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On the automorphism of C^2 with invariant axes

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0. Statement of results.

In this paper we study the biholomorphic automorphism of C^2 which leaves two coordinate axes invariant. E. Peschl investigated the automorphism of this type in [1]. We say such an automorphism is of axial type. If $F=(f(x,y), g(x,y))$ is an automorphism of axial type, then F takes the form;

$$F:\begin{cases} f=xe^{\phi(x,y)} \\ g=ye^{\psi(x,y)}, \end{cases}$$

where ϕ and ψ are holomorphic functions. We say that a function $f(x,y)$ is a component of an automorphism (of axial type) if there is a function $g(x,y)$ such that

$$T:\begin{cases} x'=f(x,y) \\ y'=g(x,y) \end{cases}$$

is an automorphism (of axial type).

Our results are as follows.

THEOREM. (1) *Let $\phi(x,y)$ be a polynomial and set $f(x,y)=xe^{\phi(x,y)}$. Then $f(x,y)$ is a component of an automorphism of axial type if and only if $\phi(x,y)$ takes the form $A(x^m y^{n+1})$, where m and n are non-negative integers and A is a polynomial of one variable.*

(2) *The transformation*

$$T:\begin{cases} x'=xe^{A(x^m y^{n+1})} \\ y'=g(x,y) \end{cases}$$

is an automorphism of axial type if and only if g takes the form

$$y\cdot\exp\left[-\frac{m}{n+1}A(x^m y^{n+1})+H(x')\right],$$

where H is a holomorphic function of one variable.

§ 1. Discriminant $D(t)$.

Let $\phi(x, y)$ be a polynomial, and set $f=x\cdot\exp[\phi(x, y)]$. We discuss the necessary condition for f becomes a component of an automorphism of axial type.

We consider the analytic set $S_c=\{(x, y): f(x, y)=c\}$. This is the inverse image of the line $x'=c$. Then S_c is non-singular and is biholomorphically equivalent to the complex plane $\boldsymbol{C}$. And S_c does not intersect the y-axis for every c, except 0.

Set $x=e^t$. The analytic set $\tilde{S}_c=\{(t, y): f(e^t, y)=c\}$ is given by the equation $\phi(e^t, y)+t=\log c$. And every branch of $\log c$ gives an irreducible component of $\tilde{S}_c$. On the other hand, the mapping

$$\pi: \begin{cases} x=e^t \\ y=y \end{cases}$$

gives the universal covering space of $\boldsymbol{C}^2-(y\text{-axis})$. Then $\tilde{S}_c$ is a covering Riemann surface of S_c and this covering has no ramifying point and has no relative boundary. Then every component of $\tilde{S}_c$ is biholomorphically equivalent to $\boldsymbol{C}$. In particular $S=\{(t, y): \phi(e^t, y)+t=0\}$ is equivalent to $\boldsymbol{C}$.

Set

$$\phi(x, y)=\phi_0(x)y^n+\phi_1(x)y^{n-1}+\cdots+\phi_{n-1}(x)y+\phi_n(x),$$

where $\phi_i(x)$ is a polynomial $(i=0, 1, \cdots, n)$.

LEMMA 1. (1) *$\phi_0(x)$ is a monomial ax^h.*

(2) *$\phi_n(x)$ is a constant.*

PROOF. (1) We consider S as a covering Riemann surface over t-space. S is equivalent to $\boldsymbol{C}$, and S has no relative boundary over any point t. This implies that $\phi_0(e^t)$ is zero-free. Consequently $\phi_0(x)$ is a monomial.

(2) If the transformation

$$F: \begin{cases} x'=xe^{\phi(x,y)} \\ y'=ye^{\phi(x,y)} \end{cases}$$

is an automorphism, it maps x-axis biholomorphically onto x'-axis. Then $x'=x\cdot\exp[\phi(x, 0)]$ is a linear function of x. Hence $\phi_n(x)$ is constant. This implies our assertion.

Now we consider the transformation

$$T: \begin{cases} x'=x\cdot\exp[-\phi_n] \\ y'=y, \end{cases}$$

then $F\circ T$ takes the form

$$\begin{cases} x'=x\cdot\exp[\phi_0'(x)y^n+\cdots+\phi_{n-1}'(x)y+0 \\ y'=y\cdot\exp[\psi'(x,y)]. \end{cases}$$

Hence we may suppose that the constant $\phi_n(x)$ is equal to 0.

Let $D(t)$ be a discriminant of $\phi(e^t, y)+t=0$ as an algebraic equation fo y. Namely;

$$D(t)=\begin{vmatrix} aX^h, \phi_1(X), \cdots\cdots, \phi_{n-1}(X), t & & \\ & aX^h, \phi_1(X), \cdots\cdots, \phi_{n-1}(X), t & \\ & \cdots\cdots\cdots\cdots & \\ & & aX^h, \phi_1(X), \cdots\cdots, t \\ naX^h, (n-1)\phi_1(X), \cdots\cdots, \phi_{n-1}(X) & & \\ & naX^h, \cdots\cdots\cdots\cdots, \phi_{n-1}(X) & \\ & \cdots\cdots\cdots\cdots & \\ & \cdots\cdots\cdots\cdots & \\ & & naX^h, \cdots\cdots, \phi_{n-1}(X) \end{vmatrix},$$

where we used the symbolical notation $X=e^t$. It is apparent that $D(t)$ is a polynomial of t and X. And $D(t)$ is not identically zero.

PROPOSITION 1. *$D(t)$ is a monomial of X.*

PROOF. We regard S as an n-fold covering Riemann surface over the t-space. Because S is non-singular in (t, y)-space, there is a ramifying point over every zero of $D(t)$. According to the relation of Riemann-Hurwitz, there must be only finitely many ramifying points, because the genus of S is finite. Set

$$D(t)=\alpha_k(t)e^{kt}+\alpha_{k-1}(t)e^{(k-1)t}+\cdots+\alpha_1(t)e^t+\alpha_0(t),$$

where $\alpha_i(t)$ $(i=0, 1, \cdots, k)$ is a polynomial of t. From the above argument $D(t)$ has only finitely many zeros. Then $D(t)$ takes the form $Q(t)\cdot\exp[\beta(t)]$, where $Q(t)$ is a polynomial of t and $\beta(t)$ is an entire function of t. Consequently we have the equality

$$\alpha_k(t)e^{kt}+\alpha_{k-1}(t)e^{(k-1)t}+\cdots+\alpha_0(t)=Q(t)e^{\beta(t)}. \quad\cdots\cdots\cdots\cdots(*)$$

The function of left hand side is of increasing order one. Then the function $\exp[\beta(t)]$ is of increasing order one also. According to the theorem of Polya in the theory of entire function, $\beta(t)$ is a linear function.

Then $\beta(t)$ takes the simple form pt. From the equality (*) we have

$$\lim_{t\to\infty}\frac{\alpha_k(t)e^{kt}+\alpha_{k-1}(t)e^{(k-1)t}+\cdots+\alpha_0(t)}{Q(t)e^{pt}}=1,$$

for positive real value t. Then we have $\operatorname{Re} p=k$. When t is a purely imaginary value we have

$$|\alpha_k(t)e^{kt}+\alpha_{k-1}(t)e^{(k-1)t}+\cdots+\alpha_0(t)|\leqq M\cdot t^N,$$

for some integer N and a positive constant value M. Then we have $\operatorname{Im} p=0$. Consequently $\beta(t)$ is equal to kt. This completes the proof.

§ 2. Necessary condition.

We consider the polynomial of two variables

$$\phi(x, y)=ax^hy^n+\phi_1(x)y^{n-1}+\cdots+\phi_{n-1}(x)y,$$

where a is a constant. And we put the following condition (A).

(A) *The discriminant $D(x, t)$ of the equation $\phi(x, y)-t=0$, as an algebraic equation for y, is a monomial of x.*

This condition is equivalent to the following condition (B).

(B) *When we regard $C_t=\{\phi(x, y)=t\}$ as a covering Riemann surface over x-plane, the ramifying point and the equivalent point (namely; the reducible point of C_t as an analytic set in (x, y)-space) of C_t are situated over $x=0$ for every t, with a finite number of exception.*

LEMMA 2. *Suppose there are a polynomial of two variables $F(x, y)$ and a polynomial of one variable G such teat $\phi(x, y)=G(F(x, y))$. If ϕ satisfies the condition* (A), *then F satisfies the condition* (A) *also.*

PROOF. Assume that ϕ satisfies the condition (B). Let $\rho_1, \rho_2, \cdots, \rho_k$ be the totality of the roots of $G(z)-t=0$. Then we have

$$C_t=\bigcup_{i=1}^{k}\{F(x, y)=\rho_i\}.$$

Consequently $F(x, y)$ satisfies the condition (B).

If $\phi(x, y)$ has no above decomposition, we say ϕ is primitive. If $\phi(x, y)$ is primitive, every C_t is irreducible and nonsingular in (x, y)-space except finite values of t.

PROPOSITION 2. *Suppose $\phi(x, y)$ satisfies the condition* (A). *Then $\phi(x, y)$ is decomposed to a polynomial of one variable and a monomial x^my^n.*

To prove this proposition we need the following lemma.

LEMMA 3. *Let $y=\xi(x)$ be an algebraic function. Suppose this function has exactly n values $\xi_1(x), \cdots, \xi_n(x)$ in $C^*=C-\{0\}$ for every x in C^*. Then we have*

$$\xi(x)=cx^{m/n},$$

where c is a complex constant and m is an integer relatively prime to n.

PROOF. Set $D_1=x$-plane$-\{0\}$. And set $D_2=y$-plane$-\{0\}$. Then $x=e^t$

realizes the universal covering of D_1. And $\xi(e^t)$ is a single valued function according to the monodromy theorem, then this function realizes the universal covering of D_2. (Because the inverse mapping ξ^{-1} gives an unramified covering D_2 over D_1 according to the assumption for ξ.) On the other hand, the universal covering of D_2 is given by the mapping $y=e^s$. Because the s-space and the t-space are biholomorphically equivalent, then we have $s=at+b$. Consequently $\xi(e^t)$ takes the form e^{at+b}. The assumption that $\xi(x)$ is an n-valued algebraic function indicates the equality $a=m/n$. This is our assertion.

PROOF OF THE PROPOSITION 2. We may assume that $\phi(x, y)$ is primitive. Let $\hat{C}_t$ be the compactification of the covering Riemann surface C_t over Riemann sphere $\boldsymbol{P}$. Let v be the sum of the degrees of ramifications of $\hat{C}_t$. Then the Euler characteristic ρ of $\hat{C}_t$ is given by $-\rho=-2n+v$. Since C_t is irreducible, we have $\rho\leqq 2$. Consequently we have $v\geqq 2n-2$. Because the ramifying point and the equivalent point of $\hat{C}_t$ are situated only over the points $x=0$ and $x=\infty$, v is at most $2n-2$. Then we have $v=2n-2$ and $\rho=2$. This implies that $\hat{C}_t$ is biholomorphically equivalent to $\boldsymbol{P}$ and that $\hat{C}_t$ has ramifying points of the degree of ramification $n-1$ over $x=0$ and $x=\infty$. Since the coefficient function of y^n in $\phi(x, y)$ is a monomial, C_t has a relative boundary over $x=0$. And every C_t, except finite, does not intersect the y-axis, then ϕ is constant there. And ϕ is constant zero on the x-axis, then ϕ is constant zero on the y-axis.

We consider $\phi(x, y)-t=0$ as an algebraic function $y=\zeta_t(x)$. Let $\tilde{C}_t$ be the Riemann surface of this algebraic function over $|x|<\infty$. Then the following properties are satisfied.

(1) C_t is irreducible, nonsingular, of order of multiplicity 1 and equal to $\tilde{C}_t$ for every t, except finite.

(2) $\phi(x, y)=0$ on $\{(x, y): xy=0\}$.

(3) $\zeta_t(x)$ has exact n values over every x except $x=0$ and $x=\infty$.

These properties ensure the assumption of Lemma 3 for $\zeta(x)$. From (2) $\phi(x, y)$ takes the form $x^{m'}y^{n'}Q(x, y)$, where m' and n' are positive integers and $Q(x, y)$ is a polynomial. By Lemma 3 we have the equality of the sets;

$$\{(x, y): x^{m'}y^{n'}Q(x, y)-t=0\}=\{(x, y): x^m y^n-c(t)=0\},$$

for general values of t. Consequently we have $m'=m$, $n'=n$ and $Q(x, y)=$ constant. This completes the proof.

§ 3. Conjugate function.

From the results of preceding arguments we know the necessary condition. Namely; if a function $f(x, y)=x\cdot\exp[\phi(x, y)]$ becomes a component of an automorphism of axial type then $\phi(x, y)$ is decomposed to a polynomial of one variable and a monomial $x^m y^n$.

In the remainder of this paper we discuss about the conjugate function $g(x, y)$ of this $f(x, y)$ such that

$$T:\begin{cases} x'=f(x, y) \\ y'=g(x, y) \end{cases}$$

becomes an automorphism of axial type. Set

$$f(x, y)=x\cdot\exp\left[c_0+c_1x^my^n+c_2(x^my^n)^2+\cdots+c_\mu(x^my^n)^\mu\right],$$

where $m\geqq 0$ and $n>0$. We consider the following automorphisms.

$$T_k:\begin{cases} x'=x\cdot\exp\left[-c_k(x^my^n)^k\right] \\ y'=y\cdot\exp\left[(m/n)c_k(x^my^n)^k\right], \qquad k=0, 1, \cdots, \mu. \end{cases}$$

Then $f(x, y)$ is reduced to the function x by the transformation $T_0\cdot T_1\cdot\cdots\cdot T_\mu$. Hence the conjugate function $g(x, y)$ is given by

$$g(x, y)=T_\mu^{-1}\cdot T_{\mu-1}^{-1}\cdots T_0^{-1}(K(x, y)),$$

where $K(x, y)$ is a conjugate function of x. If the transformation

$$T:\begin{cases} x'=f(x, y) \\ y'=g(x, y) \end{cases}$$

becomes an automorphism of axial type then the transformation

$$S:\begin{cases} \xi'=\xi \\ \eta'=K(\xi, \eta) \end{cases}$$

is an automorphism of axial type, because every T_k is an automorphism of axial type.

LEMMA 4. *The transformation*

$$S:\begin{cases} \xi'=\xi \\ \eta'=K(\xi, \eta) \end{cases}$$

is an automorphism if and only if K takes the form $(\eta+A(\xi))\cdot\exp[H(\xi)]$, where $A(\xi)$ and $H(\xi)$ are entire functions. And in particular S is an automorphism of axial type if and only if K takes the form $\eta\exp[H(\xi)]$.

PROOF. The sufficiency is trivial. Then we show the necessity. Because $K(\xi', \eta)-\eta'=0$ defines only one η for given ξ' and η', this equality is transformed to the form $\eta=G(\xi', \eta')$. And the former is linear in η', then $G(\xi', \eta')=B(\xi')\eta'-A(\xi')$. Consequently we have

$$\eta' = \frac{\eta + A(\xi)}{B(\xi)} .$$

Since $B(\xi)$ must be zero free, we have $B(\xi) = \exp[-H(\xi)]$. This implicates our assertion.

PROPOSITION 3. *Let $f(x, y)$ be a function of the form $x \cdot \exp[\phi(x, y)]$, where*

$$\phi(x, y) = c_0 + c_1(x^m y^n) + c_2(x^m y^n)^2 + \cdots + c_\mu(x^m y^n)^\mu .$$

Then the transformation

$$T: \begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$$

is an automorphism of axial type if and only if

$$g(x, y) = y \cdot \exp[-(m/n)\phi(x, y) + H(x')] ,$$

where H is an entire function.

PROOF. From the above argument, g is given by

$$g(x, y) = T_\mu^{-1} \cdot T_{\mu-1}^{-1} \cdots T_0^{-1}(y e^{H(x)}) .$$

By an elementary calculation we have the required result.

By these propositions we have the theorem stated at the beginning.

Bibliography

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