\ &= E_\nu[\exp(-\langle X(t), \phi\rangle)] \end{aligned}$$

and we have completed the proof.

By the above lemma and Theorem 2 we obtain

COROLLARY 6.1. *Let* $d \leqq 3$ *and* $\mu \in M(\boldsymbol{R}^d)$. *We assume* (2.9) *in the case* $d=2$ *or* 3. *Then, for any* $\rho>0$ *and* $\phi \in C_K(\boldsymbol{R}^d)$, *the continuous process*

$$\left(P_{\rho^2\mu(\rho^{-1}\cdot)},\ \rho^{-4}\int Y(\rho^2 t,\ x)\phi(\rho^{-1}x)dx\right)$$

is equivalent to the continuous process $(P_\mu, \int Y(t, x)\phi(x)dx)$.

Then we can prove Theorem 5 as follows. By Corollary 6.1, the process $(P_{\rho^2\mu(\rho^{-1}\cdot)}, \rho^{d-4}(\langle Y(\rho^2 t), \phi\rangle, Y(\rho^2 t, a)))$ is equivalent to the process

$$\left(P_\mu,\ \left(\int Y(t,\ \rho^{-1}x)\phi(x)dx,\ Y(t,\ \rho^{-1}a)\right)\right).$$

Then by the continuity of $Y(t, x)$, this process converges to the process $(P_\mu, (\langle\lambda, \phi\rangle, 1)Y(t, 0))$ as $\rho\to\infty$.

If $\mu\in M(\boldsymbol{R})$ is atomless then the process

$$\left(P_{\rho^2\mu(\rho^{-1}\cdot)},\ \rho^{-2}\left(\int(Y(\rho^2 t,\ x)-Y(\rho^2 t,\ 0))\phi(x)dx,\ D_x Y(\rho^2 t,\ a)\right)\right)$$

is equivalent to the process

$$\left(P_\mu,\ \left(\rho\int(Y(t,\ \rho^{-1}x)-Y(t,\ 0))\phi(x)dx,\ D_x Y(t,\ \rho^{-1}a)\right)\right).$$

By the continuity of $D_x Y(t, x)$ this process converges to the process

$$\left(P_\mu,\ \left(\int x\phi(x)dx,\ 1\right)D_x Y(t,\ 0)\right) \qquad \text{as} \quad \rho\to\infty.$$

Hence we have shown Theorem 6.

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