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Scattering theory for differential operators, II, self-adjoint elliptic operators

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The present paper is a direct continuation of Part I (reference [1] which, hereafter, will be referred to as (I)) and is concerned with the application of the results obtained in (I) to the spectral and the scattering problems for the self-adjoint elliptic differential operator

$$Hu=\sum_{|\alpha|,|\beta|\leqq m} D^{\alpha}(a_{\alpha\beta}^{(1)}+a_{\alpha\beta}(x))D^{\beta}u$$

in R^{n}. Throughout the paper the same notations as in (I) will be used. Theorems etc. given in (I) will be quoted as Theorem I.2.9 for theorems, as (I.3.7) for formulas, as [I.1] for references, etc.

Recently, S. Agmon investigated the self-adjoint elliptic operator $\sum_{|\alpha|\leqq 2m} a_{\alpha}(x)D^{\alpha}u$ by the method of limiting absorption (or by a weighted elliptic estimate) and announced the results in [I.1] and a lecture quoted in (I). The result given in the present paper considerably overlaps with Agmon's results. In particular, Theorem 1.5 given below is essentially equivalent to what is announced in [I.1]. The approach to the proof, however, is different. Our work has been carried out independently of Agmon's except for the last stage where the proof of ii) of Theorem 1.5 (or Theorem I.5.21) was completed after having been stimulated by Agmon's work.

We will treat the problem as an example to which the abstract method given in (I) can be applicable.[1] The crucial tool which makes this application possible is the trace theorem in the Sobolev spaces.

§1. Assumptions and main results.

Throughout the present paper we write $D_{j}=-i\partial/\partial x_{j}$, $x=(x_{1}, \cdots, x_{n})\in R^{n}$, and $D^{\alpha}=D_{1}^{\alpha_{1}}\cdots D_{n}^{\alpha_{n}}$, where $\alpha=(\alpha_{1}, \cdots, \alpha_{n})$ is a multi-index and the differentia-

1) As is mentioned in §1 of (I), M. Š. Birman also investigated the scattering theory for general differential operators by applying his abstract method. See, e.g., [I.2].

tion is taken in the sense of distribution. For a real number s and a non-negative integer m we put

$$L_s^2(R^n)=\{u \mid (1+|x|^2)^{s/2}u(x)\in L^2(R^n)\}\,,$$

$$H_s^m(R^n)=\{u \mid (1+|x|^2)^{s/2}D^\alpha u(x)\in L^2(R^n),\ |\alpha|\leqq m\}\,,$$

with the norm $\|u\|_{L_s^2(R^n)}=\|(1+|x|^2)^{s/2}u\|_{L^2(R^n)}$ and $\|u\|_{H_s^m(R^n)}=(\sum_{|\alpha|\leqq m}\|D^\alpha u\|^2_{L_s^2(R^n)})^{1/2}$. In particular, $L_0^2(R^n)=L^2(R^n)$ is the usual L^2-space and $H_0^m(R^n)=H^m(R^n)$ is the Sobolev space of order m. The Fourier transform will be denoted by $\mathcal{F}$:

$$(\mathcal{F}u)(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{R^n}u(x)e^{-i\xi x}dx\,,$$

where $\xi=(\xi_1,\cdots,\xi_n)\in R^n$ and $\xi x=\xi_1x_1+\cdots+\xi_nx_n$. We need to consider the Sobolev spaces of functions of the variable ξ. If necessary, we will manifest the independent variable of a function space under consideration as $L^2(R^n_\xi)$.

As the underlying Hilbert space we take $\mathfrak{H}=L^2(R^n)=L^2(R^n_x)$. Our self-adjoint operators H_1 and H_2 will be defined by quadratic forms associated with the corresponding differential operators. Let us begin with H_1. Let h_1 be the sesqui-linear form on $H^m(R^n)\times H^m(R^n)$ defined as

$$h_1[u,v]=\sum_{|\alpha|,|\beta|\leqq m}(a^{(1)}_{\alpha\beta}D^\beta u,D^\alpha v)_{L^2(R^n)}\,,\qquad u,v\in H^m(R^n)\,,\tag{1.1}$$

with constants $a^{(1)}_{\alpha\beta}$ satisfying the following assumption.

ASSUMPTION 1.1. $a^{(1)}_{\alpha\beta}=\overline{a^{(1)}_{\beta\alpha}}$ and there exists $c_1>0$ such that

$$\sum_{|\alpha|=|\beta|=m}a^{(1)}_{\alpha\beta}\,\xi^{\alpha+\beta}\geqq c_1|\xi|^{2m}\,,\qquad \xi\in R^n\,.$$

It is well-known that h_1 is a Hermitian symmetric closed form with domain $\mathfrak{D}(h_1)=H^m(R^n)$ which is bounded below. Let H_1 be the self-adjoint operator associated with h_1 in the sense of Friedrichs. Namely, H_1 is the unique self-adjoint operator such that

$$h_1[u,v]=(H_1u,v)\,,\qquad u\in\mathfrak{D}(H_1)\,,\qquad v\in\mathfrak{D}(h_1)\,.$$

Let P_1 be the polynomial given by

$$P_1(\xi)=\sum_{|\alpha|,|\beta|\leqq m}a^{(1)}_{\alpha\beta}\,\xi^{\alpha+\beta}\,,\qquad \xi=(\xi_1,\cdots,\xi_n)\,.$$

(Here and in what follows, $\alpha+\beta=(\alpha_1+\beta_1,\cdots,\alpha_n+\beta_n)$ and $\xi^\gamma=\xi_1^{\gamma_1}\cdots\xi_n^{\gamma_n}$ for $\gamma=(\gamma_1,\cdots,\gamma_n)$.) It is again well-known that $\mathfrak{D}(H_1)=H^{2m}(R^n)$ and

$$H_1u=P_1(D)u\,,\qquad u\in H^{2m}(R^n)\,.$$

Furthermore, the spectrum $\sigma(H_1)$ of H_1 is equal to $[\lambda_{\min},\infty)$, where $\lambda_{\min}=\inf_{\xi\in R^n}P_1(\xi)$.

For defining H_2 let us consider the sesqui-linear form

$$h_2[u, v] = h_1[u, v] + \sum_{|\alpha|,|\beta|\leqq m} (a_{\alpha\beta}(x)D^\beta u, D^\alpha v)_{L^2(R^n)}, \tag{1.2}$$

$$u, v \in \mathfrak{D}(h_2) = H^m(R^n).$$

On the coefficients $a_{\alpha\beta}(x)$ we first impose the following condition.

ASSUMPTION 1.2. i) $a_{\alpha\beta}(x)$ are bounded and measurable functions on R^n and all $a_{\alpha\beta}$ with $|\alpha|=|\beta|=m$ are uniformly continuous on R^n; ii) $a_{\alpha\beta}(x) = \overline{a_{\beta\alpha}(x)}$; iii) (uniform strong ellipticity) there exists $c_2 > 0$ such that

$$\sum_{|\alpha|=|\beta|=m} \{a^{(1)}_{\alpha\beta} + a_{\alpha\beta}(x)\}\xi^{\alpha+\beta} \geqq c_2|\xi|^{2m}, \qquad x \in R^n, \quad \xi \in R^n.$$

By virtue of the Gårding inequality it follows from Assumption 1.2 that h_2 is a closed Hermitian symmetric form which is bounded below. We now define H_2 to be the self-adjoint operator associated with h_2 in the sense of Friedrichs. Thus, H_2 is the self-adjoint realization of the formal differential operator $\sum_{|\alpha|,|\beta|\leqq m} D^\alpha(a^{(1)}_{\alpha\beta} + a_{\alpha\beta}(x))D^\beta u$.

The spectral measure associated with H_j, $j=1, 2$, will be denoted by E_j: $H_j = \int_{-\infty}^{\infty} \lambda E_j(d\lambda)$.

The following assumption concerning the rate of decay at infinity of the coefficients $a_{\alpha\beta}(x)$ of the perturbing operator will be the basis of the entire discussion.

ASSUMPTION 1.3. There exist $\delta > 1$ and $c_3 > 0$ such that

$$|a_{\alpha\beta}(x)| \leqq \frac{c_3}{(1+|x|)^\delta} \tag{1.3}$$

for all α, β with $|\alpha|$, $|\beta| \leqq m$ and all $x \in R^n$.

Following Agmon [I.1] we use the following notion.

DEFINITION 1.4. $\lambda \in R^1$ is said to be a critical value of the polynomial P_1 if there exists $\xi \in R^n$ such that $P_1(\xi) = \lambda$ and grad $P_1(\xi) = 0$.

By Sard's theorem the set e_1 of all critical values of P_1 is a closed set of measure zero. More strongly, it can be proved[2)] that e_1 is a finite set. As is easily seen (see § 2.2), H_1 is absolutely continuous in $R^1 - e_1$. Since $H_1 = P(D)$ does not possess eigenvalues, the finiteness of e_1 implies that H_1 itself is absolutely continuous.

We can now formulate our main theorems. The first theorem is concerned with the nature of the spectrum of H_2. As is mentioned in the introduction, this theorem is the same as a theorem announced by Agmon except that we are concerned with forms rather than differential operators.

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2) The writer is indebted to the lecture of Professor Agmon quoted on p. 76 of (I) for learning this fact.

THEOREM 1.5. *Let e_1 be the set of all critical values of P_1 and let $\lambda_{\min} = \inf_{\xi \in R^n} P_1(\xi)$ $(\lambda_{\min} \in e_1)$.* i) *The spectrum of H_2 in $(-\infty, \lambda_{\min})$ consists (if not empty) of a finite or infinite set $\{\mu_n\}$ of eigenvalues of finite multiplicity. The set $\{\mu_n\}$ is bounded from below and has no points of accumulation except possibly for $\lambda_{\min}$.* ii) *The set $\{\lambda_n\}$ of all eigenvalues of H_2 in $(\lambda_{\min}, \infty)-e_1$ has no points of accumulation in $(\lambda_{\min}, \infty)-e_1$. Each λ_n is of finite multiplicity.* iii) *H_2 restricted to $E_2((\lambda_{\min}, \infty)-(e_1 \cup \{\lambda_n\}))\mathfrak{H}$ is absolutely continuous.*

The next theorem concerning the scattering theory for the pair $\{H_1, H_2\}$ gives more information about the structure of the absolutely continuous part.

THEOREM 1.6. *H_2 restricted to $E_2((\lambda_{\min}, \infty)-(e_1 \cup \{\lambda_n\}))\mathfrak{H}$ is unitarily equivalent to H_1. This unitary equivalence is given via the wave operators*

$$W_\pm = W_\pm(H_2, H_1) = \operatorname*{s-lim}_{t \to \pm\infty} e^{itH_2} e^{-itH_1},$$

the limit on the right side being shown to exist. The principle of invariance of wave operators (cf. Theorem I.3.13) *holds as well. $W_\pm$ can be constructed in a time-independent way as in Theorems* I.3.11 *and* I.3.12.

The principle of limiting absorption which is contained implicitly in the proof may be stated as follows.

THEOREM 1.7. *Let $u \in L^2_{\delta/2}(R^n)$, where δ is the constant appearing in Assumption* 1.3, *and let v_ζ, $\operatorname{Im}\zeta \neq 0$, be the solution in $L^2(R^n)$ of the equation $(H_2-\zeta)v_\zeta = u$, or equivalently, the solution of*

$$\sum_{|\alpha|,|\beta| \leq m} (\{a^{(1)}_{\alpha\beta} + a_{\alpha\beta}(x)\} D^\beta v_\zeta, D^\alpha w) - \zeta(v_\zeta, w) = (u, w), \qquad w \in H^m(R^n).$$

Then, the correspondence $u \to v_\zeta$ determines a bounded operator $\tilde{R}_2(\zeta)$ from $L^2_{\delta/2}(R^n) \to H^m_{-\delta/2}(R^n)$. The $B(L^2_{\delta/2}, H^m_{-\delta/2})$-valued function $\tilde{R}_2$ on $\Pi^\pm = \{\zeta \mid \operatorname{Im}\zeta \gtrless 0\}$ can be extended to a locally Hölder continuous function $\tilde{R}^\pm_2 : \Pi^\pm \cup \{R^1-(e_1 \cup \{\lambda_n\} \cup \{\mu_n\})\} \to B(L^2_{\delta/2}, H^m_{-\delta/2})$. The exponent of Hölder continuity can be taken as $\min((\delta-1)/2, 1)$ *if $\delta \neq 3$ and as any number θ, $0<\theta<1$, if $\delta = 3$. In particular, for each $\lambda \in (\lambda_{\min}, \infty)-(\{\lambda_n\} \cup e_1)$ the limit $v_{\lambda \pm i0} = \lim_{\varepsilon \downarrow 0} v_{\lambda \pm i\varepsilon}$ exists in the norm of $H^m_{-\delta/2}(R^n)$. $v_{\lambda \pm i0}$ satisfies*

$$\sum_{|\alpha|,|\beta| \leq m} (\{a^{(1)}_{\alpha\beta} + a_{\alpha\beta}(x)\} D^\beta v_{\lambda \pm i0}, D^\alpha w) - \lambda(v_{\lambda \pm i0}, w) = (u, w), \tag{1.4}$$
$$w \in H^m_{\delta/2}(R^n),$$

where $(u, v) = \int u\bar{v}\,dx$, not necessarily being the L^2-inner product.

When the coefficients of H_2 are sufficiently smooth, H_2 will be a genuine differential operator defined on $H^{2m}(R^n)$. In that case $H^m_{-\delta/2}$ in the principle of limiting absorption can be replaced by $H^{2m}_{-\delta/2}$. Namely, we have the following theorem.

THEOREM 1.8. *Let H_1 and H_2 be differential operators*

$$H_1 u = \sum_{|\alpha| \leqq 2m} a_\alpha^{(1)} D^\alpha u, \qquad u \in H^{2m}(R^n),$$

$$H_2 u = \sum_{|\alpha| \leqq 2m} (a_\alpha^{(1)} + a_\alpha(x)) D^\alpha u, \qquad u \in H^{2m}(R^n).$$

We assume as before that $P_1(\xi) = \sum_{|\alpha| \leqq 2m} a_\alpha^{(1)} \xi^\alpha$ is a real elliptic polynomial so that H_1 with $\mathfrak{D}(H_1) = H^{2m}(R^n)$ is self-adjoint in $L^2(R^n)$. We assume furthermore that H_2 with $\mathfrak{D}(H_2) = H^{2m}(R^n)$ is also self-adjoint in $L^2(R^n)$[3]. Suppose that $|a_\alpha(x)| \leqq c(1+|x|)^{-\delta}$, $\delta > 1$, $|\alpha| \leqq 2m$. Then, all the conclusions of Theorems 1.5 *and* 1.6 *hold and the conclusion of Theorem* 1.7 *holds in a stronger form that $H^m_{-\delta/2}$ is replaced by $H^{2m}_{-\delta/2}$. In particular, for any compact subset K of $\boldsymbol{C}^1 - (e_1 \cup \{\lambda_n\} \cup \{\mu_n\})$ there exists a constant $c = c_K$ such that*

$$\sum_{|\alpha| \leqq 2m} \int_{R^n} (1+|x|^2)^{-\delta/2} |D^\alpha u(x)|^2 dx \tag{1.5}$$

$$\leqq c \int_{R^n} (1+|x|^2)^{\delta/2} |(H_2 - \zeta) u(x)|^2 dx$$

for any $\zeta \in K$ and $u \in H^{2m}_{\delta/2}(R^n)$. ((1.5) is exactly the estimate given by Agmon.)

§ 2. Proof of the theorems.

2.1. Definition of $\mathfrak{K}$, A, etc. We first prove Theorems 1.5, 1.6, and 1.7 in §§ 2.1–2.4. In order to apply the results obtained in (I) we will define $\mathfrak{K}$, A, B, and C suitably and verify the following sets of assumptions (A.1)—(A.4) for an arbitrary interval $I = (a, b) \subset (\lambda_{\min}, \infty)$ such that $I \cap e_1 = \emptyset$:

(A.1) Assumption I.2.1, the existence of θ, $0 \leqq \theta \leqq 1/2$, satisfying (I.2.13) and (I.2.14), and formula (I.2.15);

(A.2) Assumptions I.3.2—I.3.5;

(A.3) Assumption I.5.4 and condition (1) in Theorem I.6.1;

(A.4) Assumptions I.5.12, I.5.13, I.5.15, I.5.19, and I.5.20.

Suppose that all these assumptions have been verified. Then, as can be checked easily, conclusions ii) and iii) of Theorem 1.5 and the conclusions in Theorem 1.6 follow from Theorems I.3.12, I.3.13, and I.5.21. Assertion i) of Theorem 1.5 is well-known and may easily be deduced from Assumption I.3.4. Theorem 1.7 will be proved at the end of § 2.4.

We now proceed to the verification of the assumptions and begin with defining $\mathfrak{K}$ etc. The Hilbert space $\mathfrak{K}$ is defined to be the direct sum of copies

3) We do not need any smoothness assumption of a_α other than that for ensuring the self-adjointness of H_2 on $H^{2m}(R^n)$.

of $L^2(R^n)$ indexed by the multi-indices α and β, $0 \leqq |\alpha|, |\beta| \leqq m$: $\mathfrak{K} = \sum_{\alpha,\beta} \oplus L^2(R_n)$. A generic element of $\mathfrak{K}$ is denoted as $u = \{u_{\alpha\beta}\}$. By definition $\mathfrak{D}(A) = \mathfrak{D}(B) = L^2(R^n) \cap H^m_{-\delta/2}(R^n)$ and

$$Au = \{(1+|x|^2)^{-\delta/4} D^\alpha u\}_{\alpha\beta}, \qquad u \in \mathfrak{D}(A),$$

$$Bu = \{(1+|x|^2)^{-\delta/4} D^\beta u\}_{\alpha\beta}, \qquad u \in \mathfrak{D}(B).$$

More precisely, writing $Au = v = \{v_{\alpha\beta}\}$, for example, we define $v_{\alpha\beta}$ as $v_{\alpha\beta} = (1+|x|^2)^{-\delta/4} D^\alpha u$, $v_{\alpha\beta}$ being independent of β. Evidently, A and B are one-to-one. Finally, putting

$$c_{\alpha\beta}(x) = (1+|x|^2)^{\delta/2} a_{\alpha\beta}(x) \in L^\infty(R^n),$$

we define $C \in B(\mathfrak{K})$ as

$$Cu = \{c_{\alpha\beta} u_{\alpha\beta}\}, \qquad u = \{u_{\alpha\beta}\} \in \mathfrak{K}.$$

We record the following proposition whose proof is straightforward.

PROPOSITION 2.1.

$$\mathfrak{D}(A^*) \supset \{u = \{u_{\alpha\beta}\} \in \mathfrak{K} \mid (1+|x|^2)^{-\delta/4} u_{\alpha\beta} \in H^{|\alpha|}(R^n)\}$$

and for u belonging to the right side we have

$$A^* u = \sum_{\alpha,\beta} D^\alpha((1+|x|^2)^{-\delta/4} u_{\alpha\beta}).$$

Let us first examine those assumptions which are easily verified. First of all, it is clear that A and B are closed. Hence, Assumption I.2.1 is fulfilled. The definition of h_1 and h_2 given in § 1 implies that (I.2.13) holds with $\theta = 1/2$, $\mathfrak{D}_{1/2}$ coinciding with $H^m(R^n)$. Then, (I.2.14) is obvious and (1.2) is precisely (I.2.15). Thus, all the assumptions in group (A.1) have been verified.

In group (A.2) the last two assumptions are easy to verify. Indeed, since $\mathfrak{R}(A^*)$ is dense, Assumption I.3.5 is trivially fulfilled. To verify Assumption I.3.4 we note that the range of $R_1(\zeta)$ is $H^{2m}(R^n)$, while B is a differential operator of order m whose coefficients all tend to 0 as $|x| \to \infty$. Hence, by Rellich's theorem $BR_1(\zeta)$ is compact. (Similarly, $AR_1(\zeta)$ is compact.)

Group (A.3) is easy to handle. Assumption I.5.4 follows from the fact that $H^m(R^n)$ (or more strongly $C_0^\infty(R^n)$) is a core of A. Since $\theta = 1/2$, the validity of condition (1) of Theorem 6.1 is seen as in the proof of Proposition I.5.8.

The remaining assumptions will be verified successively in the following subsections. Hereafter, we fix $I = (a, b) \subset (\lambda_{\min}, \infty)$ such that $I \cap e_1 = \emptyset$.

2.2. Unperturbed spectral representation. Let

$$\Sigma_\lambda = \{\xi \mid P_1(\xi) = \lambda\}, \qquad \lambda \in (a, b),$$

$$\Omega=\{\xi\,|\,P_1(\xi)\in(a,b)\}=\bigcup_{a<\lambda<b}\Sigma_\lambda\,.$$

We fix $c\in I$ and write $\Sigma=\Sigma_c$. Σ is a compact real analytic surface, to which all Σ_λ, $\lambda\in I$, are diffeomorphic. Let $d\sigma$ be the surface element of Σ and let $L^2(\Sigma)$ be the space of functions on Σ square integrable with respect to $d\sigma$.

By the Fourier transform the space $E_1(I)\mathfrak{H}$ is mapped onto $L^2(\Omega)$. The latter is in turn put into one-to-one correspondence with $L^2(I;L^2(\Sigma))\cong L^2(I\times\Sigma)$ in a natural way. More precisely, this fact may be stated as follows. We denote a generic point of $I\times\Sigma$ by $(\lambda,\omega)\in I\times\Sigma$.

PROPOSITION 2.2. *There exist a C^∞-diffeomorphism ϕ from $I\times\Sigma$ onto Ω and a positive C^∞-function p on $I\times\Sigma$ satisfying the following properties*: i) *for each $\lambda\in I$, ϕ maps $\{\lambda\}\times\Sigma$ onto Σ_λ*; ii) *the mapping which sends $f\in L^2(\Omega)$ to*

$$(\Gamma f)(\lambda,\omega)=p(\lambda,\omega)f(\phi(\lambda,\omega)),\qquad(\lambda,\omega)\in I\times\Sigma\,,\tag{2.1}$$

determines a unitary operator Γ from $L^2(\Omega)$ on $L^2(I\times\Sigma)\cong L^2(I;L^2(\Sigma))$.

For the sake of completeness we shall sketch a proof of Proposition 2.2 in Appendix.

Let us now put $\mathfrak{C}=L^2(\Sigma)$ and define F as

$$F=\Gamma\mathcal{F}|_{E_1(I)\mathfrak{H}}\in B(E_1(I)\mathfrak{H},\ L^2(I;L^2(\Sigma)))\,,\tag{2.2}$$

where $\mathcal{F}|_{E_1(I)\mathfrak{H}}$ is the restriction of the Fourier transform to $E_1(I)\mathfrak{H}$. Then, it is evident by Proposition 2.2 that $\mathfrak{C}$ and F thus defined satisfy Assumption I.3.2.

2.3. Trace operators. It is convenient to verify Assumption I.3.3 simultaneously with Assumption I.5.12. Let $A_1\in B(\mathfrak{K})$ be the operator of multiplication by $(1+|x|^2)^{-\delta/4}$ in $\mathfrak{K}=\sum_{\alpha,\beta}\oplus L^2(R^n)$:

$$A_1u=\{(1+|x|^2)^{-\delta/4}u_{\alpha\beta}(x)\}\,,\qquad u=\{u_{\alpha\beta}\}\in\mathfrak{K}\,.$$

We take this operator as A_1 in Assumption I.5.12. Then it is clear that

$$\mathfrak{K}_\gamma=\sum_{\alpha,\beta}\oplus L^2_{\gamma\delta/2}(R^n)\,,$$

$$A_1^{\gamma',\gamma}u=\{(1+|x|^2)^{(\gamma-\gamma')\delta/4}u_{\alpha\beta}\}\,,\qquad u=\{u_{\alpha\beta}\}\in\mathfrak{K}_\gamma\,.$$

Let D be the operator from $\mathfrak{H}$ to $\mathfrak{K}_{-1}$ defined by

$$\begin{cases}\mathfrak{D}(D)=L^2(R^n)\cap H^m_{-\delta/2}(R^n)\,,\\ Du=\{D^\alpha u\}_{\alpha\beta}\in\sum_{\alpha,\beta}\oplus L^2_{-\delta/2}(R^n)=\mathfrak{K}_{-1}\,,\qquad u\in\mathfrak{D}(D)\,.\end{cases}\tag{2.3}$$

It is evident that Assumption I.5.12 is fulfilled with A_1 and D defined as above. If $\gamma>\delta^{-1}$ (in particular if $\gamma\geqq 1$), then there is the "trace operator" from $H^{\gamma\delta/2}(R_\xi^n)$ to $L^2(\Sigma_\lambda)$ and, roughly speaking, this operator depends Hölder continuously on λ. A precise formulation of this fact may be given as follows.

PROPOSITION 2.3. *Let $\gamma>\delta^{-1}$. Then, there exists $\Gamma(\lambda)\in B(H^{\gamma\delta/2}(R_\xi^n), L^2(\Sigma))$, $\lambda\in I$, such that:* i) *$\Gamma(\lambda)$ depends locally Hölder continuously on λ in operator norm, the exponent being* $\min((\gamma\delta-1)/2, 1)$ *if* $(\gamma\delta-1)/2\neq 1$ *and any number less than* 1 *if* $(\gamma\delta-1)/2=1$; ii) *if $f\in H^{\gamma\delta/2}(R_\xi^n)$, then*

$$\Gamma(\lambda)f=(\Gamma f|_{\Omega})(\lambda) \qquad a.e.\ in\ I, \tag{2.4}$$

where $f|_{\Omega}$ is the restriction of f to Ω. If we define

$$F(\lambda)=\Gamma(\lambda)\mathcal{F}\in B(L^2_{\gamma\delta/2}(R_x^n), L^2(\Sigma)), \qquad \lambda\in I, \tag{2.5}$$

then $F(\lambda)$ has the same kind of local Hölder continuity as $\Gamma(\lambda)$ and for any $u\in L^2_{\gamma\delta/2}(R_x^n)$ we have

$$F(\lambda)u=(FE_1(I)u)(\lambda) \qquad a.e.\ in\ I. \tag{2.6}$$

Furthermore, for any $p>2(n-1)/(\gamma\delta-1)$ one has

$$\begin{cases} \Gamma(\lambda)\in C_p(H^{\gamma\delta/2}(R_\xi^n), L^2(\Sigma)), \\ F(\lambda)\in C_p(L^2_{\gamma\delta/2}(R_x^n), L^2(\Sigma)), \end{cases} \tag{2.7}$$

where C_p is the von Neumann-Schatten class (cf. § 6.2 *of* (I)). ((2.7) *is needed only in* § 3 *where the scattering matrix is discussed.*)

PROOF. The statements involving $F(\lambda)$ follow from those for $\Gamma(\lambda)$ immediately. Let $\gamma(\lambda)$ be the trace operator from $H^{\gamma\delta/2}(R_\xi^n)$ to $L^2(\Sigma_\lambda)$: $\gamma(\lambda)f=f|_{\Sigma_\lambda}$. For any $f\in H^{\gamma\delta/2}(R_\xi^n)$ define $\Gamma(\lambda)f$ as

$$\begin{aligned}(\Gamma(\lambda)f)(\omega)&=p(\lambda,\omega)(\gamma(\lambda)f)(\phi(\lambda,w))\\ &=p(\lambda,\omega)f(\phi(\lambda,\omega)), \qquad \omega\in\Sigma,\end{aligned} \tag{2.8}$$

where p and ϕ are as given in Proposition 2.2. (Note that $\phi(\lambda,w)\in\Sigma_\lambda$ by i) of Proposition 2.2.) Then, the theorem of trace tells us that $\Gamma(\lambda)$ belongs to $B(H^{\gamma\delta/2}(R_\xi^n), L^2(\Sigma))$ and has the local Hölder continuity as prescribed in the proposition. (For the precise reasoning we cover Ω by an atlas and map each chart of the atlas into R^n so that the part of Σ_λ in that chart becomes flat.) Relation (2.4) is obvious by Proposition 2.2 and the second equality of (2.8).

In order to prove (2.7) we note that the trace operator $\Gamma(\lambda)$ actually maps $H^{\gamma\delta/2}(R_\xi^n)$ into $H^{(\gamma\delta-1-\eta)/2}(\Sigma)$ boundedly for any $\eta>0$. (For the definition of $H^s(\Sigma)$ over a surface Σ and the fact mentioned above, see, e.g., Lions and Magenes [**2**; Chapt. 1].) However, it is essentially known and can be proved easily (cf. [I.13]) that the imbedding operator: $H^s(\Sigma)\to L^2(\Sigma)$, $s>0$, belongs to $C_q(H^s(\Sigma), L^2(\Sigma))$ for any $q>(n-1)/s$. Hence, (2.7) follows. q. e. d.

Let $\mathcal{S}=\sum_{\alpha,\beta}\oplus\mathcal{S}(R_x^n)\subset\mathfrak{R}$, where $\mathcal{S}(R^n)$ is the Schwartz space of rapidly decreasing functions. ($\mathcal{S}$ will play the role of $\mathfrak{D}$ and $\mathfrak{D}'$ in Assumption I.3.3.) Let $u=\{u_{\alpha\beta}\}\in\mathcal{S}$ and write $A_1u=v=\{v_{\alpha\beta}\}$. By Proposition 2.1 we have $u\in\mathfrak{D}(A^*)$ and $A^*u=\sum_{\alpha,\beta}D^\alpha v_{\alpha\beta}\in L_s^2(R^n)$ for any $s>0$. Hence, by (2.6) and (2.5) we get

$$(FE_1(I)A^*u)(\lambda)=F(\lambda)A^*u=\Gamma(\lambda)\mathcal{F}A^*u\,. \tag{2.9}$$

Since $(\mathcal{F}A^*u)(\xi)=\sum_{\alpha,\beta}\xi^\alpha\mathcal{F}v_{\alpha\beta}$, we obtain by (2.4) and (2.1) that

$$\begin{aligned}(\Gamma(\lambda)\mathcal{F}A^*u)(\omega)&=p(\lambda,\omega)\sum_{\alpha,\beta}\phi(\lambda,\omega)^\alpha(\mathcal{F}v_{\alpha\beta})(\phi(\lambda,\omega))\\&=\sum_{\alpha,\beta}\phi(\lambda,\omega)^\alpha(\Gamma(\lambda)\mathcal{F}v_{\alpha\beta})(\omega)\\&=\sum_{\alpha,\beta}\phi(\lambda,\omega)^\alpha(F(\lambda)v_{\alpha\beta})(\omega)\,.\end{aligned} \tag{2.10}$$

Hence, if we define $\Phi(\lambda)\in B(\mathfrak{R}_1,\mathfrak{C})$ by

$$(\Phi(\lambda)v)(\omega)=\sum_{\alpha,\beta}\phi(\lambda,\omega)^\alpha(F(\lambda)v_{\alpha\beta})(\omega)\,,\qquad\omega\in\Sigma\,, \tag{2.11}$$

it follows from (2.9) and (2.10) that

$$(FE_1(I)A^*u)(\lambda)=\Phi(\lambda)v=\Phi(\lambda)A_1^{1,0}u\,,\qquad u\in\mathcal{S}\,.$$

Since $\phi(\lambda,\omega)$ is a C^∞-function, $\Phi(\lambda)$ has the same kind of local Hölder continuity as $F(\lambda)$. Thus, if we define

$$T(\lambda\,;A)=\Phi(\lambda)A_1^{1,0}\,,\qquad\lambda\in I\,, \tag{2.12}$$

then the part of Assumption I.3.3 which is concerned with $T(\lambda\,;A)$ are satisfied with $\mathfrak{D}=\mathcal{S}$. Since B has essentially the same form as A, the other part of Assumption I.3.3 is also satisfied with $T(\lambda\,;B)$ given as

$$T(\lambda\,;B)=\Phi_B(\lambda)A_1^{1,0}\,,\qquad\lambda\in I\,,$$

$$\Phi_B(\lambda)v=\sum_{\alpha,\beta}\phi(\lambda,\cdot)^\beta\Gamma(\lambda)\mathcal{F}v_{\alpha\beta}\,.$$

We will next verify Assumption I.5.15. Let $w\in\mathfrak{R}_\gamma$, $\gamma\geqq 0$. By virtue of Proposition 2.3 $\Gamma(\lambda)$ can be regarded as an operator in $B(H^{(1+\gamma)\delta/2},\mathfrak{C})$, which is locally Hölder continuous in I with exponent $\min(\{(\gamma+1)\delta-1\}/2,1)$ with the usual modification in the case $(\gamma+1)\delta-1=2$. Noting $\{(\gamma+1)\delta-1\}/2\geqq\gamma/2+(\delta-1)/2>\gamma/2+(\delta-1)/4$, we put $\rho_0=(\delta-1)/4$. Then, $\Gamma(\lambda)$ (resp. $\Phi_B(\lambda)$) is Hölder continuous as a $B(H^{(1+\gamma)\delta/2},\mathfrak{C})$-valued (resp. $B(L^2_{(1+\gamma)\delta/2},\mathfrak{C})$-valued) function with exponent θ given in Assumption I.5.15. Since $w\in\mathfrak{R}_\gamma=L^2_{\gamma\delta/2}$

implies $C^*w\in L^2_{\gamma\delta/2}$, the first part of Assumption I.1.15 follows. Using (I.5.13) we obtain (I.5.18) as follows: $\|\{T(\lambda;B)-T(\lambda';B)\}C^*w\|_{\mathfrak{C}}\leqq c|\lambda-\lambda'|^{\theta}\|A_1^{1,0}C^*w\|_{\mathfrak{K}_{1+\gamma}}$ $=c|\lambda-\lambda'|^{\theta}\|C^*w\|_{\mathfrak{K}_\gamma}\leqq c|\lambda-\lambda'|^{\theta}\|w\|_{\mathfrak{K}_\gamma}$.

2.4. Verification of other assumptions. i) Assumption I.5.13. By (2.3) D maps $H^m(R^n)$ boundedly into $\mathfrak{K}_0$. Because of the ellipticity of H_1, however, the graph norm of $|H_1|^{1-\theta}=|H_1|^{1/2}$ is equivalent to the norm of $H^m(R^n)$. Hence (I.5.15) follows.

ii) Assumption I.5.19. That A is one-to-one is obvious. It is easy to see that the space $\mathfrak{Y}$ defined in § I.5.2 is exactly $H^m_{-\delta/2}(R^n)$ and the operator $\boldsymbol{A}$ has the form

$$\boldsymbol{A}u=\{(1+|x|^2)^{-\delta/4}D^\alpha u\}_{\alpha\beta}\,,\qquad u\in H^m_{-\delta/2}(R^n)\,.$$

Therefore, if $\boldsymbol{A}u\in\mathfrak{K}_1$ (i. e. $(1+|x|^2)^{-\delta/4}D^\alpha u\in L^2_{\delta/2}(R^n)$), then $D^\alpha u\in L^2(R^n)$ and hence $u\in H^m(R^n)=\mathfrak{D}_{1/2}$. Furthermore, $\|u\|_{L^2}\leqq\|u\|_{H^m}=c\|\boldsymbol{A}u\|_{\mathfrak{K}_1}$, so that (I.5.27) holds.

iii) Assumption I.5.20. By the definition of $\mathfrak{S}$ (Definition I.5.3) and the fact that $\mathfrak{Y}=H^m_{-\delta/2}(R^n)$ it is evident that $H^m_{\delta/2}(R^n)\subset\mathfrak{S}$. Therefore, Assumption I.5.20 is obviously satisfied.

We have thus verified all the assumptions in (A.1)-(A.4). As is mentioned earlier, this proves Theorems 1.5 and 1.6.

PROOF OF THEOREM 1.7. We use the notation in Theorem I.6.1 and take $\mathfrak{K}'=\{u=\{u_{\alpha\beta}\}\mid u_{\alpha\beta}=0$ unless $\alpha=\beta=0\}$. Then, evidently $\mathfrak{X}=\mathfrak{X}_0=L^2_{\delta/2}(R^n)$. Since $\mathfrak{Y}=H^m_{-\delta/2}$, Theorem 1.7 follows from Theorem I.6.1.

2.5. PROOF OF THEOREM 1.8. In this case we use an unsymmetric factorization $H_2=H_1+B^*C^*A$ by shifting all the differentiation to A. More precisely, we define $\mathfrak{K}$ to be the direct sum of copies of $L^2(R^n)$ indexed by multi-indices α, $0\leqq|\alpha|\leqq 2m$: $\mathfrak{K}=\sum_{0\leqq|\alpha|\leqq 2m}\oplus L^2(R^n)$. Writing $a_\alpha(x)=(1+|x|^2)^{-\delta/2}c_\alpha(x)$, $c_\alpha\in L^\infty(R^n)$, we define A, B, and C as follows: $\mathfrak{D}(A)=L^2(R^n)\cap H^{2m}_{-\delta/2}(R^n)$, $B\in B(\mathfrak{H},\mathfrak{K})$, $\mathfrak{H}=L^2(R^n)$,

$$Au=\{(1+|x|^2)^{-\delta/4}D^\alpha u\}_\alpha\,,\qquad u\in\mathfrak{D}(A)\,,$$

$$Bu=\{(1+|x|^2)^{-\delta/4}u\}_\alpha\,,\qquad u\in\mathfrak{H}\,,$$

$$Cu=\overline{\{c_{\alpha\beta}u_{\alpha\beta}\}}\,.$$

Then, it is clear that $H_2=H_1+B^*C^*A$ and (I.2.15) holds with $\theta=0$ (cf. Example I.2.10). Sets of assumptions (A.1)-(A.4) except for (1) in Theorem I.6.1 can be verified in essentially the same way as before. Thus, the conclusions of Theorems 1.5 and 1.6 hold. In order to prove the principle of limiting absorption in the desired form, we put $\mathfrak{K}'=\{u=\{u_\alpha\}\mid u_\alpha=0$ unless $\alpha=0\}$.

Then, $\mathfrak{X}=L^2_{\delta/2}$. On the other hand we have $\mathfrak{Y}=H^{2m}_{-\delta/2}$. Hence, $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy condition (2) of Theorem I.6.1. Thus, Theorem I.6.1 is applicable and yields the desired result. q. e. d.

§ 3. Scattering matrix.

Let $\lambda_1=\lambda_{\min}<\lambda_2<\cdots<\lambda_r$ be the enumeration of all critical values of P_1 and let $\lambda_{r+1}=\infty$. Let $I_k=(\lambda_k, \lambda_{k+1})$, $k=1, \cdots, r$. Then, obviously $\mathfrak{H}=\sum_{k=1}^{r}\oplus E_1(I_k)\mathfrak{H}$, $\mathfrak{H}=L^2(R^n)$. Let

$$F_k=\Gamma_k\mathcal{F}\,|_{E_1(I_k)\mathfrak{H}}\in B(E_1(I_k)\mathfrak{H},\ L^2(I_k\,;\,L^2(\Sigma^{(k)})))$$

be constructed as in (2.2), where $\Sigma^{(k)}=\Sigma_{\nu_k}$ for some $\nu_k\in I_k$. Then, $F=\sum_{k=1}^{r}\oplus F_k\in B(\mathfrak{H}, \sum\oplus L^2(I_k\,;\,L^2(\Sigma^{(k)})))$ gives a spectral representation of H_1. We may and shall identify $\sum\oplus L^2(I_k\,;\,L^2(\Sigma^{(k)}))$ with the direct integral space

$$\int_{\lambda_{\min}}^{\infty}\oplus\mathfrak{H}_\lambda d\lambda\,,\qquad \mathfrak{H}_\lambda=L^2(\Sigma^{(k)})\qquad \text{if } \lambda\in I_k\,.$$

Then, by means of the spectral representation F the scattering operator $S=W_+^*W_-$ is converted to a decomposable operator. Namely, $\tilde{S}=FSF^{-1}$ is written as

$$(\tilde{S}\phi)(\lambda)=\mathcal{S}(\lambda)\phi(\lambda)\,,\qquad \phi\in\int_{\lambda_{\min}}^{\infty}\oplus\mathfrak{H}_\lambda d\lambda$$

where $\mathcal{S}(\lambda)$ is a unitary operator in $\mathfrak{H}_\lambda$.

THEOREM 3.1. *Let C_r be the von Neumann-Schatten class introduced in* I.6.2. *Then,*

$$\mathcal{S}(\lambda)-1\in C_\eta(L^2(\Sigma^{(k)}))\,,\qquad \lambda\in I_k\,,$$

for any $\eta>(n-1)/(\delta-1)$. In particular, $\mathcal{S}(\lambda)-1$ is of Hilbert-Schmidt class if $\delta>(n+1)/2$ and of trace class if $\delta>n$.

PROOF. Theorem follows from Corollary I.6.4, (2.7), (2.11), and (2.12) at once.

Appendix.

PROOF OF PROPOSITION 2.2.

Consider the initial value problem

$$\text{(A.1)}\qquad \frac{d}{d\lambda}f(\lambda\,;\,\omega)=\frac{(\operatorname{grad}P)(f(\lambda\,;\,\omega))}{|(\operatorname{grad}P)(f(\lambda,\omega))|^2}\,,$$

$$\text{(A.2)}\qquad f(c\,;\,\omega)=\omega\in\Sigma=\Sigma_c$$

for the vector function $f(\lambda\,;\,\omega)=(f_1(\lambda\,;\,\omega), \cdots, f_n(\lambda\,;\,\omega))$ of the real variable λ.

Then, $\xi=f(\lambda;\omega)$ determines a curve l_ω in Ω with parameter λ. Since $(d/d\lambda)P(f(\lambda;\omega))=\operatorname{grad}P(f(\lambda;\omega))\cdot(d/d\lambda)f(\lambda;\omega)=1$ and $P(f(c;\omega))=P(\omega)=c$, we have $\lambda=P(f(\lambda;\omega))$. Namely, the value of the parameter λ at a point on the curve coincides with the value of the polynomial P at that point. On the other hand, the denominator of (A.1) is not zero so far as $f(\lambda;\omega)$ stays in Ω. We can therefore conclude that the solution of the initial value problem (A.1), (A.2) exists for $\lambda\in I$.

Let us now define the mapping $\phi: I\times\Sigma\to\Omega$ by

$$\phi(\lambda, \omega)=f(\lambda;\omega), \qquad (\lambda, \omega)\in I\times\Sigma .$$

Then, it is immediately verified that ϕ is a one-to-one C^∞-mapping from $I\times\Sigma$ *onto* Ω having property i) of Proposition 2.2.

In order to see that ϕ^{-1} is also C^∞, we show that the Jacobian of ϕ expressed in a local coordinate does not vanish in $I\times\Sigma$. The Jacobian does not vanish on $\{c\}\times\Sigma$ because ϕ is essentially the identity on $\{c\}\times\Sigma$ by (A.2) and the "normal derivative" $(\partial/\partial\lambda)\phi(\lambda, \omega)=(\partial/\partial\lambda)f(\lambda;\omega)$ does not vanish by (A.1). Next, take an arbitrary $(\lambda_1, \omega_1)\in I\times\Sigma$ and consider a sufficiently small neighbourhood U of (λ_1, ω_1) of the type $(a', b')\times V$, where $a'<\lambda_1<b'$ and V is a neighbourhood of ω_1 in Σ. Let $U'=(a'', b'')\times V$ with $a''=a'+(c-\lambda_1)$, $b''=b'+(c-\lambda_1)$. Then, $\phi(U)$ is obtained by translating $\phi(U')$ along the curves determined by (A.1). More precisely, the mapping $\psi: \phi(U')\to\Omega$ determined by

$$\psi f(\lambda;\omega)=f(\lambda-(c-\lambda_1);\omega), \qquad (\lambda, \omega)\in U',$$

maps $\phi(U')$ onto $\phi(U)$ and is one-to-one. Since ψ is constructed by solving differential equation (A.1), which can be solved either forward or backward, it is clear that ψ is a C^∞-diffeomorphism. Finally, let $\tau: U'\to U$ be defined by $\tau(\lambda, \omega)=(\lambda-(c-\lambda_1), \omega)$. Evidently, $\psi\circ\phi|_{U'}=\phi|_U\circ\tau$. In this formula, all the mappings except for $\phi|_U$ have the non-vanishing Jacobian. Hence, the Jacobian of ϕ does not vanish on U. Thus, we have shown that ϕ is a C^∞-diffeomorphism.

Finally, we construct p satisfying the requirement ii) of Proposition 2.2. First, p is defined locally. Using a local coordinate $(y_1, \cdots, y_{n-1})$ of Σ, the surface element $d\sigma$ is written as $d\sigma=g(y_1, \cdots, y_{n-1})dy_1\cdots dy_{n-1}$, where g is a positive C^∞-function. In terms of the same local coordinate, let ϕ be expressed as $\xi_j=\phi_j(\lambda;y_1, \cdots, y_{n-1})$, $j=1, \cdots, n$, and define p as

$$p(\lambda;y_1, \cdots, y_{n-1})=\left|\frac{\partial(\phi_1, \cdots, \phi_n)}{\partial(\lambda;y_1, \cdots, y_{n-1})}\right|^{1/2} g(y_1, \cdots, y_{n-1})^{-1/2}.$$

Then, it can be checked easily that p is a C^∞-function which does not depend on the choice of local coordinate and that

(A.3) $$d\xi_1 \cdots d\xi_n = p(\lambda, y_1, \cdots, y_{n-1})^2 d\sigma d\lambda.$$

By pasting these locally defined p together, we get a C^∞-function $p: I \times \Sigma \to (0, \infty)$. The unitarity of Γ defined by (2.1) follows from (A.3) at once.

q. e. d.

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