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Extremal length and univalent functions
II. Integral estimates of strip mappings

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Introduction.

In 1971, Jenkins and Oikawa [2] used extremal length techniques to obtain quantitative results on the boundary behavior of conformal mappings. This became the starting point for further investigations in that direction (e.g. [3], [5], [6], [7], and [8]); these have led to completely new theorems as well as to improvements over classical proofs and results.

At the core of [7] are two general theorems in which the real and imaginary parts of the mapping function of a strip onto a parallel strip are approximated by quantities related to extremal length. The general theory developed in [7] has applications to many areas of conformal mapping; several of these were explored in [7] itself. The authors are devoting a series of subsequent papers to further applications of that general theory. In [8], the first paper of the series, the results of [7] are applied to the problem of the angular derivative. In the present paper we apply [7] to J. Lelong-Ferrand's extensions of the Ahlfors Fundamental Inequalities [1] as well as her asymptotic formula for the strip mapping presented in her book [4, pp. 187-204]. Among the interesting applications of her theorems is a series of criteria for the existence of the angular derivative [4, pp. 205-215]. By our methods we are able to derive sharper versions of her results and to compare her asymptotic formula with others.

Our proof of Ferrand's extension of the Ahlfors First Inequality by use of extremal length techniques is included in [8], Proposition 3. (We present there only the lower estimate of the module of the pertinent quadrilateral; in view of [2, p. 665] this is all that is needed to obtain the estimate involving the real part of the mapping function). In the present paper we deal therefore only with her extension of the Second Inequality.

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The particular form of the integral involved in the error terms of Ferrand's theorems leads also to a simple proof of the Jenkins-Oikawa inequality [**2**, Theorem 2], which underlies their version of Ahlfors' Second Inequality (see § 4).

1. A module estimate in terms of the Ferrand integral.

The central result is the following

THEOREM 1. *Suppose Q is the quadrilateral $\{w=u+iv \mid \varphi_-(u)<v<\varphi_+(u),\ a<u<b\}$ where $\varphi_+(u)$ and $\varphi_-(u)$ are absolutely continuous functions and $\theta(u)=\varphi_+(u)-\varphi_-(u)\geqq l>0$ for $u\in[a, b]$. If $\lambda(a, b)$ denotes the extremal distance between the vertical sides of Q, then*

$$\lambda(a, b)\leqq\int_a^b \frac{du}{\theta(u)}+\frac{5}{l}\left[\int_a^b \frac{(\varphi_+')^2}{1+|\varphi_+'|}du+\int_a^b \frac{(\varphi_-')^2}{1+|\varphi_-'|}du\right] \tag{1}$$

provided each of the integrals in the brackets is smaller than $l/8$.

In the special case when $\varphi_+(u)\geqq l/2$ and $\varphi_-(u)\leqq -l/2$ in $[a, b]$ the latter restriction is not needed and (1) *holds with the constant* $5/l$ *replaced by* $5/2l$.

REMARK. Theorem 1 is only of interest when φ_+' or φ_-' (or both) are not bounded in $[a, b]$. Otherwise the inequality (1) follows immediately from [**7**, (13.4)] with the constant 5 replaced by another constant which depends on the bounds for $|\varphi_+'|$ and $|\varphi_-'|$.

PROOF. 1. We show first that we may assume $\varphi_+, \varphi_-\in C^1[a, b]$. We write φ for either φ_+ or φ_-. Since $\varphi'\in L^1[a, b]$ there exists a sequence $\{g_n\}$ of continuous functions in $[a, b]$ such that

$$\|\varphi'-g_n\|=\int_a^b |\varphi'(u)-g_n(u)|du<\frac{1}{n}, \quad n=1, 2, \cdots.$$

Then

$$\begin{aligned}\left\|\frac{g_n^2}{1+|g_n|}-\frac{\varphi'^2}{1+|\varphi'|}\right\| &= \left\|\frac{g_n^2(1+|\varphi'|)-\varphi'^2(1+|g_n|)}{(1+|g_n|)(1+|\varphi'|)}\right\| \\ &\leqq \|g_n-\varphi'\|+\left\|\frac{|\varphi' g_n|\,|g_n-\varphi'|}{(1+|g_n|)(1+|\varphi'|)}\right\| \\ &\leqq 2\|g_n-\varphi'\|<\frac{2}{n}.\end{aligned} \tag{2}$$

Let

$$\psi_n(u)=\varphi(a)+\int_a^u g_n(t)\,dt, \quad \psi_n'(u)=g_n(u), \quad u\in(a, b)$$

so that

$$|\psi_n(u)-\varphi(u)| \leqq \|g_n-\varphi'\| < \frac{1}{n}, \quad n=1, 2, \cdots, \quad u \in [a, b].$$

Suppose ψ_n^+ and ψ_n^- are the approximations to φ_+ and φ_-, respectively, determined in the above manner. Define

$$\varphi_n^+(u)=\psi_n^+(u)-\frac{1}{n} \quad \text{and} \quad \varphi_n^-(u)=\psi_n^-(u)+\frac{1}{n}.$$

Then

$$0 \leqq \varphi_+(u)-\varphi_n^+(u) < \frac{2}{n} \quad \text{and} \quad 0 \geqq \varphi_-(u)-\varphi_n^-(u) \geqq -\frac{2}{n}$$

and

$$\theta(u) \geqq \theta_n(u)=\varphi_n^+(u)-\varphi_n^-(u) \geqq \theta(u)-\frac{4}{n} \geqq l-\frac{4}{n}. \tag{3}$$

Let $n>4/l$. The quadrilateral $Q_n=\{a<u<b, \varphi_n^-(u)<v<\varphi_n^+(u)\}$ is contained in Q. If $\lambda_n(a, b)$ denotes the extremal distance between the vertical sides of Q_n we have therefore

$$\lambda(a, b) \leqq \lambda_n(a, b). \tag{4}$$

Suppose we proved that (for all sufficiently large n)

$$\lambda_n(a, b) \leqq \int_a^b \frac{du}{\theta_n(u)}+\frac{5}{l-4/n}\left[\int_a^b \frac{(\varphi_n^{+\prime})^2 du}{1+|\varphi_n^{+\prime}|}+\int_a^b \frac{(\varphi_n^{-\prime})^2 du}{1+|\varphi_n^{-\prime}|}\right].$$

Then we obtain (1) from (4), (2), and (3) by letting $n \to \infty$.

2. We now assume that $\varphi_+, \varphi_- \in C^1[a, b]$ and proceed with the following construction. For the moment we write φ in place of φ_+.

Let $M \geqq 2$ be a constant and let $\{\bar{J}_n\}$, $n=1, 2, \cdots$ denote the intervals in $[a, b]$ in which $|\varphi'(u)| \geqq M$. In each $\bar{J}_n$ we have either $\varphi'(u) \geqq M$ or $\varphi'(u) \leqq -M$. If $\varphi'(u) \geqq M$ in $\bar{J}_n=[\alpha_n, \beta_n]$ then clearly $\varphi(u) \geqq \varphi(\alpha_n)+(u-\alpha_n)M=g(u)$ there. We define $\varphi_n(u)=g(u)$ in the largest interval $\bar{J}_n^*: \alpha_n \leqq u \leqq \beta_n^* \leqq b$ in which $g(u) \leqq \varphi(u)$. We have $\beta_n \leqq \beta_n^*$ and the point $(\beta_n^*, g(\beta_n^*))$ is the first point of intersection of the ray $v=g(u)$, $u \geqq \alpha_n$, with $v=\varphi(u)$ or with the line $u=b$, whichever is met first. For $u \in [a, b]$, $u \notin \bar{J}_n^*$ we set $\varphi_n(u)=\varphi(u)$.

If $\varphi'(u) \leqq -M$ in $\bar{J}_n$ the definitions of φ_n and $\bar{J}_n^*$ are analogous; if $\bar{J}_n=[\alpha_n, \beta_n]$ then $\bar{J}_n^*=[\alpha_n^*, \beta_n]$ where $a \leqq \alpha_n^* \leqq \alpha_n \leqq \beta_n$.

In both cases $\varphi_n(u) \leqq \varphi(u)$ and φ_n has a continuous derivative, except possibly at two points where it has left and right hand derivatives, bounded for all n in $[a, b]$. While the intervals $\bar{J}_n$ are disjoint the $\bar{J}_n^*$ may overlap. We denote by J_n and J_n^* the interiors of $\bar{J}_n$ and $\bar{J}_n^*$, respectively.

3. Before proceeding further we need several inequalities:

$$|\varphi'(u)| \leqq \frac{\varphi'(u)^2}{1+|\varphi'(u)|}+1$$

and therefore for any measurable set $E \subset [a, b]$

$$V(E)=\int_E |\varphi'(u)| du \leqq \int_E \frac{\varphi'^2(u)}{1+|\varphi'(u)|} du+m(E). \tag{5}$$

Hence

$$m(J_n^*) \leqq \frac{V(J_n^*)}{M} \leqq \frac{1}{M} \int_{J_n^*} \frac{\varphi'^2}{1+|\varphi'|} du+\frac{1}{M} m(J_n^*),$$

$$\left(1-\frac{1}{M}\right) m(J_n^*) \leqq \frac{1}{M} \int_{J_n^*} \frac{\varphi'^2}{1+|\varphi'|} du$$

or

$$m(J_n^*) \leqq \frac{1}{M-1} \int_{J_n^*} \frac{\varphi'^2}{1+|\varphi'|} du. \tag{6}$$

From (5) and (6) we obtain

$$V(J_n^*) \leqq \frac{M}{M-1} \int_{J_n^*} \frac{\varphi'^2}{1+|\varphi'|} du. \tag{7}$$

Also, if $\tilde{J}_n^*$ denotes a subinterval of J_n^*, which contains J_n, then we deduce similarly

$$m(\tilde{J}_n^*) \leqq \frac{1}{M-1} \int_{\tilde{J}_n^*} \frac{\varphi'^2}{1+|\varphi'|} du. \tag{8}$$

Finally, we note that, for $u \in [a, b]$

$$\varphi(u)-\varphi_n(u) \leqq V(J_n^*) \leqq \frac{M}{M-1} \int_{J_n^*} \frac{\varphi'^2}{1+|\varphi'|} du.$$

By hypothesis $\int_a^b \varphi'^2(1+|\varphi'|)^{-1} du \leqq \frac{l}{8}$; since $M \geqq 2$ we obtain

$$\varphi(u)-\varphi_n(u) \leqq 2\int_a^b \frac{\varphi'^2}{1+|\varphi'|} du \leqq 2 \cdot \frac{l}{8}=\frac{l}{4}. \tag{9}$$

Moreover

$$\min_{u \in [a,b]} \varphi_n(u) \geqq \min_{u \in [a,b]} \varphi(u). \tag{10}$$

In the following we shall consider the set $E_n=\bigcup_{m=1}^{n} J_m^*$. Note that if $J_k\subset J_m^*$ and $J_m\subset J_k^*$ then either $J_k^*\subset J_m^*$ or $J_k^*\supset J_m^*$. It is then easily seen that we can pass to a suitable disjoint family $\{\tilde{J}_m^*\}$ of subintervals of the J_m^*, and apply (8) to obtain

$$\begin{aligned} m(E_n)=m\Big(\bigcup_{m=1}^{n}{}' \tilde{J}_m^*\Big)&\leqq \sum_{m=1}^{n}{}' \frac{1}{M-1}\int_{\tilde{J}_m^*}\frac{\varphi'^2}{1+|\varphi'|}du \\ &=\frac{1}{M-1}\int_{E_n}\frac{\varphi'^2}{1+|\varphi'|}du \end{aligned} \tag{11}$$

where $\bigcup'$ and $\sum'$ denote union and summation over the subset of $m=1,2,\cdots,n$. If $E=\bigcup_{m=1}^{\infty} J_m^*$, then

$$m(E)\leqq\frac{1}{M-1}\int_E\frac{\varphi'^2}{1+|\varphi'|}du. \tag{12}$$

Furthermore we note that $m(E-E_n)\to 0$ as $n\to\infty$.

4. We assume $M\geqq 2$ and define for $u\in[a,b]=\bar{I}$

$$f_n(u)=\min(\varphi_1(u),\varphi_2(u),\cdots,\varphi_n(u)).$$

From (9) we have for all n

$$\varphi(u)\geqq f_n(u)\geqq\varphi(u)-\frac{l}{4}. \tag{13}$$

Furthermore, f_n is continuous in $\bar{I}$ and for $u\in I-\bar{E}_n$: $f_n(u)=\varphi(u)$, $f'_n(u)=\varphi'(u)$. There are at most finitely many points in $\bar{E}_n$ where $f'_n(u)$ does not exist and at all other points of E_n we have $f'_n(u)=\pm M$. At the exceptional points f_n has a right and left hand derivative bounded by $B=\max_{u\in\bar{I}}|\varphi'(u)|$. Thus $f_n\in\mathrm{Lip}(1)$ in $\bar{I}$ with the Lipschitz constant B.

We estimate now

$$\begin{aligned} \int_a^b f_n'^2(u)du&=\int_{E_n} f_n'^2du+\int_{I-E} f_n'^2du+\int_{E-E_n} f_n'^2du \\ &=M^2 m(E_n)+\int_{I-E}\varphi'^2du+\int_{E-E_n} f_n'^2du. \end{aligned} \tag{14}$$

On $I-E$ we have $|\varphi'(u)|\leqq M$ and there $1\leqq\dfrac{1+M}{1+|\varphi'|}$. Hence

$$\int_{I-E}\varphi'^2du\leqq(1+M)\int_{I-E}\frac{\varphi'^2}{1+|\varphi'|}du$$

$$=(M+1)\int_a^b\frac{\varphi'^2}{1+|\varphi'|}du-(M+1)\int_E\frac{\varphi'^2}{1+|\varphi'|}du.$$

Using this inequality in (14) and (11) to estimate $m(E_n)$ we have

$$\int_a^b f_n'^2du\leqq(M+1)\int_a^b\frac{\varphi'^2}{1+|\varphi'|}du+\frac{M^2}{M-1}\int_{E_n}\frac{\varphi'^2}{1+|\varphi'|}du$$

$$-(M+1)\int_E\frac{\varphi'^2}{1+|\varphi'|}du+B^2m(E-E_n)$$

$$\leqq(M+1)\int_a^b\frac{\varphi'^2}{1+|\varphi'|}du+\frac{1}{M-1}\int_E\frac{\varphi'^2}{1+|\varphi'|}du+B^2m(E-E_n).$$

We choose now $M=2$ and obtain with $\varepsilon_n=B^2m(E-E_n)$

$$\int_a^b f_n'^2du\leqq4\int_a^b\frac{\varphi'^2}{1+|\varphi'|}du+\varepsilon_n\qquad\left(\lim_{n\to\infty}\varepsilon_n=0\right).\tag{15}$$

5. We return now to our original notation $\varphi=\varphi_+$ and write $f_n=f_n^+$. Next we perform the construction just described with $\varphi=-\varphi_-$ and obtain for all $n=1, 2, \cdots$ a function $f_n=-f_n^-$ such that, for all $u\in[a, b]$ and all n,

$$\varphi_-(u)\leqq f_n^-(u)\leqq\varphi_-(u)+\frac{l}{4}<\varphi_+(u)-\frac{l}{4}\leqq f_n^+(u)\leqq\varphi_+(u)\tag{16}$$

and

$$\int_a^b(f_n^{-\prime}(u))^2du\leqq4\int_a^b\frac{\varphi_-'^2}{1+|\varphi_-'|}du+\varepsilon_n'\qquad\left(\lim_{n\to\infty}\varepsilon_n'=0\right).\tag{17}$$

The quadrilateral $Q_n=\{a<u<b, f_n^-(u)<v<f_n^+(u)\}$ is contained in Q. If $\lambda_n(a,b)$ denotes the extremal distance between the vertical sides of Q_n then we have with $\theta_n(u)=f_n^+(u)-f_n^-(u)$ (cf. [7, (13.4)]; a careful estimate in (13.3) shows that $e(s, s_2)\leqq\frac{1}{2}\int_{s_1}^{s_2}(\varphi_+'^2+\varphi_-'^2)\frac{ds}{\theta(s)}$)

$$\lambda(a, b)\leqq\lambda_n(a, b)\leqq\int_a^b\frac{du}{\theta_n(u)}+\frac{1}{2}\int_a^b\frac{(f_n^{+\prime})^2+(f_n^{-\prime})^2}{\theta_n(u)}du.$$

Since by (16) $\theta_n(u)\geqq\theta(u)-\frac{l}{2}\geqq\frac{l}{2}$ we obtain from (15) and (17)

$$\lambda(a, b) \leqq \int_a^b \frac{du}{\theta(u)} + \int_a^b \left(\frac{1}{\theta_n} - \frac{1}{\theta}\right) du$$

$$+ \frac{4}{l} \left\{ \int_a^b \frac{\varphi_+'^2}{1+|\varphi_+'|} du + \int_a^b \frac{\varphi_-'^2}{1+|\varphi_-'|} du \right\} + \frac{2(\varepsilon_n + \varepsilon_n')}{l}.$$

Noting that $\theta_n(u) = \theta(u)$ except for $u \in E^+ \cup E^-$, where E^+ denotes the set E used in the construction of f_n^+ and E^- the corresponding set for f_n^-, we have by (12)

$$\left| \int_a^b \left(\frac{1}{\theta_n} - \frac{1}{\theta}\right) du \right| \leqq \frac{1}{l} (m(E^+) + m(E^-))$$

$$\leqq \frac{1}{l} \left\{ \int_a^b \frac{\varphi_+'^2}{1+|\varphi'|} du + \int_a^b \frac{\varphi_-'^2}{1+|\varphi_-'|} du \right\}.$$

Thus we find

$$\lambda(a, b) \leqq \int_a^b \frac{du}{\theta(u)} + \frac{5}{l} \left\{ \int_a^b \left(\frac{\varphi_+'^2}{1+|\varphi_+'|} + \frac{\varphi_-'^2}{1+|\varphi_-'|} \right) du \right\} + \frac{2(\varepsilon_n + \varepsilon_n')}{l}. \tag{18}$$

Letting $n \to \infty$ we obtain (1).

In the case where $\varphi_+(u) \geqq \frac{l}{2}$, $\varphi_-(u) \leqq -\frac{l}{2}$ for $u \in [a, b]$, it follows from (10) that $f_n^+(u) \geqq \frac{l}{2}$ and $f_n^-(u) \leqq -\frac{l}{2}$. Thus $\theta_n(u) \geqq l$ and the factor $\frac{1}{l}$ in (18) and (1) may be replaced by $\frac{1}{2l}$. Also the restriction $\int_a^b \frac{\varphi_\pm'^2}{1+|\varphi_\pm'|} du < \frac{l}{8}$ (used in (9) to ensure $\theta_n(u) \geqq \frac{l}{2}$) is no longer necessary.

REMARK. It is of interest to compare the estimate of the module $\lambda(a, b)$ in (1) in terms of the Ferrand integral $\int_a^b \frac{\varphi'^2}{1+|\varphi'|} du$ with the corresponding estimate in terms of the integral $\int_a^b \frac{\varphi_+'^2 + \varphi_-'^2}{\theta} du$ as given e.g. in (13.4) and (13.3) of [7] (for $t_1 = 0$, $t_2 = 1$). There are elementary examples for φ_+ and φ_- for which the Ferrand integral yields a smaller error term than the latter integral and conversely. On the other hand, we obtained the estimate in (1) by use of the estimate of the module for an approximating (inscribed) domain in terms of the *second* integral. Because the error term in (1) is insensitive to $\theta(u)$ it appears that the estimate of the module in terms of $\int_a^b (\varphi_+'^2 + \varphi_-'^2) \frac{du}{\theta}$ is the stronger result.

The Ferrand integral is of interest since it combines two cases: it converges when (a) $\int^{\infty}\varphi'^2 du<\infty$ or when (b) $\int^{\infty}|\varphi'|du<\infty$, i.e. the total variation of φ is bounded, for $\varphi=\varphi_+$ and $\varphi=\varphi_-$. (Cf. § 4).

2. Extension of the second fundamental inequality.

Our first application of our Theorem 1 is the proof of the following theorem which is an improvement of J. Lelong-Ferrand's result in [4], p. 201, (3) and (4).

THEOREM 2. *Suppose R is a simply connected region with the following property*: $R\supset R_0=\{w=u+iv \mid -\infty<u<\infty,\ \varphi_-(u)<v<\varphi_+(u)\}$ *where φ_+ and φ_- are absolutely continuous, $\theta(u)=\varphi_+(u)-\varphi_-(u)$, and the integrals*

$$\int^{\infty}\frac{\varphi_+'^2}{1+|\varphi_+'|}du<\infty \quad \text{and} \quad \int^{\infty}\frac{\varphi_-'^2}{1+|\varphi_-'|}du<\infty. \tag{19}$$

Let σ_u denote the crosscut of R which lies on $\operatorname{Re} w=u$ *and intersects R_0 and $\sigma(u)$ $(\leqq\infty)$ the length of σ_u.*

Let F be a univalent analytic function which maps R onto the parallel strip $S=\{z=x+iy \mid -\infty<x<\infty,\ 0<y<1\}$ such that $\operatorname{Re}F(w)\to\pm\infty$ *as* $\operatorname{Re}w\to\pm\infty$, *respectively, for $w\in R_0$. Let*

$$\bar{x}(u)=\sup_{w\in\sigma_u}\operatorname{Re}F(w),\quad \underline{x}(u)=\inf_{w\in\sigma_u}\operatorname{Re}F(w).$$

Then for $u_2>u_1$, for all sufficiently large u_1,

$$\underline{x}(u)-\bar{x}(u_1)\leqq\int_{u_1}^{u_2}\frac{du}{\theta(u)}+\frac{5}{l}\int_{u_1}^{u_2}\left(\frac{\varphi_+'^2}{1+|\varphi_+'|}+\frac{\varphi_-'^2}{1+|\varphi_-'|}\right)du=\Psi(u_1,u_2). \tag{20a}$$

If the length of σ_u satisfies $\sigma(u)\leqq L$ for all $u\geqq u_0$ (for some u_0) then

$$\bar{x}(u_2)-\underline{x}(u_1)\leqq\int_{u_1}^{u_2}\frac{du}{\theta(u)}+\frac{4L}{l}+\frac{5}{l}\int_{u_1-2L}^{u_2+2L}\left(\frac{\varphi_+'^2}{1+|\varphi_+'|}+\frac{\varphi_-'^2}{1+|\varphi_-'|}\right)du \tag{20b}$$

for all sufficiently large u_1.

PROOF. (a) Let $Q_R(u_1,u_2)$ denote the component of $R-\{\sigma_{u_1}\cup\sigma_{u_2}\}$ which contains the quadrilateral $Q_{R_0}(u_1,u_2)=Q(u_1,u_2)=\{u_1<u<u_2,\ \varphi_-(u)<v<\varphi_+(u)\}$. If $\lambda_R(u_1,u_2)$ and $\lambda(u_1,u_2)$ are the extremal distances between the vertical sides in Q_R and Q, respectively, we have $\lambda_R(u_1,u_2)\leqq\lambda(u_1,u_2)$. Since $\underline{x}(u_2)-\bar{x}(u_1)\leqq\lambda_R(u_1,u_2)$ the first inequality, (20a), is an immediate consequence of Theorem 1 if u_1 is taken so large that

$$\int_{u_1}^{\infty}\frac{(\varphi_+')^2}{1+|\varphi_+'|}du<\frac{l}{8} \quad \text{and} \quad \int_{u_1}^{\infty}\frac{(\varphi_-')^2}{1+|\varphi_-'|}du<\frac{l}{8}. \tag{21}$$

(b) To obtain (20b) we proceed as follows. The inverse F^{-1} of F maps the crosscuts $\{\operatorname{Re} z=\underline{x}(u_1),\ 0<y<1\}$ and $\{\operatorname{Re} z=\bar{x}(u_2),\ 0<y<1\}$ onto (generalized) crosscuts γ_1 and γ_2 of R, respectively. Let $R^*=\{w=u+iv \mid w\in\sigma_u,\ -\infty<u<\infty\}$, the Ostrowski kernel of R, and let

$$\underline{u}_1=\inf_{w\in\gamma_1\cap R^*}\operatorname{Re} w,\quad \bar{u}_2=\sup_{w\in\gamma_2\cap R^*}\operatorname{Re} w.$$

Then $\underline{u}_1<u_1<u_2<\bar{u}_2$. From the Ahlfors distortion theorem (with the constant 2 in place of 4, cf. [**2**, p. 665]) we obtain (even if, e.g. $\gamma_1\cap\sigma_{\underline{u}_1}=\emptyset$)

$$\int_{\underline{u}_1}^{u_1}\frac{du}{\sigma(u)}\leqq 2,\quad \int_{u_2}^{\bar{u}_2}\frac{du}{\sigma(u)}\leqq 2,$$

and since $\sigma(u)\leqq L$ we have

$$u_1-\underline{u}_1\leqq 2L,\quad \bar{u}_2-u_2\leqq 2L. \tag{22}$$

We repeat now the reasoning applied above with $Q_R(\underline{u}_1, \bar{u}_2)$ and $Q_{R_0}(\underline{u}_1, \bar{u}_2)$ and obtain

$$\bar{x}(u_2)-\underline{x}(u_1)\leqq\lambda_R(\underline{u}_1, \bar{u}_2)\leqq\lambda(\underline{u}_1, \bar{u}_2)\leqq\int_{\underline{u}_1}^{\bar{u}_2}\frac{du}{\theta(u)}+\frac{5}{l}\int_{\underline{u}_1}^{\bar{u}_2}\left(\frac{\varphi_+'^2}{1+|\varphi_+'^2|}+\frac{\varphi_-'^2}{1+|\varphi_-'|}\right)du$$

provided (21) is satisfied with u_1 replaced by u_1-2L. By (22)

$$\int_{\underline{u}_1}^{u_1}\frac{du}{\theta(u)}\leqq\frac{2L}{l}\quad\text{and}\quad\int_{u_2}^{\bar{u}_2}\frac{du}{\theta(u)}\leqq\frac{2L}{l}.$$

Using these inequalities and the fact that $\underline{u}_1\geqq u_1-2L$, $\bar{u}_2\leqq u_2+2L$ we obtain (20b).

REMARKS. The assumption that $\sigma(u)$ be bounded is not essential for an estimate of $\bar{x}(u_2)-\underline{x}(u_1)$ such as in (20b). It is sufficient to have a bound for the oscillation $\omega(u)=\bar{x}(u)-\underline{x}(u)$, since by (20a)

$$\bar{x}(u_2)-\underline{x}(u_1)\leqq\Psi(u_1, u_2)+\omega(u_1)+\omega(u_2).$$

There are geometrical conditions on the boundary of R which will ensure this and do not require the boundedness of $\sigma(u)$, e.g. [**9**, Lemma 1], Obrock [**5**, Theorem 1′, p. 200].

If $R\equiv R_0$ then $\omega(u)\to 0$ as $u\to\infty$: this follows from (24) below and Theorem 1 in [**7**], and we have $\bar{x}(u_2)-\underline{x}(u_1)\leqq\Psi(u_1, u_2)+o(1)$ as $u_1\to\infty$.

3. Asymptotic expression for the mapping function.

We now prove the asymptotic formula for the mapping function in [4, Théorème VI, 13a, p. 204]. It should be noted that our formulation, Theorem 3 below, does not require the assumption that $\theta(u)$ be bounded above made in Ferrand's theorem. Theorem 3 below may be compared with our Theorem 8 in [7]. In Theorem 8 no upper or lower bound is placed on $\theta(u)$; in addition cf. also the Remark to Theorem 1 of the present paper.

Other types of asymptotic expressions for the strip mapping function are given in [9, Theorem IX], [7, Section IV], Obrock [5, Theorem 2].

THEOREM 3. *Suppose R is the domain $\{w=u+iv: -\infty<u<\infty, \varphi_-(u)<v<\varphi_+(u)\}$ where $\varphi_+(u)$ and $\varphi_-(u)$ are absolutely continuous, $\theta(u)=\varphi_+(u)-\varphi_-(u)\geqq l>0$ and the integrals* (19) *converge. Let F map R conformally onto the parallel strip $S=\{z=x+iy: -\infty<x<\infty, 0<y<1\}$ such that $\lim_{u\to\pm\infty} \operatorname{Re} F(w)=\pm\infty$. Then for $w\in R$*

$$F(w)=\int_{u_0}^{u}\frac{dt}{\theta(t)}+i\frac{v-\varphi_-(u)}{\theta(u)}+C+o(1), \quad as \quad u\to+\infty \tag{23}$$

where C is a real constant.

PROOF. (a) Since by a well known inequality and by Theorem 1, $u_1<u_2$,

$$\int_{u_1}^{u_2}\frac{du}{\theta(u)}\leqq\lambda(u_1, u_2)\leqq\int_{u_1}^{u_2}\frac{du}{\theta(u)}+o(1) \quad \text{as} \quad u_1\to\infty, \tag{24}$$

it follows from Theorem 1 or Theorem 3 of [7] that

$$\operatorname{Re} F(w)=\int_{u_0}^{u}\frac{dt}{\theta(t)}+C+o(1) \quad \text{as} \quad u\to\infty .$$

(b) To prove the conclusion on $\operatorname{Im} F(w)$ we follow the reasoning of [7, Section 13]. In the notation of [7] we choose for V_s the Ahlfors crosscut θ_u $(s=u)$ and for H_t the arcs $\{w=u+iv, v=t\varphi_+(u)+(1-t)\varphi_-(u)\}$ and set $\lambda(V_{s_1}, V_{s_2}; H_{t_1}, H_{t_2})=\lambda(s_1, s_2; t_1, t_2)$. Let $0\leqq t_1<t_2\leqq 1$, let $\Phi_i(u)=t_i\varphi_+(u)+(1-t_i)\varphi_-(u)$, $i=1, 2$, and let $Q(t_1, t_2)$ denote the quadrilateral $\{a<u<b, \Phi_1(u)<v<\Phi_2(u)\}$. We assume $t_2-t_1\geqq\delta$, where $0<\delta<1/2$. Then $\Phi_2(u)-\Phi_1(u)=(t_2-t_1)\theta(u)\geqq l(t_2-t_1)$ $\geqq l\delta$, and by Theorem 1

$$\lambda(a, b; t_1, t_2)\leqq\frac{1}{t_2-t_1}\int_a^b\frac{du}{\theta(u)}+\frac{5}{(t_2-t_1)l}\left[\int_a^b\frac{\Phi_2'^2}{1+|\Phi_2'|}du+\int_a^b\frac{\Phi_1'^2}{1+|\Phi_1'|}du\right] \tag{25}$$

provided each of the integrals in the brackets is $\leqq\frac{l\delta}{8}\left(\leqq\frac{l}{8}(t_2-t_1)\right)$. Since

$$|t_i\varphi'_+ + (1-t_i)\varphi'_-| \leqq (\varphi'^2_+ + \varphi'^2_-)^{1/2}$$

and since the function $x^2(1+x)^{-1}$ for $x>0$ increases with x we have

$$\frac{\Phi'^2_i}{1+|\Phi'_i|} \leqq \frac{\varphi'^2_+ + \varphi'^2_-}{1+\sqrt{\varphi'^2_+ + \varphi'^2_-}} \leqq \frac{\varphi'^2_+}{1+|\varphi'_+|} + \frac{\varphi'^2_-}{1+|\varphi'_-|}, \quad i=1,\, 2.$$

Hence, by (19) each of the integrals in the brackets in (25) will be $\leqq l\delta/8$ for all sufficiently large a and all $b>a$.

We also note that

$$\int_a^b \frac{du}{\theta(u)} \leqq \lambda(a,\, b) = \lambda(a,\, b\,;\, 0,\, 1).$$

Now we determine as in [7, Section 13] $d=d(a)$ such that $\lambda(a,\, d(a))=1$ which implies $\int_a^{d(a)} \frac{du}{\theta(u)} \leqq 1$. By (25)

$$\lambda(H_{t_1},\, H_{t_2}\,;\, V_a,\, V_{d(a)}) \geqq \frac{t_2-t_1}{\int_a^{d(a)} \frac{du}{\theta(u)} + \frac{5}{l}\left[2\int_a^{d(a)} \left(\frac{\varphi'^2_+}{1+|\varphi'_+|} + \frac{\varphi'^2_-}{1+|\varphi'^2_-|}\right) du\right]}$$

$$\geqq \frac{t_2-t_1}{1+\int_a^{d(a)} \frac{10}{l}\left(\frac{\varphi'^2_+}{1+|\varphi'_+|} + \frac{\varphi'^2_-}{1+|\varphi'_-|}\right) du}.$$

Hence

$$\lambda(H_{t_1},\, H_{t_2}\,;\, V_a,\, V_{d(a)}) - (t_2-t_1) \geqq -\frac{\eta}{1+\eta}$$

where η denotes the integral in the denominator, and is arbitrarily small for all sufficiently large a ($\geqq S_{II}$). This verifies Condition II of [7] uniformly for all t with $\delta \leqq t \leqq 1-\delta$.

Hence, by Theorem 2 of [7]

$$\operatorname{Im} F(w) = \frac{v-\varphi_-(u)}{\theta(u)} + o(1) \quad \text{as} \quad u \to \infty \tag{26}$$

uniformly for $w=u+iv \in R$ with $\delta \leqq \frac{v-\varphi_-(u)}{\theta(u)} \leqq 1-\delta$.

Since $0 \leqq \operatorname{Im} F(w) \leqq 1$ for $w \in R$ and δ may be chosen arbitrarily small, it follows by a simple argument that (26) holds uniformly for $w \in R$.

4. Boundaries of bounded variation.

Finally we apply Theorem 1 to quadrilaterals Q where φ_+ and φ_- are of bounded variation. This application is suggested by the inequality which holds when φ is absolutely continuous:

$$\int_a^b \frac{\varphi'^2}{1+|\varphi'|}du=\int_a^b |\varphi'|\frac{|\varphi'|}{1+|\varphi'|}du\leqq\int_a^b|\varphi'|du=V(a, b), \tag{27}$$

where $V(a, b)$ is the total variation of φ in $[a, b]$. This fact can be used to deduce from our Theorem 1 the theorem of Jenkins and Oikawa [**2**, Theorem 2] used in the proof of their version of Ahlfors' Second Fundamental Inequality.

THEOREM 4. *Suppose* $Q=\{-\theta_1(u)<v<\theta_2(u),\ a<u<b\}$ *where* θ_1, θ_2 *are positive functions and have respective finite total variations* V_1, V_2 *on* $[a, b]$. *Suppose* $\theta_i(u)\geqq\theta^{(m)}$ *on* $[a, b]$. *If* $\lambda(a, b)$ *denotes the extremal distance between the vertical sides of* Q *then, with* $\theta(u)=\theta_1(u)+\theta_2(u)$,

$$\lambda(a, b)\leqq\int_a^b\frac{du}{\theta(u)}+\frac{5}{4\theta^{(m)}}(V_1+V_2). \tag{28}$$

REMARK. Theorem 2 in [2] is stated for the case that Q (in our Theorem 4) is the (Ostrowski) kernel of the quadrilateral considered whose module is $\lambda(a, b)$. Since the module for the kernel is an upper bound for the module of the quadrilateral, (28) is all that is needed. The bound $\theta_i(u)\leqq L$ is not used in our formulation.

PROOF. Let $a=u_0<u_1<u_2<\cdots<u_{n+1}=b$ be a partition of $[a, b]$. In each interval $[u_\nu, u_{\nu+1}]$ let $m_\nu=\min\limits_{u\in[u_\nu, u_{\nu+1}]}\theta_2(u)$. If $m_\nu<m_{\nu+1}$, $\nu=0, 1, \cdots, n-1$, connect the point $(u_{\nu+1}, m_\nu)$ with the point $\left(\frac{u_{\nu+1}+u_{\nu+2}}{2}, m_{\nu+1}\right)$ by a straight line segment $s_{\nu+1}$; if $m_\nu>m_{\nu+1}$, connect $(u_{\nu+1}, m_{\nu+1})$ with $\left(\frac{u_\nu+u_{\nu+1}}{2}, m_\nu\right)$ by a straight line segment s'_ν. Then for $\nu=0, 1, \cdots, n$ we define a function $\varphi_+^{(n)}(u)$ in $[u_\nu, u_{\nu+1}]$ as the ordinate of the point on a connecting segment s_ν or s'_ν whose abscissa is u if such segments occur; otherwise define it to be m_ν. The function $\varphi_+^{(n)}(u)$ is clearly absolutely continuous in $[a, b]$, $\theta^{(m)}\leqq\varphi_+^{(n)}(u)\leqq\theta_2(u)$ and its total variation in $[a, b]$ is $\leqq V_2$.

Using θ_1 instead of θ_2 we construct in the same manner an absolutely continuous function, $-\varphi_-^{(n)}$, such that $\theta^{(m)}\leqq-\varphi_-^{(n)}(u)\leqq\theta_1(u)$ and its total variation in $[a, b]$ is $\leqq V_1$.

The quadrilateral $Q_n=\{a<u<b,\ \varphi_-^{(n)}(u)<v<\varphi_+^{(n)}(u)\}\subset Q$; hence the extremal distance between the vertical sides of Q_n satisfies $\lambda_n(a, b)\geqq\lambda(a, b)$.

Let $\theta_n(u)=\varphi_+^{(n)}(u)-\varphi_-^{(n)}(u)$. Applying Theorem 1 to Q_n (in the case $\varphi_+(u)\geqq l/2$, $\varphi_-(u)\leqq -l/2$) and using (27) we obtain

$$\lambda(a, b)\leqq\lambda_n(a, b)\leqq\int_a^b \frac{du}{\theta_n(u)}+\frac{5}{4\theta^{(m)}}(V_1+V_2).$$

If we let $n\to\infty$ so that the norm of the partition tends to 0 the integral on the right hand side converges to $\int_a^b \frac{du}{\theta(u)}$ and (28) follows.

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