J. Math. Soc. Japan
Vol. 31, No. 2, 1979

On a conjecture of Nakai on Picard principle

By Michihiko KAWAMURA*)

(Received Nov. 30, 1977)

A nonnegative locally Hölder continuous function $P(z)$ on the punctured closed unit disk $0<|z|\leqq 1$ will be referred to as a *density* on Ω : $0<|z|<1$. We view Ω as interior of the bordered surface: $0<|z|\leqq 1$; so we consider the boundary $\partial\Omega$ to be the circle: $|z|=1$. The *elliptic dimension* of a density P on Ω at $z=0$, $\dim P$ in notation, is defined to be the dimension of the half module $\mathcal{P}_P$ of nonnegative solutions of the equation $\Delta u=Pu$ $(\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2)$ on Ω with vanishing boundary values on $\partial\Omega$: $|z|=1$. After Bouligand we say that the *Picard principle* is valid for P at $z=0$ if $\dim P=1$. We are interested in determining those densities P on Ω for which the Picard principle is valid. We observe the following example of Nakai [**26, 29**]: Let $P_\mu(z)=|z|^{-\mu}$ and $P_{-\infty}(z)=0$. Then

$$\dim P_\mu=\begin{cases} 1 & (\mu\in[-\infty, 2]) \\ \mathfrak{c} & (\mu\in(2, +\infty)) \end{cases} \tag{1}$$

where $\mathfrak{c}$ is the cardinal number of continuum. In this connection Nakai *conjectured* that the Picard principle is valid for general densities $P(z)$ on Ω with $P(z)=\mathcal{O}(|z|^{-2})$ $(z\to 0)$. The *purpose* of this paper is to prove this conjecture in the *affirmative*. Namely we shall prove the following

MAIN THEOREM. *The Picard principle is valid for any density $P(z)$ on Ω with $P(z)=\mathcal{O}(|z|^{-2})$ $(z\to 0)$.*

The proof of this theorem will be given in Section 1. A formulation of the Harnack inequality by S. Itô [**11**] will play an essential role in our proof. The author is grateful to Professor Itô for his advice on this inequality. In Section 2 we will discuss the order of the generator $g(z)$ of $\mathcal{P}_P$ as $z\to 0$ for $P(z)=\mathcal{O}(|z|^{-2})$ $(z\to 0)$. We will establish the following inequality:

$$C_1 \log\frac{1}{|z|}\leqq g(z)\leqq C_2|z|^{-c} \tag{2}$$

on Ω where C_1 and C_2 are positive constants and $c=\sup_\Omega |z|P(z)^{1/2}$. In the final Section 3 we will mention two generalizations of the main theorem. We will show that the condition $P(z)\leqq \text{const.}\,|z|^{-2}$ on Ω in the main theorem can be relaxed to $P(z)\leqq\text{const.}\,|z|^{-2}$ only on a sequence of disjoint concentric annuli A_n

*) The author is grateful to Professor Nakai for the valuable discussions with him.

in Ω converging to $z=0$ such that $\inf_n \operatorname{mod} A_n>0$. In this form, the result will further be generalized to certain ends of Riemann surfaces.

§1. General densities $P(z)=\mathcal{O}(|z|^{-2})$ $(z\to 0)$.

1.1. A *density* $P(z)$ on $\Omega: 0<|z|<1$ is, by definition, a locally Hölder continuous function $P(z)$ on $0<|z|\leqq 1$. We denote by $\mathfrak{P}_P$ the half module of nonnegative solutions of $\Delta u=Pu$ on Ω with vanishing boundary values. We are interested in characterizing those densities P on Ω for which the *Picard principle* is valid, i.e. $\mathfrak{P}_P$ has a single generator. For *rotation free* densities $P(z)$, i.e. densities P with $P(z)=P(|z|)$ on Ω, the problem may be viewed as being almost brought to its completion (cf. Brelot [2, 3], Nakai [26, 29], Kawamura-Nakai [14], Godefroid [6] etc.). For rotation free densities, we know ([29], [14]) that if two densities P_1 and P_2 are related as $P_2(z)\leqq P_1(z)$ or $P_2(z)=\text{const.}\,P_1(z)$ on Ω, then the Picard principle is valid for P_2 if it does for P_1. Hence for rotation free densities $P(z)$ with $P(z)=\mathcal{O}(|z|^{-2})$ $(z\to 0)$ the Picard principle is valid since it does for $|z|^{-2}$. What happens to general densities $P(z)$ with $P(z)=\mathcal{O}(|z|^{-2})$ $(z\to 0)$? For general densities $P(z)$, we only know two fragmentary results:
If

$$\int_{\Omega-E} P(z)\log\frac{1}{|z|}\,dx\,dy<\infty$$

for a closed subset E of Ω thin at $z=0$, then the Picard principle is valid (Nakai [28]);
If

$$\int_{\Omega} P(z)\,dx\,dy<\infty\,,$$

then the Picard principle is valid (Nakai [32], Kawamura [13]). None of these two results can take care of densities $P(z)=\mathcal{O}(|z|^{-2})$ $(z\to 0)$ and we need some completely new device to establish the following

THEOREM. *For a general densities $P(z)$ on Ω satisfying*

$$P(z)\leqq c^2|z|^{-2} \tag{3}$$

with a suitable constant c the Picard principle is valid.

1.2. *The duality theorem.* The unique bounded solution $e=e_P$ of

$$L_P u\equiv \Delta u-Pu=0 \tag{4}$$

on $\Omega: 0<|z|<1$ with continuous boundary values 1 on $\partial\Omega: |z|=1$ will be referred to as the *P-unit* (cf. Singer [40]). Consider the *associated operator* $\hat{L}_P$ with L_P which is introduced by Nakai [31, 32]:

(5) $$\hat{L}_P u \equiv \Delta u + 2\nabla(\log e_P)\cdot\nabla u$$

for $u\in C^2(\Omega)$ where e_P is the P-unit on $\bar{\Omega}=\Omega\cup\partial\Omega$, and $\nabla f=(\partial f/\partial x, \partial f/\partial y)$. We say that the *Riemann theorem* is valid for $\hat{L}_P$ at $z=0$ if $\lim_{z\to 0}u(z)$ exists for every bounded solution u of

(6) $$\hat{L}_P u=0$$

on Ω and continuous on $\partial\Omega$. Nakai ([**31, 32**]; cf. also Heins [**8**], Hayashi [**7**], Nakai [**28**]) showed the following *duality theorem: The Picard principle is valid for an operator L if and only if the Riemann theorem is valid for the associated operator* $\hat{L}$.

1.3. *The maximum-minimum principle.* Let Ω_t be $0<|z|<t$ and $\partial\Omega_t$ be $|z|=t$ for $t\in(0,1]$. And consider classes $\mathscr{F}_t$ and $\mathscr{B}_t$ of nonnegative and nonnegative bounded solution of (4) on Ω_t with boundary values zero and continuous boundary values on $\partial\Omega_t$, respectively. Then it is known that (cf. Brelot [**2**], Ozawa [**33**], Royden [**36**], Nakai [**28**])

(7) $$\mathscr{F}_t\cap\mathscr{B}_t=\{0\},$$

i.e. $z=0$ is of *parabolic character.*

For a nonnegative solution u of (4) the maximum principle is valid. On the other hand, Nakai [**28**] showed the following *maximum-minimum principle* for the solution of the associated equation (6): *Let v be a bounded nonnegative solution of* (6) *on* Ω_s ($s\in(0,1]$). *Then*

(8) $$\sup_{z\in\Omega_t}v(z)=\max_{z\in\partial\Omega_t}v(z),\quad \inf_{z\in\Omega_t}v(z)=\min_{z\in\partial\Omega_t}v(z)$$

is valid for any $t\in(0,s)$.

1.4. *Itô's form of the Harnack inequality.* We denote by $\Delta(z_0,R)$ the open disk: $|z-z_0|<R$ and $\overline{\Delta(z_0,R)}=(|z-z_0|)\leqq R)$. Suppose that a density $P(z)$ is given on Ω, $\overline{\Delta(z_0,R)}\subset\Omega$, and $0\leqq P(z)\leqq c$ on $\overline{\Delta(z_0,R)}$ for some $c>0$. Then S. Itô [**11**] showed the following:

The inequality

(9) $$u(z)\leqq 4e^{R\sqrt{c}}u(z_0)$$

is valid for any $z\in\overline{\Delta(z_0,R/2)}$ *and for any nonnegative solution u of* (4) *on* $\overline{\Delta(z_0,R)}$.

We include here an outline of the proof of Itô [**11**] for the sake of convenience to the reader. Let u be a nonnegative solution of (4) on $\overline{\Delta(z_0,R)}$ and let $e_{c,\rho}$ be the c-unit of $\Delta u=cu$ on $\overline{\Delta(z_0,\rho)}$ for $0<\rho\leqq R$. We first show the following inequality

(10) $$e_{c,R}(z_0)\cdot(2\pi)^{-1}\int_0^{2\pi}u(\zeta)\rho\,d\theta\leqq u(z_0)\rho$$

where $\zeta=z_0+\rho e^{i\theta}$. Let v be the solution of $\Delta u=cu$ with $u=v$ on $\partial\Delta(z_0,\rho)$, and let $G(z,\zeta)$ be the Green's function of $\Delta u=cu$ on $\Delta(z_0,\rho)$. Since the constant function c is rotation free, $\partial G(z_0,\zeta)/\partial n_\zeta$ (where n denotes inner normal) is also rotation free, we have that

$$\begin{aligned} v(z_0)&=(2\pi)^{-1}\int_0^{2\pi} v(\zeta)\rho(\partial G(z_0,\zeta)/\partial n_\zeta)d\theta \\ &=\rho(2\pi)^{-1}(\partial G(z_0,\zeta)/\partial n_\zeta)\int_0^{2\pi} u(\zeta)d\theta \\ &=e_{c,\rho}(z_0)\cdot(2\pi)^{-1}\int_0^{2\pi} u(\zeta)d\theta\,. \end{aligned}$$

Since $0\leqq P(z)\leqq c$ and $u=v$ on $\partial\Delta(z_0,\rho)$, the comparison principle assures that $v(z_0)\leqq u(z_0)$. Observing that $e_{c,\rho}(z_0)\geqq e_{c,R}(z_0)$ we deduce (10).

Let z be any point in $\overline{\Delta(z_0,R/2)}$ and set $|z-z_0|=r$. Integration of both sides of (10) by ρ on $(0,2r)$ yields

$$\iint_{\Delta(z_0,2r)} u\,dxdy\leqq 4\pi r^2(e_{c,R}(z_0))^{-1}u(z_0)\,. \tag{11}$$

Since u is subharmonic in $\overline{\Delta(z,r)}\subset\overline{\Delta(z_0,2r)}$, we have that

$$u(z)\leqq(\pi r^2)^{-1}\iint_{\Delta(z,r)} u\,dxdy\leqq(\pi r^2)^{-1}\iint_{\Delta(z_0,2r)} u\,dxdy\,. \tag{12}$$

On the other hand c-unit $e_{c,R}(z)$ is represented by the modified Bessel function of the first kind of zero order I_0 (cf. Bowman [1]), i.e. $e_{c,R}(z)=I_0(\sqrt{c}\,|z-z_0|)/I_0(\sqrt{c}\,R)$, where

$$I_0(x)=1+2^{-2}x^2+2^{-2}\cdot4^{-2}x^4+2^{-2}\cdot4^{-2}\cdot6^{-2}x^6+\cdots\leqq e^x\quad(0\leqq x)\,.$$

By (11), (12) and the above fact, we obtain (9).

1.5. *Uniform Harnack inequality I.* For a density $P(z)$ on $\Omega: 0<|z|<1$ with $P(z)=O(|z|^{-2})$ we shall reform Itô's form of the Harnack inequality as follows. We may assume that $0\leqq P(z)\leqq c^2|z|^{-2}$ on $0<|z|\leqq 1$ for some nonnegative number c. Let a and s be real numbers such that $a\in(0,1)$ and $s\in(0,1]$. We maintain the following concentric circle form of the Harnack inequality:

LEMMA. *There exists a positive number K_1 depending only on a and c such that*

$$u(w)\leqq K_1u(z) \tag{13}$$

for any nonnegative solution u of $L_Pu=0$ on Ω_s and for any z and w with $|z|=|w|=t<s/(1+a)$.

To prove this assertion take arbitrary points z_0 and w on the circle $\Gamma_0: |z|=t$, $(t<s/(1+a))$. Let Γ be a subarc of Γ_0 with the initial point z_0 and the terminal point $-z_0$ such that $w\in\Gamma$. Consider a finite sequence of the

closed disks $\overline{\Delta(z_k, at/2)}$ such that $z_k \in \Gamma \cap \partial\Delta(z_{k-1}, at/2)$ and $z_k \neq z_{k-2}$. We observe that

$$\Gamma \subset \bigcup_{k=0}^{p} \overline{\Delta(z_k, at/2)} \subset \Omega_s ,$$

if $p=1+[\pi/2 \arcsin(a/4)]$, where [ ] denotes the Gauss symbol. Observe that the maximum of $c^2|z|^{-2}$ on $\overline{\Delta(z_k, at)}$ is attained at the point $(1-a)z_k$ and that the maximum value equals $c^2(1-a)^{-2}t^{-2}$. Thus we have that $P(z) \leqq c^2(1-a)^{-2}t^{-2}$ on $\overline{\Delta(z_k, at)}$ for each k $(k=0, 1, 2, \cdots, p)$. By applying Itô's form of the Harnack inequality to the disk $\overline{\Delta(z_k, at)}$, we deduce that

$$u(z) \leqq K_0 u(z_k)$$

if $z \in \overline{\Delta(z_k, at/2)}$ for any nonnegative solution u of (4) on Ω_s, where $K_0=4 \exp \{ca/(1-a)\}$. In particular, by setting $z=z_{k+1}$ on the above inequality and using these inequalities k times $(k \leqq p)$, we have that $u(w) \leqq K_0^k u(z_0)$. Since $K_0>1$, we deduce that $u(w) \leqq K_0^p u(z_0)$. We stress that K_0^p depends only on a and c. By setting $K_1=K_0^p$, we have the desired conclusion.

1.6. *Uniform Harnack inequality II.* Consider the associated equation (6) with (4). We shall prove that a similar inequality as in 1.5 is valid for nonnegative solutions of (6) on Ω_s. This inequality will play an essential role in the proof of the theorem.

Let a, c and s be real numbers as mentioned in 1.5. Then we have the following

LEMMA. *There exists a positive number K depending only on a and c such that*

$$v(w) \leqq K v(z) \tag{14}$$

for any nonnegative solution v of $\hat{L}_P v=0$ on Ω_s, and for any z and w with $|z|=|w|=t<s/(1+a)$.

To prove this take an arbitrary nonnegative solution v of (6) on Ω_s and let e_P be the P-unit of (4) on Ω and set $u=ve_P$. Observe that e_P and u are both nonnegative solutions of (4) on Ω_s. By Lemma in 1.5, we have that

$$e_P(z) \leqq K_1 e_P(w)$$

and

$$v(w)e_P(w) \leqq K_1 v(z) e_P(z)$$

for any z and w such that $|z|=|w|=t<s/(1+a)$. Since e_P is positive on Ω_s, the above two inequalities imply that $v(w) \leqq K_1^2 v(z)$. By setting $K=K_1^2$ we have the desired conclusion.

1.7. *Proof of the theorem.* In view of the duality theorem in 1.2, we only have to show that the Riemann theorem is valid for the associated operator

$\hat{L}_P$ at $z=0$, i.e. $\lim_{z\to 0}v(z)$ exists for every bounded solution v of (6) on Ω with continuous boundary values on $\partial\Omega$.

Let v be an arbitrary bounded solution of (6) on Ω and take any $s\in(0,1]$. Set

$$m(s)=\inf_{z\in\Omega_s}v(z)$$

and

$$v_s(z)=v(z)-m(s).$$

Since the associated operator $\hat{L}_P$ is linear and a constant function is a solution of (6), the function v_s is a nonnegative bounded solution of (6) on Ω_s. Thus the maximum-mimimum principle in 1.3 is applicable to v_s on Ω_s. Then, there exist two points w_t and z_t on $\partial\Omega_t$ such that

$$\sup_{z\in\Omega_t}v_s(z)=\max_{z\in\partial\Omega_t}v_s(z)=v_s(w_t)$$

and

$$\inf_{z\in\Omega_t}v_s(z)=\min_{z\in\partial\Omega_t}v_s(z)=v_s(z_t)$$

for each t in $(0,s)$. From Lemma in 1.6, we have that $v_s(w_t)\leqq Kv_s(z_t)$ for any t with $t<s/(1+a)$, where a and K are as in 1.6, i.e.

$$\sup_{z\in\Omega_t}v(z)-m(s)\leqq K(\inf_{z\in\Omega_t}v(z)-m(s)). \tag{15}$$

For a fixed s, on letting $t\to 0$ in the above inequality, we have that

$$\limsup_{z\to 0}v(z)-m(s)\leqq K(\liminf_{z\to 0}v(z)-m(s)).$$

Since $\lim_{s\to 0}m(s)=\lim\inf_{z\to 0}v(z)$, again letting $s\to 0$ in the above inequality we deduce that $\lim\sup_{z\to 0}v(z)-\lim\inf_{z\to 0}v(z)\leqq 0$, i.e. $\lim_{z\to 0}v(z)$ exists.

The proof of the theorem is herewith complete.

§2. Order of generator of $\mathcal{P}_P$.

2.1. In view of Theorem in 1.1, if the density $P(z)$ on Ω: $0<|z|<1$ satisfies that $P(z)=\mathcal{O}(|z|^{-2})$ $(z\to 0)$, then $\dim P=1$, i.e. the half module $\mathcal{P}_P$ of nonnegative solutions of (4) on Ω with vanishing boundary values on $\partial\Omega$ has a single generator g, i.e. $\mathcal{P}_P=\boldsymbol{R}^+g$. We are interested in the question to determine the rate of growth of g as $z\to 0$. For a density $P(z)=\mathcal{O}(|z|^{-2})$, we first estimate the growth from the above as follows:

THEOREM. *If a density $P(z)$ satisfies $P(z)\leqq c^2|z|^{-2}$ on Ω, then the generator g of $\mathcal{P}_P$ has the order $|z|^{-c}$,* i.e. *$g(z)=\mathcal{O}(|z|^{-c})$ as $z\to 0$, where $c>0$.*

2.2. To prove the theorem in 2.1 it is sufficient to show that there exist a positive solution $g\in\mathcal{P}_P$ and an $r\in(0,1)$ such that $g(z)\leqq v(z)\equiv|z|^{-c}-|z|^c$ on $\bar{\Omega}_r$ since $\dim P=1$. We observe that v is a rotation free positive solution of $\Delta v=c^2|z|^{-2}v$ on Ω and $v=0$ on $\partial\Omega$. Again observe that for a fixed $w\in\Omega$ there exists a positive solution $u\in\mathfrak{F}_P$ with $u(w)<v(w)$. Set $D=\{z\in\Omega;\ u(z)<v(z)\}$. The set D is an open set in Ω containing w. Let G be a connected component of D containing w. Then we have that any circle C: $|z|=t$ $(0<t<|w|)$ intersects G. In fact, contrary to the assertion assume that there exists a circle C: $|z|=t$ $(0<t<|w|)$ such that $C\cap G=\emptyset$. Since G is a connected component of D containing w, G is contained in an annulus: $t<|z|<1$, i.e. $\bar{G}$ is compact in $\bar{\Omega}$. Since $u=v$ on ∂G and $P(z)\leqq c^2|z|^{-2}$, by the comparison principle, we have that $u\geqq v$ in G. This contradicts the definition of G. Hence $C\cap G\neq\emptyset$.

Let z_t be a point in $C\cap G$ for $t\in(0,|w|)$. Take an a such that $0<a<\min(1,(1-|w|)/|w|)$ and determine the number K_1 of (13) in 1.5 for a and c. Set $g=u/K_1$. Then, g is the required function. In fact, by (13) we have that $g(z)\leqq K_1 g(z_t)=u(z_t)<v(z_t)=t^{-c}-t^c=v(z)$ for z with $|z|=t$ and for any $t\in(0,r)$ where $r=|w|/(1+a)$.

2.3. We have discussed in 2.1 and 2.2 about the upper order of a generator g of $\mathfrak{F}_P$ and obtained that $g(z)=\mathcal{O}(|z|^{-c})$ $(z\to 0)$ if $P(z)\leqq c^2|z|^{-2}$. We next proceed to the determination of the *lower* order of g as $z\to 0$.

Before doing this, we mention the following unpublished remark of Hideo Imai: For any density P on Ω and any nonzero $u\in\mathfrak{F}_P$,

$$\liminf_{z\to 0}\left(\frac{1}{2\pi}\int_0^{2\pi}u(ze^{i\theta})d\theta\right)\Big/\log\frac{1}{|z|}>0\,. \tag{16}$$

In particular, if P is rotation free with $\dim P=1$,

$$\liminf_{z\to 0}\ u(z)\Big/\log\frac{1}{|z|}>0\,. \tag{17}$$

A typical example is the generator $\log(1/|z|)$ of $\mathfrak{F}_0$, i.e. $\mathfrak{F}_P$ with $P\equiv 0$. In view of these one might feel that for any density P on Ω with $\dim P=1$ (17) is always valid. It may be of some interest, in this connection, to recall the following result of Nakai [**27**] (cf. also Kawamura [**12**]): There exists a *finite* density P (i.e. $\int_\Omega P(z)dxdy<+\infty$) such that $\liminf_{z\to 0}v(z)=0$ for *any* positive solution v of (4) on Ω. As already mentioned the Picard principle is valid for any finite, and hence for the above, P (Nakai [**32**]). Therefore (17) may not be valid for the generator g of $\mathfrak{F}_P$ even for P quite close to 0 in some sense such as finiteness. However, we are fortunate enough to obtain the following:

THEOREM. *If a density $P(z)$ satisfies $P(z)\leqq c^2|z|^{-2}$ on Ω, then*

$$\liminf_{z\to 0}\ g(z)/\log\frac{1}{|z|}>0 \tag{18}$$

for the generator g of $\mathfrak{F}_P$. In particular $\lim_{z\to 0}u(z)=+\infty$ for any nonzero $u\in\mathfrak{F}_P$.

2.4. The proof of the theorem in 2.3 goes on analogous way as in 2.2.

Take a fixed point $w\in\Omega$. Then there exists a $u\in\mathfrak{F}_P$ such that $\log(1/|w|)<u(w)$. Set $D=\{z\in\Omega;\ \log(1/|z|)<u(z)\}$ and let G be a connected component of D containing w. Then we have that any circle C: $|z|=t$ $(0<t<|w|)$ intersects G as in 2.2. There exist an r $(0<r<|w|)$ and K_1 as in (13) such that $u(z_t)\leqq K_1u(z)$ for $|z|=|z_t|=t$, $0<t<r$, where $z_t\in C\cap G$. By setting $g=K_1u$, we have that $\log(1/|z|)<g(z)$ for $|z|<r$ as in 2.2. Since g is a generator of $\mathfrak{F}_P$, we have the desired conclusion.

§ 3. Generalizations of the main theorem.

3.1. Reexamining the proof in 1.7 of the theorem in 1.1 we naturally come to being aware of the following: In the implication of

$$\limsup_{z\to0} v(z)-m(s)\leqq K(\liminf_{z\to0} v(z)-m(s)) \tag{19}$$

in 1.7 from (15) in 1.7, i.e.

$$\sup_{z\in\Omega_t}v(z)-m(s)\leqq K(\inf_{z\in\Omega_t}v(z)-m(s))\,, \tag{15}$$

we only have to take the inferior limit in (15) as $t\to0$ instead of taking, as was done in 1.7, the limit in (15) as $t\to0$. Therefore having (15) for *every* $t\in(0,1)$ is superfluous for establishing (19). We only have to have (15) for a decreasing zero sequence $\{t_n\}$ with $t_n<s/(1+a)$. Supposing the validity of (15) for $t=t_n$ $(n=1,2,\cdots)$, and letting $n\to\infty$, we deduce (19). Examination of the proof of (15) in 1.7, we at once see that (15) is valid for $t=t_n$ $(n=1,2,\cdots)$ by *only* assuming $P(z)\leqq c^2|z|^{-2}$ on annuli

$$A_n:\ a_n\equiv(1-a)t_n\leqq|z|\leqq(1+a)t_n\equiv b_n\,, \tag{20}$$

$n=1,2,\cdots$. Here observe that $\operatorname{mod}A_n=\log(b_n/a_n)=\log((1+a)/(1-a))$. Conversely, let $\{A_n\}$ be a sequence of closed disjoint concentric annuli A_n in Ω such that A_{n+1} separates $z=0$ from A_n. In order to obtain the above mentioned situation, i.e. A_n is represented as in (20), it is necessary and sufficient for $\{A_n\}$ to satisfy $\inf_n\operatorname{mod}A_n>0$. Such a sequence $\{A_n\}$ will be referred to as satisfying the *condition* [A]. We then obtain the following generalization of the main theorem:

THEOREM. *If a density $P(z)$ on Ω: $0<|z|<1$ satisfies $P(z)\leqq c^2|z|^{-2}$ on $A=\bigcup_{n=1}^{\infty}A_n$, where c is a nonnegative constant and $\{A_n\}$ is a sequence of annuli in Ω with the condition* [A], *then the Picard principle is valid for P.*

3.2. Consider a parabolic *end* Ω of a Riemann surface in the sense of Heins [8], i.e. the relative boundary $\partial\Omega$ of Ω is a single analytic Jordan curve and Ω has an isolated single ideal boundary point δ of parabolic character. A *density* $P=P(z)dxdy$ $(z=x+iy)$ on Ω is a 2-form on $\bar\Omega=\Omega\cup\partial\Omega$ with nonnegative locally Hölder continuous coefficients $P(z)$. The *elliptic dimension* of a density at δ, $\dim P$ in notation, is defined (Nakai [**31**, **32**]) to be the dimension of the half module $\mathcal{P}_P$ of nonnegative solutions of the equation

$$L_P u \equiv \Delta u - Pu = 0 \quad (\text{i.e. } d^*du - uP = 0) \tag{21}$$

on Ω with the vanishing boundary values on $\partial\Omega$. The elliptic dimension of the particular density $P \equiv 0$ at δ, i.e. $\dim 0$, is called the *harmonic dimension* of δ. As in Section 1, we say that the *Picard principle* is valid for a density P at δ if $\dim P = 1$.

3.3. Consider an end Ω with $\dim 0 = 1$ at δ. To further generalize the result in 3.1, we take an *Evans harmonic function* l on Ω (cf. e.g. Nakai [**23**, **24**], Sario-Nakai [**37**], Sario-Noshiro [**38**]), i.e.

(a) $l \in \mathcal{P}_0$

(b) $\lim_{z\to\delta} l(z) = +\infty$

(c) $\int_{\partial\Omega} {}^*dl = -2\pi$.

Since $\dim 0 = 1$ at δ, such an l is unique on Ω and l is a generator of $\mathcal{P}_0$. Using l we introduce a *polar coordinate differential* $(dr, d\theta)$ with its center δ:

$$\begin{cases} dr/r = -dl \\ d\theta = -{}^*dl\,. \end{cases}$$

Then $r = e^{-l}$ is a single valued function on Ω but θ is a multi-valued function on Ω. We may use $re^{i\theta}$ as local parameters at each point of Ω except for the isolated set of Ω where $rdrd\theta = rdr_\wedge d\theta = 0$. We denote by E the set of $r \in (0, 1)$ such that the level line $\{z \in \Omega\,;\ r(z) = r\}$ is an analytic Jordan curve in Ω. Clearly E is an open subset of $(0, 1)$ with $0 \in \bar{E}$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $(0, 1)$ such that $a_{n+1} < b_{n+1} < a_n < b_n$ and $[a_n, b_n] \subset E$ $(n = 1, 2, \cdots)$. Then

$$A_n = \{z \in \Omega\,;\ a_n \leqq r(z) \leqq b_n\} \tag{22}$$

is an annulus with $\operatorname{mod} A_n = \log(b_n/a_n)$. We say that $\{A_n\}$ satisfies the *condition* [A] if

$$\inf_n \operatorname{mod} A_n > 0\,. \tag{23}$$

The existence of such a sequence $\{A_n\}$ is a property of Ω, which may not hold in general. A typical example of such an Ω is the punctured unit disk $0 < |z| < 1$.

3.4. Consider a density P on Ω. Then $P/rdrd\theta$ is a single valued function on Ω with values in $[0, +\infty]$ finite on Ω except for the set where $rdrd\theta = 0$. We can represent as

$$P = P(r, \theta) r\,dr\,d\theta = P(re^{i\theta}) r\,dr\,d\theta\,. \tag{24}$$

In particular, $re^{i\theta}$ is a genuine polar coordinate on each of A_n. Then we have the following generalization of Theorem in 3.1 and, of course, of the main theorem:

THEOREM. *Suppose there exists a sequence* $\{A_n\}$ *of annuli as in* (22) *with the condition* [A] *in* Ω. *If a density* $P=P(re^{i\theta})rdrd\theta$ *on* Ω *satisfies* $P(re^{i\theta})\leqq c^2r^{-2}$ *on* $\bigcup_{n=1}^{\infty}A_n$, *then the Picard principle is valid for* P *at* δ.

In fact, since the duality theorem and the maximum-minimum principle are also valid on Ω (cf. [**31**]) and since the Harnack principle is only used on $\bigcup_{n=1}^{\infty}A_n$, the proof for the theorem in 3.1 can be verbatim applied to the present one.

3.5. We remark that our first assumption $\dim 0=1$ in 3.3 is used to obtain the uniqueness of the Evans harmonic function, but actually the condition [A] (i.e. (23)) is stronger than the condition $\dim 0=1$. Heins [8] showed that $\dim 0=1$ if Ω satisfies the *condition* [H]: There exists a sequence $\{A_n\}$ of disjoint annuli with analytic Jordan boundaries on Ω satisfying the condition that for each n, A_{n+1} separates A_n from the ideal boundary δ, and A_1 separates the relative boundary $\partial\Omega$ from the ideal boundary and

$$\sum_{n=1}^{\infty} \operatorname{mod} A_n=\infty\,. \tag{25}$$

We shall show that we cannot replace the condition [A] (i.e. (23)) by the Heins condition [H], i.e. (25), in the theorem in 3.4, by a counter example: there exist a rotation free density P on $0<|z|<1$ and a sequence of annuli $\{A_n\}$ with (25) such that $P(z)=0$ on $\bigcup_{n=1}^{\infty}A_n$ but $\dim P\neq 1$ (hence $\dim P=\mathfrak{c}$ (cf. Nakai [**26**])).

3.6. To construct an example mentioned above, we recall the P-unit criterion for the rotation free densities $P(z)$ (Kawamura-Nakai [**14**]). The *associated function* $Q(t)$ to $P(r)$ is the function on $[0,\infty)$ defined by $Q(t)=e^{-2t}P(e^{-t})$. The *Riccati component* a_Q of Q is the unique nonnegative solution of the equation

$$-a'(t)+a(t)^2=Q(t) \tag{26}$$

on $[0,\infty)$. It is known (cf. [**14**]) that $\dim P=1$ if and only if

$$\int_0^{\infty}\frac{dt}{a_Q(t)+1}=+\infty\,. \tag{27}$$

Take three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ such that $a_n=1+2\sum_{k=1}^{n-1}(1/k)$, $(a_1=1)$, $b_n=a_n+1/n$, $c_n=b_n+1/n$, $(n=1,2,\cdots)$. With the aid of these sequences we define a continuous function $a(t)$ on $[0,\infty)$ by $a(t)=1/(c_n-t)$ if $t\in[a_n,b_n]$, $a(t)$ is a polygonal line if $t\in[b_n,a_{n+1}]$, and $a(t)=1/2$ if $t\in[0,1]$. We observe that $-a'(t)+a(t)^2\geqq 0$ for $t\neq a_n, b_n$ and the equality is valid in (a_n,b_n). We compute

$$\sum_{n=1}^{\infty}\int_{b_n}^{a_{n+1}}\frac{dt}{a(t)+1}\leqq 2\sum_{n=1}^{\infty}\frac{1}{n(n+3)}<+\infty$$

and

$$\sum_{n=1}^{k}\int_{a_n}^{b_n}\frac{dt}{a(t)+1}=\sum_{n=1}^{k}\left(\frac{1}{n}+\log\frac{n+1}{n+2}\right)$$
$$=C+\log 2+\log(k/(k+2))+d_k$$

where C is the Euler constant and $d_k\to 0$ $(k\to\infty)$. Therefore we deduce that $\int_0^{\infty}(a(t)+1)^{-1}dt<+\infty$. We replace $a(t)$ by the parabola on sufficiently small neighbourhoods of a_n and b_n which are tangent to $a(t)$ so that resulting function $a_Q(t)$ is of class C^1 on $[0,\infty)$. We successively set $Q(t)=-a_Q'(t)+a_Q(t)^2$ and $P(r)=r^{-2}Q(-\log r)$. This $P(r)$ is a required density.

References

[1] F. Bowman, Introduction to Bessel Function, Longmans, Green and Co. Ltd., 1938.

[2] M. Brelot, Étude de l'équation de la chaleur $\Delta u=c(M)u(M)$, $c(M)\geqq 0$, au voisinage d'un point singulier du coefficient, Ann. Sci. École Norm. Sup., **48** (1931), 153-246.

[3] M. Brelot, Sur le principe des singularités positives et la notion de source pour l'équation (1) $\Delta u(M)=c(M)u(M)$, $(c\geqq 0)$, Ann. Univ. Lyon Sci. Math. Astro., **11** (1948), 9-19.

[4] C. Constantinescu and A. Cornea, Über einige Problem von M. Heins, Rev. Roumaine Math. Pures Appl., **4** (1959), 277-281.

[5] C. Constantinescu and A. Cornea, Ideale Ränder Riemannscher Flächen, Springer, 1963.

[6] M. Godefroid, Sur un article de Kawamura et Nakai à propos du principe de Picard, Bull. Sci. Math., **102** (1978), 295-303.

[7] K. Hayashi, Les solutions positives de l'équation $\Delta u=Pu$ sur une surface de Riemann, Kōdai Math. Sem. Rep., **13** (1961), 20-24.

[8] M. Heins, Riemann surfaces of infinite genus, Ann. of Math., **55** (1952), 296-317.

[9] S. Itô, On existence of Green function and positive superharmonic functions for linear elliptic operators of second order, J. Math. Soc. Japan, **16** (1964), 299-306.

[10] S. Itô, Martin boundary for linear elliptic differential operator of second order in a manifold, J. Math. Soc. Japan, **16** (1964), 307-334.

[11] S. Itô, Partial Differential Equations, Baifûkan, 1966 (in Japanese).

[12] M. Kawamura, Denseness of singular densities, Proc. Japan Acad. **50** (1974), 818-820.

[13] M. Kawamura, Picard principle for finite densities on some end, Nagoya Math. J., **67** (1977), 35-40.

[14] M. Kawamura and M. Nakai, A test of Picard principle for rotation free densities II, J. Math. Soc. Japan, **28** (1976), 232-342.

[15] Z. Kuramochi, An example of a null-boundary Riemann surface, Osaka J. Math., **6** (1954), 83-91.

[16] A. Lahtinen, On the solutions of $\Delta u=Pu$ for acceptable densities on open Riemann surfaces, Ann. Acad. Sci. Fenn., **515** (1972).
[17] A. Lahtinen, On the equation $\Delta u=Pu$ and the classification of acceptable densities on Riemann surfaces, Ann. Acad. Sci. Fenn., **553** (1973).
[18] A. Lahtinen, On the existence of singular solutions of $\Delta u=Pu$ on Riemann surfaces, Ann. Acad. Fenn., **546** (1973).
[19] R. Martin, Minimal positive harmonic functions, Trans. Amer. Math. Soc., **49** (1941), 137-172.
[20] L. Myrberg, Über die Integration der Differentialgleichung $\Delta u=c(P)u$ auf offenen Riemannschen Flächen, Math. Scand., **2** (1954), 142-152.
[21] L. Myrberg, Über die Existenz der Greenschen Funktion der Gleichung $\Delta u=c(P)u$ auf Riemannschen Flächen, Ann. Sci. Fenn., **170** (1954).
[22] M. Nakai, The space of nonnegative solutions of the equation $\Delta u=Pu$ on a Riemann surface, Kōdai Math. Sem. Rep., **12** (1960), 151-178.
[23] M. Nakai, On Evans potential, Proc. Japan Acad., **38** (1962), 624-629.
[24] M. Nakai, Evans' harmonic functions on Riemann surfaces, Proc. Japan Acad., **39** (1963), 74-78.
[25] M. Nakai, Order comparisons on canonical isomorphisms, Nagoya Math. J. **50** (1973), 67-87.
[26] M. Nakai, Martin boundary over an isolated singularity of rotation free density, J. Math. Soc. Japan, **26** (1974), 483-507.
[27] M. Nakai, Densities without Evans solutions, Tôhoku Math. J., **26** (1974), 363-370.
[28] M. Nakai, A test for Picard principle, Nagoya Math. J., **56** (1974), 105-119.
[29] M. Nakai, A test of Picard principle for rotation free densities, J. Math. Soc. Japan, **27** (1975), 412-431.
[30] M. Nakai, A remark on Picard principle II, Proc. Japan Acad., **51** (1975).
[31] M. Nakai, Picard principle and Riemann theorem, Tôhoku Math. J., **28** (1976), 277-292.
[32] M. Nakai, Picard principle for finite densities, Nagoya Math. J., **70** (1978) 7-14.
[33] M. Ozawa, Classification of Riemann surfaces, Kōdai Math. Sem. Rep., **4** (1952), 63-76.
[34] M. Ozawa, Some classes of positive solutions of $\Delta u=Pu$ on Riemann surfaces I, Kōdai Math. Sem. Rep., **6** (1954), 121-126.
[35] M. Ozawa, Some classes of positive solutions of $\Delta u=Pu$ on Riemann surfaces II, Kōdai Math. Sem. Rep., **7** (1955), 15-20.
[36] H. Royden, The equation $\Delta u=Pu$, and the classification of open Riemann surfaces, Ann. Acad. Sci. Fenn., **271** (1959).
[37] L. Sario and M. Nakai, Classification Theory of Riemann Surfaces, Springer, 1970.
[38] L. Sario and K. Noshiro, Value Distribution Theory, Van Nostrand, 1966.
[39] I. Singer, Dirichlet finite solutions of $\Delta u=Pu$, Proc. Amer. Math. Soc., **32** (1972), 464-468.
[40] I. Singer, Boundary isomorphism between Dirichlet finite solutions and harmonic functions, Nagoya Math. J., **50** (1973), 7-20.
[41] M. Šur, The Martin boundary for a linear elliptic second order operator, Izv. Akad. Nauk SSSR, **27** (1963), 45-60, (in Russian).

Michihiko KAWAMURA
Department of Mathematics
Faculty of Education
Fukui University
Bunkyo, Fukui, Japan

Current Address
Department of Mathematics
Faculty of Education
Gifu University
Nagara, Gifu
Japan