J. Math. Soc. Japan
Vol. 24, No. 3, 1972

On the global existence of real analytic solutions of linear differential equations (I)*

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(Received Dec. 22, 1971)

Contents

§ 0. Introduction.

The purpose of this paper is to present global existence theorems for linear differential equations with constant coefficients which satisfy suitable regularity conditions. (Cf. conditions (1.1), (1.2) and Definition 4.2.) Our proof relies on the existence of good elementary solutions, the meaning of which is clarified in § 1 and § 4. Sato's theory of sheaf C (Sato [2]~[5]) also plays an essential role in the course of the proof.

We remark that the problem of the global existence of real analytic solutions has remained unsolved because the topological structure of the space of real analytic functions on an open set Ω in $\boldsymbol{R}^n$ is a complicated one. (Cf. Ehrenpreis [1], Martineau [1].) In fact there has been no general result even when Ω is convex; the only results hitherto known seem to be Theorems α and β, which we list up below for the reader's convenience. We also note that during the preparation of this paper Professors E. De Giorgi and L. Cattabriga have informed the author that they have obtained the affirmative answer by the method of a priori estimate when $\Omega=\boldsymbol{R}^2$. (Cf. E. De

* Work supported in part by the Sakkokai Foundation.

Giorgi and L. Cattabriga [1] and [2]. See also Theorem 5.7 and the remark following it.)

THEOREM α (Malgrange [1]). *Let a linear differential operator* $P(D)$ *be elliptic, i.e., have a principal symbol* $P_m(\xi)$ *never vanishing for any non-zero real cotangent vector* ξ. *Then for any open set* Ω *in* $\boldsymbol{R}^n$, $P(D)u=f$ *has a real analytic solution* $u(x)$ *for any real analytic function* $f(x)$ *defined on* Ω.

THEOREM β (Ehrenpreis, Malgrange and Komatsu, see Komatsu [1] Theorem 3.1). *Let* K *be a compact convex set in* $\boldsymbol{R}^n$. *Then* $P(D)u=f$ *has a solution* $u(x)$ *in* $\mathcal{A}(K)$ *for any* $f(x)$ *in* $\mathcal{A}(K)$. *Here* $\mathcal{A}(K)$ *denotes the space of real analytic functions on* K, *i.e.,* $\mathcal{A}(K)=\varinjlim_{V\supset K}\mathcal{O}(V)$, *where* V *runs over the fundamental system of open neighbourhoods of* K *in* $\boldsymbol{C}^n$, *and* $\mathcal{O}(V)$ *denotes the space of holomorphic functions defined on* V.

The extension of the results of this paper to the linear differential operators with real analytic coefficients and also to the overdetermined systems of linear differential equations will be given in our forthcoming papers. See also Kawai [7], [8].

Throughout this paper we use the same notations as in Kawai [3]~[8] unless otherwise stated. For example, we denote by $P(x, \xi)$ the symbol of the m-th order linear differential operator $P(x, D_x)$, where ξ_j stands for $\partial/\partial x_j$, and by $P_m(x, \xi)$ its principal symbol, i.e., the homogeneous part of $P(x, \xi)$ of order m with respect to ξ.

The results of this paper have been announced in Kawai [6], [7], [8].

The author expresses his hearty thanks to Professor Sato, Professor Komatsu and Mr. Kashiwara for the stimulating conversations with them.

§1. Cnstruction of good elementary solutions —the case of real principal symbol—

In this section we construct good elementary solutions for linear differential operators $P(D)$ with constant coefficients satisfying conditions (1.1) and (1.2) below. The definition of good elementary solutions for those operators is given in Definition 1.1.

(1.1) $P_m(\xi)$, the principal symbol of $P(D)$, is real for real cotangent vector ξ.

(1.2) The operator $P(D)$ is of simple characteristics, i.e., $\mathrm{grad}_\xi P_m(\xi)$ never vanishes whenever $P_m(\xi)=0$, where ξ is a non-zero real cotangent vector.

The results of this section are essentially given in Kawai [3]. See also Andersson [1], where the analytic singular support of elementary solutions of linear differential equations with constant coefficients is investigated. However, since we have treated general linear differential operators with variable coefficients there, the proof of the global existence of the elementary solutions for linear differential operator with constant coefficients is not explicitly given in that paper. By this reason we supplement our preceding paper here by giving the global estimate concerning the convergence of the asymptotic expansions given in the form (1.6) below. We also remark that conditions (1.1) and (1.2) are relaxed in § 4, but in this section we restrict ourselves to the consideration of the operators satisfying these conditions for the sake of simplicity, partly because the operator satisfying conditions (1.1) and (1.2) is the most natural and classical one, i. e., its treatment directly concerns with the classical notion of bicharacteristics, though the operator treated in § 4 does not.

DEFINITION 1.1. (Good elementary solutions for the linear differential operator satisfying conditions (1.1) and (1.2).) A hyperfunction $E(x)$ satisfying the equation

$$P(D)E(x)=\delta(x) \tag{1.3}$$

is called a good elementary solution of the linear differential operator $P(D)$ with constant coefficients satisfying conditions (1.1) and (1.2) if it satisfies either condition $(1.4)_+$ or $(1.4)_-$.

$(1.4)_+$ $\quad$ $\text{S.S.}\,E(x)\subset\{(x,\xi)\in S^*\boldsymbol{R}^n \mid x=0 \text{ or } x=t\ \mathrm{grad}_\xi P_m(\xi),$ where $t\geqq 0$ and $P_m(\xi)=0\}$.

$(1.4)_-$ $\quad$ $\text{S.S.}\,E(x)\subset\{(x,\xi)\in S^*\boldsymbol{R}^n \mid x=0 \text{ or } x=t\ \mathrm{grad}_\xi P_m(\xi),$ where $t\leqq 0$ and $P_m(\xi)=0\}$.

Here S.S. $E(x)$ denotes the support of $E(x)$ which is regarded as a section of sheaf $\mathcal{C}$, i. e., the support of $\beta(E(x))$, where β denotes the cannonical surjection from the sheaf of germs of hyperfunctions to the sheaf $\pi_*\mathcal{C}$. Here π denotes the cannonical projection from the cotangential sphere bundle to the base space and $\pi_*\mathcal{C}$ denotes the direct image of sheaf $\mathcal{C}$ under the mapping π. In the sequel we call a section of sheaf $\mathcal{C}$ a micro-hyperfunction or a microfunction for short. Concerning the definition and properties of sheaf $\mathcal{C}$ we refer to Sato [2]~[5] and Sato, Kawai and Kashiwara [1]. Here we emphasize the fact that the employment of microfunctions makes clear the role of the bicharacteristic *strips* (cf. Kawai [3], [4], Sato, Kawai and Kashiwara [1]), and even when we are treating linear differential operators with constant coefficients as in this paper, specifying the "cotangential component of the singularity" of hyperfunctions is very important as is shown later.

THEOREM 1.2. *Let a linear differential operator $P(D)$ with constant coeffi-*

cients satisfy conditions (1.1) *and* (1.2). *Then there exist two good elementary solutions* $E_+(x)$ *and* $E_-(x)$ *of* $P(D)$ *such that* $E_+(x)$ *satisfies condition* $(1.4)_+$ *and* $E_-(x)$ *satisfies condition* $(1.4)_-$.

PROOF. We first construct $E_+(x)$. For that purpose we introduce the following functions $\Phi_j(\tau)$ for the sake of simplicity of notations.

$$\Phi_j(\tau)=\begin{cases}(-1)^{j-1}(-j-1)!\,\tau^j & (j<0)\\ \dfrac{1}{j!}\tau^j\log\tau-\dfrac{1}{j!}\Big(1+\cdots+\dfrac{1}{j}\Big)\tau^j & (j\geqq 0).\end{cases}\tag{1.5}$$

Taking into account the following well known formula which expands δ-function into plane waves

$$\delta(x)=\frac{(n-1)!}{(-2\pi\sqrt{-1})^n}\int_{|\xi|=1}\frac{\omega(\xi)}{(\langle x,\xi\rangle+\sqrt{-1}\,0)^n},\tag{1.6}$$

we want to obtain the solution $E_+(x)$ in the form

$$E_+(x)=\frac{1}{(-2\pi\sqrt{-1})^n}\int_{|\xi|=1}\sum_{j\geqq 0}\frac{c_j(\xi)}{(P_m(\xi)-\sqrt{-1}\,0)^j}\Phi_{mj-n}(\langle x,\xi\rangle+\sqrt{-1}\,0)\omega(\xi),\tag{1.7}$$

where n is the space dimension and $\omega(\xi)$ denotes the volume element on the unit sphere, i.e.,

$$\omega(\xi)=\sum_{j=1}^{n}(-1)^j\xi_j d\xi_1\wedge\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\wedge\cdots\wedge d\xi_n.$$

Of course the above expression of $E_+(x)$ does not make sense unless we give the precise meaning of the formal series

$$\sum_{j\geqq 0}\frac{c_j(\xi)}{(P_m(\xi)-\sqrt{-1}\,0)^j}\Phi_{mj-n}(\langle x,\xi\rangle+\sqrt{-1}\,0).$$

We give the meaning of this series as a hyperfunction defined by the boundary value of the holomorphic function

$$\sum_{j\geqq 0}\frac{c_j(\zeta)}{(P_m(\zeta))^j}\Phi_{mj-n}(\langle z,\zeta\rangle)\tag{1.8}$$

from the domain $\Omega=\{(z,\zeta)\in \boldsymbol{C}^{2n}\mid \operatorname{Im}P_m(\zeta)<0,\ \operatorname{Im}\langle z,\zeta\rangle>0$ and (z,ζ) is sufficiently close to $S^*\boldsymbol{R}^n\cong \boldsymbol{R}^n\times S^{n-1}\}$.

Now we go on to the proof of the convergence of the series (1.8) in Ω. The coefficients $c_j(\zeta)$ in the series (1.8) should be defined so that

$$P(D_z)\Big(\sum_{j\geqq 0}\frac{c_j(\zeta)}{(P_m(\zeta))^j}\Phi_{mj-n}(\langle z,\zeta\rangle)\Big)=\frac{1}{(-2\pi\sqrt{-1})^n}\Phi_{-n}(\langle z,\zeta\rangle)\tag{1.9}$$

holds. For that purpose it is sufficient to choose $c_j(\zeta)$ so that

(1.10) $$\sum_{j\geq 0}\frac{c_j(\zeta)}{(P_m(\zeta))^j}\Phi_{mj-n}(\langle z,\zeta\rangle)=\sum_{j\geq 0}(P_m(D_z)-P(D_z))^j\Big(\frac{\Phi_{mj+m-n}(\langle z,\zeta\rangle)}{(P_m(\zeta))^{j+1}}\Big)$$

holds. In fact if we can prove that the right hand side of (1.10) converges absolutely and locally uniformly in Ω, then we easily find by differentiation term by term that it satisfies the equation (1.9) and it is also clear that $c_j(\zeta)$ can be defined successively by the relation (1.10) so that it becomes homogeneous of order 0 with respect to ζ.

Therefore what remains to prove is the convergence of the series in the right hand side of (1.10). In the sequel we denote by $Q(\zeta)$ the lower order terms of $P(\zeta)$ multiplied by (-1), i.e., $Q(\zeta)=P_m(\zeta)-P(\zeta)$. We also denote by $Q_k(\zeta)$ the homogeneous part of degree k of $Q(\zeta)$. For the sake of simplicity of notations we abbreviate in the sequel

$$\sum_{k=0}^{m-1}j_k \quad\text{to}\quad |(j_k)| \qquad\text{and}\qquad \sum_{k=0}^{m-1}kj_k \quad\text{to}\quad \|(j_k)\|\,.$$

Using these notations the right hand side of (1.10) is clearly re-written in the following way:

(1.11) $$\begin{aligned}&\sum_{j\geq 0}\Big(\sum_{k=0}^{m-1}Q_k(D)\Big)^j\Big(\frac{\Phi_{mj+m-n}(\langle z,\zeta\rangle)}{(P_m(\zeta))^{j+1}}\Big)\\ &=\sum_{j\geq 0}\Big(\sum_{\substack{|(j_k)|=j\\ j_k\geq 0}}\frac{j!}{j_0!\cdots j_{m-1}!}Q_0^{j_0}(D)\cdots Q_{m-1}^{j_{m-1}}(D)\Big)\Big(\frac{\Phi_{mj+m-n}(\langle z,\zeta\rangle)}{(P_m(\zeta))^{j+1}}\Big)\\ &=\sum_{j\geq 0}\Big(\sum_{\substack{|(j_k)|=j\\ j_k\geq 0}}\frac{j!}{j_0!\cdots j_{m-1}!}Q_0^{j_0}(\zeta)\cdots Q_{m-1}^{j_{m-1}}(\zeta)\frac{\Phi_{mj+m-n-\|(j_k)\|}(\langle z,\zeta\rangle)}{P_m(\zeta)^{j+1}}\Big)\,.\end{aligned}$$

In the sequel we denote z by $x+\sqrt{-1}\,y$ and ζ by $\xi+\sqrt{-1}\,\eta$, where x, y, ξ and η are real vectors.

It is obviously sufficient to consider the convergence of the series (1.10) assuming that $P_m(\xi)=0$, since the convergence of this series when $P_m(\xi)\neq 0$ is proved in the course of the proof in the case when $P_m(\xi)=0$, as the following proof shows. On the other hand, if $P_m(\xi)=0$, then conditions (1.1) and (1.2) show that

(1.12) $P_m(\xi+\sqrt{-1}\,\eta)=O(|\eta|)$ in a domain $V\subset \boldsymbol{C}^n$ which touches the cone $\{\zeta\in\boldsymbol{C}^n\mid\langle \mathrm{Im}\,\zeta,\ \mathrm{grad}_\xi P_m(\xi)\rangle<0\}$ up to the second order of $|\eta|$ along the real axis,

since $P_m(\xi+\sqrt{-1}\,\eta)=\langle\sqrt{-1}\,\eta,\ \mathrm{grad}_\xi P_m(\xi)\rangle+O(|\eta|^2)$. We want to estimate (1.10) in $\Omega=\{(z,\zeta)\mid \mathrm{Im}\,P_m(\zeta)<0,\ \mathrm{Im}\,\langle z,\zeta\rangle>0,\ |\mathrm{Im}\,\zeta|\ll 1\}$. In the sequel we denote $|P_m(\zeta)|$ by t, $|\langle z,\zeta\rangle|$ by s and $|Q_j(\zeta)|$ by a_j. We first consider the case $s\leq 1$. Then we have

$$(1.13)\quad \left|\sum_{j\geq 0}\Big(\sum_{\substack{|(j_k)|=j\\ j_k\geq 0}}\frac{j\,!}{j_0\,!\cdots j_{m-1}\,!}\,Q_0^{j_0}(\zeta)\cdots Q_{m-1}^{j_{m-1}}(\zeta)\frac{\Phi_{mj+m-n-\|(j_k)\|}(\langle z,\zeta\rangle)}{P_m(\zeta)^{j+1}}\Big)\right|$$

$$\leqq\sum_{j\geq 0}\frac{1}{t^{j+1}}\Big(\sum_{\substack{|(j_k)|=j\\ j_k\geq 0\\ mj+m-n-\|(j_k)\|\geq 0}}\frac{j\,!}{j_0\,!\cdots j_{m-1}\,!}\,a_0^{j_0}\cdots a_{m-1}^{j_{m-1}}\Big\{\frac{s^{mj+m-n-\|(j_k)\|}|\log s|}{(mj+m-n-\|(j_k)\|)\,!}$$

$$+\frac{\sum_{p=1}^{mj+m-n-\|(j_k)\|}\frac{1}{p}}{(mj+m-j-\|(j_k)\|)\,!}\,s^{mj+m-n-\|(j_k)\|}\Big\}+\sum_{\substack{|(j_k)|=j\\ j_k\geq 0\\ mj+m-n-\|(j_k)\|<0}}\frac{j\,!}{j_0\,!\cdots j_{m-1}\,!}\times$$

$$\times a_0^{j_0}\cdots a_{m-1}^{j_{m-1}}(-mj-m+n+\|(j_k)\|-1)\,!\,s^{mj+m-n-\|(j_k)\|}\Big)$$

$$\leqq\sum_{j\geq 0}\frac{1}{t^{j+1}}\Big(\sum_{\substack{|(j_k)|=j\\ j_k\geq 0}}\frac{j\,!}{j_0\,!\cdots j_{m-1}\,!}\,a_0^{j_0}\cdots a_{m-1}^{j_{m-1}}\Big\{\frac{s^{m-n+j}|\log s|}{(m-n+j)\,!}-\frac{(2s)^{m-n+j}}{(m-n+j)\,!}\Big\}\Big)$$

$$+\sum_{\substack{j\geq 0\\ m-n+j<0}}\frac{1}{t^{j+1}}\Big(\sum_{\substack{|(j_k)|=j\\ j_k\geq 0}}\frac{j\,!}{j_0\,!\cdots j_{m-1}\,!}\,a_0^{j_0}\cdots a_{m-1}^{j_{m-1}}(n-m-j-1)\,!\,s^{m-n+j}\Big),$$

since $\|(j_k)\|=\sum_{k=0}^{m-1}kj_k$ attains its maximum $(m-1)j$ under the condition $|(j_k)|=\sum_{k=0}^{m-1}j_k=j$ and $j_k\geqq 0$ when $(j_0,\cdots,j_{m-2},j_{m-1})=(0,\cdots,0,j)$ and $\sum_{p=1}^{l}\frac{1}{p}\leqq\log l+1\leqq 2^l$ holds. As the last term in (1.13)

$$\sum_{\substack{j\geq 0\\ m-n+j<0}}\frac{1}{t^{j+1}}\Big(\sum_{\substack{|(j_k)|=j\\ j_k\geq 0}}\frac{j\,!}{j_0\,!\cdots j_{m-1}\,!}\,a_0^{j_0}\cdots a_{m-1}^{j_{m-1}}(n-m-j-1)\,!\,s^{m-n+j}\Big)$$

is only a finite sum, this does not give rise to any difficulty concerning the convergence of the series (1.11). On the other hand we have the following estimate (1.14) concerning the other term of (1.13):

$$(1.14)\quad \sum_{\substack{j\geq 0\\ m-n+j\geq 0}}\frac{1}{t^{j+1}}\sum_{\substack{|(j_k)|=j\\ j_k\geq 0}}\frac{j\,!}{j_0\,!\cdots j_{m-1}\,!}\,a_0^{j_0}\cdots a_{m-1}^{j_{m-1}}\Big\{\frac{s^{m-n+j}|\log s|}{(m-n+j)\,!}+\frac{(2s)^{m-n+j}}{(m-n+j)\,!}\Big\}$$

$$\leqq\sum_{\substack{j\geq 0\\ m-n+j\geq 0}}\frac{|\log s|\,(3s)^{m-n+j}(a_0+\cdots+a_{m-1})^j}{t^{j+1}(m-n+j)\,!}$$

$$\leqq|\log s|\,t^{m-n-1}(a_0+\cdots+a_{m-1})^{n-m}\times$$

$$\times\sum_{\substack{j\geq 0\\ m-n+j\geq 0}}\frac{1}{(m-n+j)\,!}\Big(\frac{3(a_0+\cdots+a_{m-1})s}{t}\Big)^{m-n+j}$$

$$\leqq|\log s|\,t^{m-n-1}(a_0+\cdots+a_{m-1})^{n-m}\exp\Big(\frac{3(a_0+\cdots+a_{m-1})s}{t}\Big).$$

Thus we have proved the convergence of the series (1.11) as far as $|P_m(\zeta)|\neq 0$ and $0<|\langle z,\zeta\rangle|\leqq 1$. Even if $|\langle z,\zeta\rangle|>1$, we can perform an analogous estimate of the series (1.11). In fact we can estimate the series (1.11) except for a finite sum by

$$(1.15)\qquad \sum_{\substack{j\geqq 0\\ m-n+j\geqq 0}} \frac{1}{t^{j+1}} \sum_{\substack{|(j_k)|=j\\ j_k\geqq 0}} \frac{j!}{j_0!\cdots j_{m-1}!}\times$$

$$\times a_0^{j_0}\cdots a_{m-1}^{j_{m-1}}\left\{\frac{s^{mj+m-n}}{(m-n+j)!}|\log s|+\frac{2^{m-n+j}s^{mj+m-n}}{(m-n+j)!}\right\},$$

since $mj+m-n-\|(j_k)\|$ attains its maximum under the condition $|(j_k)|=j$ and $j_k\geqq 0$ when $(j_0, j_1, \cdots, j_{m-1})=(j, 0, \cdots, 0)$. Then it is clear that the quantity given by (1.15) can be majorized as follows:

$$(1.16)\qquad \sum_{\substack{j\geqq 0\\ m-n+j\geqq 0}} \frac{|\log s|\,(3s)^{mj+m-n}(a_0+\cdots+a_{m-1})^j}{t^{j+1}(m-n+j)!}$$

$$=|\log s|\, t^{m-n-1}(a_0+\cdots+a_{m-1})^{n-m}\times$$

$$\times \sum_{\substack{j\geqq 0\\ m-n+j\geqq 0}} \frac{1}{(m-n+j)!}\frac{(a_0+\cdots+a_{m-1})^{m-n+j}}{t^{m-n+j}}((3s)^m)^{j+\frac{m-n}{m}}$$

$$\leqq |\log s|\, t^{m-n-1}(a_0+\cdots+a_{m-1})^{n-m}\exp\left(\frac{3^m(a_0+\cdots+a_{m-1})s^m}{t}\right).$$

Thus we have proved the convergence of the series (1.11) as far as $|P_m(\zeta)|\neq 0$ and $|\langle z, \zeta\rangle|\neq 0$. Therefore we take the boundary value of the holomorphic function defined by the series (1.11) and obtain a hyperfunction $F(x, \xi)$ in (x, ξ). This is the precise meaning of the (formal) series

$$\sum_{j\geqq 0}\frac{c_j(\xi)}{(P_m(\xi)-\sqrt{-1}\,0)^j}\Phi_{mj-n}(\langle x, \xi\rangle+\sqrt{-1}\,0).$$

Now what remains to be investigated is the study of the location of singularities of the hyperfunction $\int_{|\xi|=1} F(x, \xi)\omega(\xi)$. For that purpose we first investigate S.S. $F(x, \xi)$ on $S^*(S^*\boldsymbol{R}^n)$ and next apply Sato's lemma on the regularity of integrals along fibers, which we quote below as Lemma 1.3.

In the sequel we denote by (ζ_x, ζ_ξ) a cotangent vector at (x, ξ). Taking (1.12) into account we easily find the following relation (1.17) by the definition, since the series (1.11) defines a holomorphic function in Ω and $F(x, \xi)$ is a hyperfunction defined by the boundary value of the series (1.11) from the domain Ω.

$$(1.17)\qquad \text{S.S.}\,F(x, \xi)$$

$$\subset\{(x, \xi\,;\,\zeta_x, \zeta_\xi)\,|\,\langle x, \xi\rangle=0,\ P_m(\xi)=0,\ \zeta_x=\xi,$$

$$\zeta_\xi=-\alpha\,\mathrm{grad}_\xi P_m(\xi)+\beta x,\ \text{where } \alpha+\beta=1,\ \alpha, \beta\geqq 0\}$$

$$\cup\{(x, \xi\,;\,\zeta_x, \zeta_\xi)\,|\,\langle x, \xi\rangle=0,\ P_m(\xi)\neq 0,\ \zeta_x=\xi,\ \zeta_\xi=x\}$$

$$\cup\{(x, \xi\,;\,\zeta_x, \zeta_\xi)\,|\,\langle x, \xi\rangle\neq 0,\ P_m(\xi)=0,\ \zeta_x=0,\ \zeta_\xi=-\mathrm{grad}_\xi P_m(\xi)\}.$$

On the other hand Sato's lemma on the regularity of integrals along fibers is the following

LEMMA 1.3. (Sato [3] § 6, [4] § 6.5.) *Let $f: N \to M$ be a real analytic mapping from an $(n+r)$-dimensional real analytic manifold N to an n-dimensional real analytic manifold M with maximal rank. Denote by dy the fundamental r-form along fiber. Suppose that a hyperfunction $\mu(x)$ defined on N satisfies the following condition:*

(1.18) $$f \text{ is a proper mapping over } \mathrm{S.S.}\,\mu(x)\,.$$

Then the integral along fibers $\int_{f^{-1}} \mu(x)dy$ is well-defined. (*See Sato* [4] § 6.5 *about the definition of the integral of microfunctions along fibers which is compatible with that of hyperfunctions* (*Sato* [1] § 10).) *Moreover*

(1.19) $$\mathrm{S.S.} \int_{f^{-1}} \mu(x)dy \subset \sigma_f(\mathrm{S.S.}\,\mu(x) \cap S^*M \times_M N)\,,$$

where $S^*M \times_M N$ denotes the fiber product of S^*M and N over M and σ_f denotes the natural homomorphism from $S^*M \times_M N$ to S^*M induced by the mapping f.

Now we apply Lemma 1.3 to the integral $\int_{|\xi|=1} F(x, \xi)\omega(\xi)$. Then (1.19) combined with (1.17) obviously implies that

(1.20) $$\mathrm{S.S.} \int_{|\xi|=1} F(x, \xi)\omega(\xi)$$
$$\subset \{(x, \xi) \in S^*M \mid x=0 \text{ or } x=t\,\mathrm{grad}_\xi P_m(\xi)\ (t \geqq 0) \text{ with } P_m(\xi)=0\}\,,$$

since the third term in the right hand side of (1.17) does not give any contributions to $\mathrm{S.S.} \int_{|\xi|=1} F(x, \xi)\omega(\xi)$ by condition (1.2).

Therefore $E_+(x) = \dfrac{1}{(-2\pi\sqrt{-1})^n} \int_{|\xi|=1} F(x, \xi)\omega(\xi)$ satisfies condition $(1.4)_+$, and it is also clear by (1.6) and (1.9) that $E_+(x)$ satisfies condition (1.3), i.e., $P(D)E_+(x)=\delta(x)$. Thus we have constructed a good elementary solution $E_+(x)$ of $P(D)$.

On the other hand, if we consider the series (1.11) in the domain $\Omega_- = \{(z, \zeta) \mid \mathrm{Im}\,P_m(\zeta) > 0 \text{ and } \mathrm{Im}\,\langle z, \zeta \rangle > 0\}$ and define a hyperfunction $F_-(x, \xi)$ by the boundary value of the holomorphic function defined in Ω_-, then we have

(1.17′) $$\mathrm{S.S.}\,F_-(x, \xi)$$
$$\subset \{(x, \xi; \zeta_x, \zeta_\xi) \mid \langle x, \xi \rangle = 0,\ P_m(\xi)=0,\ \zeta_x=\xi,$$
$$\zeta_\xi = \alpha\,\mathrm{grad}_\xi P_m(\xi) + \beta x, \text{ where } \alpha+\beta=1,\ \alpha, \beta \geqq 0\}$$
$$\cup \{(x, \xi; \zeta_x, \zeta_\xi) \mid \langle x, \xi \rangle = 0,\ P_m(\xi) \neq 0,\ \zeta_x=\xi,\ \zeta_\xi = x\}$$
$$\cup \{(x, \xi; \zeta_x, \zeta_\xi) \mid \langle x, \xi \rangle \neq 0,\ P_m(\xi)=0,\ \zeta_x=0,\ \zeta_\xi=\mathrm{grad}_\xi P_m(\xi)\}\,.$$

Therefore applying Lemma 1.3 to the integral along fibers

$$\frac{1}{(-2\pi\sqrt{-1})^n}\int_{|\xi|=1} F_-(x, \xi)\omega(\xi),$$

we easily find that this integral defines a good elementary solution of $P(D)$ satisfying $(1.4)_-$. Thus we have also constructed $E_-(x)$. This completes the proof of Theorem 1.2.

REMARK. We can prove Theorem 1.2 by suitably modifying the celebrated reasonings of John [**1**], where only the elliptic operators are treated. One essential point in the proof is that non-characteristic Cauchy problem in the complex domain has a unique entire solution as far as all the data given are entire functions, the linear differential operator under consideration is of constant coefficients and the initial hypersurface is a hyperplane. (See e.g. Leray [**1**], Lemma 9.1.)

But we preferred the proof given above because we can treat more general class of linear differential operators, even the operators with variable coefficients, by this asymptotic expansion method without much modification. See for example § 4.

§ 2. Global existence of real analytic solutions —compact case—

In this section we prove the global existence of real analytic solution of the equation $P(D)u(x)=f(x)$ where the differential operator $P(D)$ satisfies conditions (1.1) and (1.2) and the known function $f(x)$ is real analytic on a compact set $K\subset \boldsymbol{R}^n$ specified later on. In this section and also in later sections we denote by $\mathcal{A}(K)$ the space of real analytic functions on K, i.e., $\mathcal{A}(K)=\varinjlim_{V\supset K}\mathcal{O}(V)$, where V runs over the open neighbourhoods of K in $\boldsymbol{C}^n$ and $\mathcal{O}(V)$ denotes the space of holomorphic functions defined on V. For an open set Ω $(\subset \boldsymbol{R}^n)$ we also denote by $\mathcal{A}(\Omega)$ the space of real analytic functions defined on Ω. Since we do not use the topological structure of the space $\mathcal{A}(\Omega)$, we do not discuss it here.

LEMMA 2.1. *Assume that the compact set K is the closure of the open set $\Omega=\{x\in \boldsymbol{R}^n \mid \varphi(x)<0\}$, where $\varphi(x)$ is a real valued real analytic function defined in a neighbourhood of K satisfying* $\mathrm{grad}_x\varphi(x)\neq 0$ *on the boundary $\partial\Omega$ of Ω. Suppose that the differential operator $P(D)$ satisfies conditions* (1.1) *and* (1.2) *and that the compact set K satisfies the following geometrical condition* (2.1). *Then for any $f(x)$ in $\mathcal{A}(K)$ we can find $u(x)$ in $\mathcal{A}(\Omega)$ such that $P(D)u(x)=f(x)$ holds in Ω.*

(2.1) *For any x_0 in $\partial\Omega$ where $P_m(\mathrm{grad}_x\varphi(x)|_{x=x_0})=0$ holds, the bicharacteristic curve $b_{(x_0,\,\mathrm{grad}_x\varphi(x)|_{x=x_0})}$ of $P(D)$ issuing from $(x_0, \mathrm{grad}_x\varphi(x)|_{x=x_0})$ never intersects Ω.*

PROOF. We make the essential use of the good elementary solution constructed in Theorem 1.2. For that purpose we first extend $f(x)$ to whole space $\boldsymbol{R}^n$ by $f(x)\theta(-\varphi(x))$, here $\theta(t)$ denotes the 1-dimensional Heaviside function. We also denote $f(x)\theta(-\varphi(x))$ by $\tilde{f}(x)$. It is clear that the n-dimensional hyperfunction $\theta(-\varphi(x))$ is well-defined, since $\mathrm{grad}_x\,\varphi(x)\neq 0$ when $\varphi(x)=0$. Moreover the hyperfunction $\tilde{f}(x)$ has compact support K by the definition.

Now consider one of the good elementary solutions of $P(D)$, e.g., $E_+(x)$ and denote it by $E(x)$ for short. Then we define a hyperfunction $\tilde{u}(x)$ by the following convolution of $\tilde{f}(x)$ and $E(x)$ i.e., a special kind of integral along fibers of hyperfunctions (Sato [1] § 6.5), which makes sense, since $\tilde{f}(x)$ has compact support.

$$\tilde{u}(x)=\int E(x-y)\tilde{f}(y)dy\,. \tag{2.2}$$

Then it is clear by (1.3) that $P(D)\tilde{u}(x)=\tilde{f}(x)$ holds on $\boldsymbol{R}^n$.

Next we investigate the regularity property of $\tilde{u}(x)$. Here we make full use of the theory of microfunctions, especially Sato's lemma on the regularity of integrals along fibers. For that purpose we first determine where the singularities of $2n$-variable hyperfunction $E(x-y)\tilde{f}(y)$ locate. First note that

$$\mathrm{S.S.}\,\tilde{f}(x)\subset\{(x,\xi)\in S^*\boldsymbol{R}^n\mid\varphi(x)=0,\ \xi=\pm\mathrm{grad}_x\,\varphi(x)\} \tag{2.3}$$

and

$$\mathrm{S.S.}\,E(x)\subset\{(x,\xi)\in S^*\boldsymbol{R}^n\mid x=0\ \text{or}\ x=t\,\mathrm{grad}_\xi\,P_m(\xi)\ (t\geqq 0),\ P_m(\xi)=0\} \tag{2.4}$$

hold by their definitions. Then taking (2.3) and (2.4) into account it is easily verified by the definition of the multiplication of two hyperfunctions (Sato [4] § 6.4) that

$$\mathrm{S.S.}\,E(x-y)\tilde{f}(y)\subset\bigcup_{j=1}^{5}S_j\,, \tag{2.5}$$

where

(i) $S_1=\{(x,y;\xi,\eta)\in S^*\boldsymbol{R}^{2n}\mid x=y,\ \varphi(y)=0,\ (\xi,\eta)=(\alpha\zeta,-\alpha\zeta\pm\beta\,\mathrm{grad}_y\varphi(y))$, where ζ is any non-zero real cotangent vector and $\alpha+\beta=1$ with $\alpha,\beta\geqq 0\}$,

(ii) $S_2=\{(x,y;\xi,\eta)\in S^*\boldsymbol{R}^{2n}\mid x=y,\ \varphi(y)\neq 0,\ \xi=-\eta\neq 0\}$,

(iii) $S_3=\{(x,y;\xi,\eta)\in S^*\boldsymbol{R}^{2n}\mid x=y+t\,\mathrm{grad}_\zeta\,P_m(\zeta)\ (t\geqq 0),\ P_m(\zeta)=0,\ \varphi(y)=0,$ $(\xi,\eta)=(\alpha\zeta,-\alpha\zeta\pm\beta\,\mathrm{grad}_y\,\varphi(y))$, where ζ is a non-zero real cotangent vector and $\alpha+\beta=1$ with $\alpha,\beta\geqq 0\}$,

(iv) $S_4=\{(x,y;\xi,\eta)\in S^*\boldsymbol{R}^{2n}\mid x=y+t\,\mathrm{grad}_\zeta\,P_m(\zeta)\ (t\geqq 0),\ P_m(\zeta)=0,\ \varphi(y)\neq 0,$ $\xi=-\eta=\zeta\neq 0\}$

and

(v) $S_5=\{(x, y;\xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid \varphi(y)=0,\ (\xi, \eta)=(0, \pm\operatorname{grad}_y\varphi(y)),\ x\neq y+t\operatorname{grad}_\zeta P_m(\zeta)$ for any $t\geqq 0$ and real non-zero cotangent vector ζ satisfying $P_m(\zeta)=0\}$.

After these preparatory investigation of singularities of the integrand of the integral (2.2) along fibers we apply Lemma 1.3 to the integral. Then we have

(2.6) $\quad \mathrm{S.S.}\,\tilde{u}(x)\subset\{(x, \xi)\in S^*\boldsymbol{R}^n \mid$ there exists y such that $\varphi(y)=0$, $\xi=\pm\operatorname{grad}_y\varphi(y)$ and $x=y+t\operatorname{grad}_\xi P_m(\xi)\ (t\geqq 0)\}$,

since the sets S_2, S_4, S_5 do not give any contributions to $\mathrm{S.S.}\,\tilde{u}(x)$ by Lemma 1.3 and since the effect to $\mathrm{S.S.}\,\tilde{u}(x)$ of the sets S_1 and S_3 after the integration is obviously contained in the right hand side of (2.6) by Lemma 1.3 also.

Combining the geometrical condition on Ω (2.1) and the above relation (2.6) we have proved that $S^*\Omega\cap\mathrm{S.S.}\,\tilde{u}(x)=\emptyset$. Especially this implies by the definition that the hyperfunction $\tilde{u}(x)$ is real analytic in Ω. Therefore denoting the restriction of $\tilde{u}(x)$ to Ω by $u(x)$, we have the required real analytic solution $u(x)$ of the equation $P(D)u(x)=f(x)$ in Ω. This completes the proof of Lemma 2.1.

Next we improve this lemma a little by using *both* good elementary solutions of $P(D)$. Not only the result given in the following Lemma 2.2, which plays an essential role in later sections, but also its method of the proof is of much importance, since the method of the proof of this lemma will be regarded as a model of the proofs of the theorems in this section and also in later sections.

LEMMA 2.2. *Assume that a compact set K is the closure of the open set $\Omega=\{x \mid \varphi(x)<0\}$, where $\varphi(x)$ is a real valued real analytic function defined in a neighbourhood of K satisfying* $\operatorname{grad}_x\varphi(x)\neq 0$ *on $\partial\Omega$. Suppose that the differential operator $P(D)$ satisfies conditions* (1.1) *and* (1.2) *and that K satisfies the following geometrical condition* (2.7). *Then for any $f(x)$ in $\mathcal{A}(K)$ we can find $u(x)$ in $\mathcal{A}(\Omega)$ such that $P(D)u(x)=f(x)$ holds in Ω.*

(2.7) *For any characteristic boundary point x_0, i.e., the boundary point where $P_m(\operatorname{grad}_x\varphi(x)|_{x=x_0})=0$ holds, the bicharacteristic curve $b_{(x_0, \operatorname{grad}_x\varphi(x)|_{x=x_0})}$ of $P(D)$ issuing from $(x_0, \operatorname{grad}_x\varphi(x)|_{x=x_0})$ intersects Ω in an open interval.*

REMARK. It is clear by condition (2.7) and the definition of bicharacteristics that the bicharacteristic curve $b_{(x_0, -\operatorname{grad}_x\varphi(x)|_{x=x_0})}$ of $P(D)$ issuing from $(x_0, -\operatorname{grad}_x\varphi(x)|_{x=x_0})$ also intersects Ω in an open interval.

PROOF OF LEMMA 2.2. As in the proof of Lemma 2.1 we extend $f(x)$ to $\boldsymbol{R}^n$ by $\tilde{f}(x)=f(x)\theta(-\varphi(x))$. On the other hand by condition (2.7) we can decompose the set $N=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x\in\partial\Omega,\ P_m(\xi)=0,\ \xi=\pm\operatorname{grad}_x\varphi(x)\}$ into the form $N_+\cup N_-$, where $N_+=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x\in\partial\Omega,\ P_m(\xi)=0,\ \xi=\pm\operatorname{grad}_x\varphi(x)$

and the positive half side of the bicharacteristic curve $b^{+}_{(x,\xi)}: x+t\,\mathrm{grad}_\xi P_m(\xi)$ $(t\geqq 0)$ issuing from (x, ξ) does not intersect $\Omega\}$ and $N_- = \{(x, \xi)\in S^*\boldsymbol{R}^n \mid x\in\partial\Omega$, $P_m(\xi)=0$, $\xi=\pm\mathrm{grad}_x\,\varphi(x)$ and the negative half side of the bicharacteristic curve $b^{-}_{(x,\xi)}: x+t\,\mathrm{grad}_\xi P_m(\xi)$ $(t\leqq 0)$ issuing from (x, ξ) does not intersect $\Omega\}$. Since the boundary of Ω is smooth, we easily see that the sets N_+ and N_- are closed. In fact, assume that $(x_n, \xi_n)\in N_+$ converges to $(x, \xi)\in N$ but that $(x, \xi)\notin N_+$, then the positive half side of the bicharacteristic curve of $P(D)$ through (x, ξ) intersects Ω. Hence the smoothness of the boundary of Ω assures that a convex cone with vertex at x containing the positive half side of the bicharacteristic curve through (x, ξ) is contained near x in Ω. Therefore the half line $x+t\,\mathrm{grad}_\xi P_m(\xi)\,|_{\xi=\xi_n}$ $(t\geqq 0)$ intersects Ω for sufficiently large n by the continuity of $\mathrm{grad}_\xi P_m(\xi)$. This immediately implies that $(x_n, \xi_n)\notin N_+$ for sufficiently large n. This is a contradiction and we have proved that the set N_+ is closed. We also conclude that N_- is closed by the same reasoning.

We use this decomposition of the set N as follows. Since sheaf $\mathcal{C}$ is flabby, the vanishing of $H^1(\boldsymbol{R}^n, \mathcal{A})$ proves the existence of hyperfunctions $\tilde{f}_+(x)$ and $\tilde{f}_-(x)$ such that

(2.8) $\quad \mathrm{S.S.}\,(\tilde{f}(x)-\tilde{f}_+(x)-\tilde{f}_-(x))=\emptyset$,

(2.9) $\quad \mathrm{S.S.}\,\tilde{f}_+(x)\cap N\subset N_+$ and $\mathrm{S.S.}\,\tilde{f}_-(x)\cap N\subset N_-$,

(2.10) $\quad \mathrm{S.S.}\,\tilde{f}_+(x)$, $\mathrm{S.S.}\,\tilde{f}_-(x)\subset S^*_{\partial\Omega}\boldsymbol{R}^n$, where $S^*_{\partial\Omega}\boldsymbol{R}^n$ denotes the conormal bundle $\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x\in\partial\Omega,\ \xi=\pm\mathrm{grad}_x\,\varphi(x)\}$.

(About the flabbiness of sheaf $\mathcal{C}$ we refer to Kashiwara [**1**]. See also Sato, Kawai and Kashiwara [1], where the shortest proof using the theory of elliptic pseudo-differential operators is given.)

Using these two hyperfunctions $\tilde{f}_+(x)$ and $\tilde{f}_-(x)$, we define $v(x)$ by the following sum of two integrals along fibers, which make sense as those of microfunctions. (Cf. Sato [4] § 6.5.)

(2.11)
$$v(x)=\int E_+(x-y)\tilde{f}_+(y)\,dy+\int E_-(x-y)\tilde{f}_-(y)\,dy\,.$$

Now we investigate the regularity property of $v(x)$, i.e., the support of the microfunction $v(x)$. It is performed in a similar way to the proof of Lemma 2.1 as follows. In fact we easily see that

(2.12)
$$\mathrm{S.S.}\,E_+(x-y)\tilde{f}_+(y)\subset\bigcup_{j=1}^{8}S_{j,+}\,,$$

where

(i) $S_{1,+}=\{(x, y\,;\xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid x=y,\ (y, \theta)\in\mathrm{S.S.}\,\tilde{f}_+(y),\ (\xi, \eta)=(\alpha\zeta, -\alpha\zeta+\beta\theta)$, where ζ is any non-zero real cotangent vector, $\theta=\mathrm{grad}_y\,\varphi(y)$ or $-\mathrm{grad}_y\,\varphi(y)$, and $\alpha+\beta=1$ with $\alpha, \beta\geqq 0\}$,

(ii) $S_{2,+}=\{(x, y; \xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid x=y,\ \varphi(y)\neq 0,\ \xi=-\eta\}$,

(iii) $S_{3,+}=\{(x, y; \xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid x=y+t\,\mathrm{grad}_\zeta P_m(\zeta)\ (t\geqq 0),\ P_m(\zeta)=0,\ (y, \theta)\in \mathrm{S.S.}\tilde{f}_+(y),\ (\xi, \eta)=(\alpha\zeta, -\alpha\zeta+\beta\theta)$, where ζ is a non-zero real cotangent vector, $\theta=\mathrm{grad}_y\varphi(y)$ or $-\mathrm{grad}_y\varphi(y)$, and $\alpha+\beta=1$ with $\alpha, \beta\geqq 0\}$,

(iv) $S_{4,+}=\{(x, y; \xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid x=y+t\,\mathrm{grad}_\zeta P_m(\zeta)\ (t\geqq 0),\ P_m(\zeta)=0,\ \varphi(y)\neq 0,\ \xi=-\eta=\zeta\neq 0\}$

and

(v) $S_{5,+}=\{(x, y; \xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid \xi=0,\ (y, \eta)\in \mathrm{S.S.}\tilde{f}_+(y),\ x\neq y+t\,\mathrm{grad}_\zeta P_m(\zeta)$ for any $t\geqq 0$ and real cotangent vector ζ satisfying $P_m(\zeta)=0\}$,

and

$$\mathrm{S.S.}\,E_-(x-y)\tilde{f}_-(y)\subset \bigcup_{j=1}^{5} S_{j,-}\,, \tag{2.13}$$

where the sets $S_{j,-}$ are defined by changing the sign of t and replacing $\mathrm{S.S.}\tilde{f}_+(y)$ by $\mathrm{S.S.}\tilde{f}_-(y)$ in the definition of the sets $S_{j,+}$.

Combining Lemma 1.3 with (2.12) and (2.13), we conclude that

$$\begin{aligned}(2.14)\quad &\mathrm{S.S.}\,v(x)\\ &\subset \{(x, \xi)\in S^*\boldsymbol{R}^n \mid x=y+t\,\mathrm{grad}_\xi P_m(\xi)\ (t\geqq 0),\ P_m(\xi)=0,\ (y, \xi)\in N_+\}\\ &\cup \{(x, \xi)\in S^*\boldsymbol{R}^n \mid x=y+t\,\mathrm{grad}_\xi P_m(\xi)\ (t\leqq 0),\ P_m(\xi)=0,\ (y, \xi)\in N_-\}\,.\end{aligned}$$

This immediately implies by the definitions of the sets N_+ and N_- that

$$\mathrm{S.S.}\,v(x)\cap S^*\Omega=\emptyset\,. \tag{2.15}$$

On the other hand the relations (2.8) and (1.3) imply that

$$P(D)v(x)=\tilde{f}(x) \tag{2.16}$$

holds as an equation between microfunctions. Choose $V(x)\in\mathcal{B}(\boldsymbol{R}^n)$ so that $\beta(V(x))=v(x)$. The existence of such a hyperfunction $V(x)$ is assured again by the following exact sequence (2.18) and the well known fact that $H^1(\boldsymbol{R}^n, \mathcal{A})=0$. Then (2.16) means the following relation:

$$P(D)V(x)=\tilde{f}(x)+g(x)\,, \tag{2.17}$$

where $g(x)\in\mathcal{A}(\boldsymbol{R}^n)$, since we have the following exact sequence (2.18) as one of the fundamental properties of sheaf C.

$$0\longrightarrow \mathcal{A}\longrightarrow \mathcal{B}\xrightarrow{\ \beta\ }\pi_*C\longrightarrow 0\,, \tag{2.18}$$

where π denotes the cannonical surjection from the cotangential sphere bundle to the base space and π_*C denotes the direct image of sheaf C under this mapping. (See Sato [2]~[5].) Note that the relation $\beta(V(x))=v(x)$ and the relation (2.15) imply that $V(x)$ is real analytic in Ω.

Restricting $g(x)$ to a sufficiently large ball B containing K in its interior we can apply Lemma 2.1 and find a real analytic function $w(x)$ defined in the interior of B, which satisfies $P(D)w(x)=g(x)$ there. Subtracting $w(x)$ from $V(x)$ we obtain by (2.17) the required real analytic function $u(x)$ satisfying the equation $P(D)u(x)=f(x)$ in Ω, since $V(x)$ is real analytic in Ω. This completes the proof of the lemma.

Now the following theorem, which is only a restatement of Lemma 2.2, is clear.

THEOREM 2.3. *Suppose that the differential operator* $P(D)$ *satisfies conditions* (1.1) *and* (1.2). *Assume that a compact set* $K\subset \boldsymbol{R}^n$ *has a fundamental system of neighbourhoods with smooth boundary* Ω_ν, *where* Ω_ν *is represented as* $\{x\in \boldsymbol{R}^n \mid \varphi_\nu(x)<0\}$ *by a real valued real analytic function* $\varphi_\nu(x)$ *defined in a neighbourhood of* $\bar{\Omega}_\nu$ *and* Ω_ν *satisfies the following geometrical condition* (2.19). *Then* $P(D)\mathcal{A}(K)=\mathcal{A}(K)$ *holds.*

(2.19) *At any characteristic boundary point* x_0 *of* Ω_ν, *the bicharacteristic curve of* $P(D)$ *issuing from* $(x_0, \mathrm{grad}_x\, \varphi_\nu|_{x=x_0})$ *intersects* K *in a closed interval.*

The proof is similar to that of Lemma 2.2 and we omit the details.

REMARK. If K is a compact convex set in $\boldsymbol{R}^n$, then condition (2.19) is clearly satisfied.

Now we consider the case where $\partial\Omega$ has some singularities in Lemma 2.1, namely we investigate the case where the compact set K is the closure of an open set Ω of the form $\bigcap_{j=1}^{p}\Omega_j$, where each Ω_j has smooth boundary and Ω_j and Ω_k intersects in general position.

THEOREM 2.4. *Suppose that the differential operator* $P(D)$ *satisfies conditions* (1.1) *and* (1.2). *Assume that a compact set* $K\subset \boldsymbol{R}^n$ *is the closure of an open set* $\Omega=\bigcap_{j=1}^{p}\Omega_j$, *where each* Ω_j *satisfies the following regularity conditions* (2.20) *and* (2.21). *Moreover we assume that the compact set* K *satisfies the following geometrical condition* (2.22). *Then for any* $f(x)$ *in* $\mathcal{A}(K)$ *we can find* $u(x)$ *in* $\mathcal{A}(\Omega)$ *such that* $P(D)u(x)=f(x)$ *holds in* Ω.

(2.20) Ω_j *is represented as* $\{x\in \boldsymbol{R}^n \mid \varphi_j(x)<0\}$ *by a real valued real analytic function* $\varphi_j(x)$ *defined in a neighbourhood of* $\bar{\Omega}_j$.

(2.21) $\{\mathrm{grad}_x\, \varphi_{j_q}(x)\}_{q=1}^{k}$ *are linearly independent as far as* $\varphi_{j_1}(x)=\cdots=\varphi_{j_k}(x)=0$.

(2.22) *If* x_0 *satisfies* $\varphi_{j_1}(x)=\cdots=\varphi_{j_k}(x)=0$ *then for any non-zero* ϑ *that is a linear combination of* $\mathrm{grad}_x\, \varphi_{j_1}(x)|_{x=x_0}, \cdots, \mathrm{grad}_x\, \varphi_{j_k}(x)|_{x=x_0}$ *and that satisfies* $P_m(\vartheta)=0$, *the bicharacteristic curve of* $P(D)$ *through* (x_0, ϑ) *intersects* Ω *in an open interval.*

PROOF. By using conditions (2.20) and (2.21) we extend $f(x)$ to $\boldsymbol{R}^n$ by $\tilde{f}(x)=f(x)\cdot\theta(-\varphi_1(x))\cdot\cdots\cdot\theta(-\varphi_p(x))$. This multiplication is possible by condition (2.21). By this definition we have

$$\text{(2.23)}\qquad \mathrm{S.S.}\tilde{f}(x)\subset\bigcup_{q=1}^{p}\{(x,\xi)\in S^*\boldsymbol{R}^n \mid \varphi_j(x)\leqq 0\ (j=1,\cdots,p),\ \varphi_{j_1}(x)=\cdots=\varphi_{j_q}(x)$$
$$=0,\ \xi \text{ belongs to the vector space spanned by } \{\pm\mathrm{grad}_x\,\varphi_{j_k}(x)\}_{k=1}^{q}\}\,.$$

In the sequel we denote the right hand side of (2.23) by M for short. We also denote by N the set $\{(x,\xi)\in M \mid P_m(\xi)=0\}$. Then condition (2.22) assures us that we can decompose the set N into the union of two closed set N_+ and N_- just in the same way as in the proof of Lemma 2.1 using the orientation of the bicharacteristic curves of $P(D)$ through (x,ξ).

Now we can proceed in a similar way to the proof of Lemma 2.1. We first decompose $\tilde{f}(x)$ into the form $\tilde{f}_+(x)+\tilde{f}_-(x)$ so that

$$\text{(2.24)}\qquad \mathrm{S.S.}\,(\tilde{f}(x)-\tilde{f}_+(x)-\tilde{f}_-(x))=\emptyset\,,$$

$$\text{(2.25)}\qquad \mathrm{S.S.}\tilde{f}_+(x)\cap N\subset N_+ \quad\text{and}\quad \mathrm{S.S.}\tilde{f}_-(x)\cap N\subset N_-\,,$$

$$\text{(2.26)}\qquad \mathrm{S.S.}\tilde{f}_+(x),\ \mathrm{S.S.}\tilde{f}_-(x)\subset M\,.$$

Secondly we define a microfunction $v(x)$ by the following integral:

$$\text{(2.27)}\qquad v(x)=\int E_+(x-y)\tilde{f}_+(y)dy+\int E_-(x-y)\tilde{f}_-(y)dy\,.$$

Then condition (2.24) assures us that $P(D)v(x)=\tilde{f}(x)$ holds as an equation between microfunctions. Conditions (2.25) and (2.26) also prove that

$$\text{(2.28)}\qquad \mathrm{S.S.}\,v(x)\cap S^*\Omega=\emptyset\,.$$

Therefore we have a hyperfunction $V(x)$ which is real analytic in Ω and satisfies

$$\text{(2.29)}\qquad P(D)V(x)=\tilde{f}(x)+g(x)$$

for some $g(x)$ in $\mathcal{A}(\boldsymbol{R}^n)$. Applying Lemma 2.1 as in the proof of Lemma 2.2 to eliminate $g(x)$ from (2.29), we obtain a required real analytic solution $u(x)$ satisfying $P(D)u(x)=f(x)$ in Ω. This completes the proof of Theorem 2.4.

REMARK. We can also restate Theorem 2.4 in a form analogous to Theorem 2.3 so that it assures the existence of solutions in the space $\mathcal{A}(K)$. However, the modification is clear, so we omit the details.

We can also weaken the regularity condition of the boundary of K as in the following theorem, which is better to be stated after Theorem 3.1 logically. However we have preferred to state it here since it gives a good existence theorem for the space $\mathcal{A}(K)$.

THEOREM 2.5. *Suppose that the differential operator $P(D)$ satisfies conditions*

(1.1) *and* (1.2). *Assume that the compact set K has a fundamental system of open neighbourhoods* $\{\Omega_j\}_{j=1}^{\infty}$ *which satisfies the following conditions* (2.30) *and* (2.31). *Then* $P(D)\mathcal{A}(K)=\mathcal{A}(K)$ *holds.*

(2.30) *For any x_0 in $\partial\Omega_j$ we can insert an open convex cone with vertex at x_0 into Ω_j locally.*

(2.31) *Any bicharacteristic curve of $P(D)$ issuing from (x, ξ), where $x\in\partial\Omega_j$ and $P_m(\xi)=0$, intersects Ω_k in an open interval for some k.*

The proof of this theorem is given by the aid of the flabbiness of the sheaf of germs of hyperfunctions. It is just the same as that of Theorem 3.1 in the next section, hence we omit the detailed proof here. We also refer the reader to Theorem 6.1 in § 6, where no regularity conditions on the boundary of K are assumed.

§ 3. Global existence of real analytic solutions (II) —the open case—

In this section we prove the global existence of real analytic solutions of the equation $P(D)u(x)=f(x)$ where the differential operator $P(D)$ satisfies conditions (1.1) and (1.2) and the known function $f(x)$ is real analytic in a relatively compact open set $\Omega\subset \boldsymbol{R}^n$, whose conditions are specified later on. The main idea of our proof is as follows: firstly we prove the surjectivity of the mapping $P(D): \mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega})\to\mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega})$ and secondly we apply Lemma 2.2 to obtain the surjectivity of the mapping $P(D): \mathcal{A}(\Omega)\to\mathcal{A}(\Omega)$. The surjectivity of the mapping $P(D): \mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega})\to\mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega})$ is proved in an analogous way to the proof of Theorem 2.4. (Moreover concerning the first part of the proof we can sometimes prove the surjectivity of the mapping $P(D): \mathcal{A}(\Omega)/\mathcal{A}(K)\to\mathcal{A}(\Omega)/\mathcal{A}(K)$ for sufficiently large compact convex set K or the surjectivity of the mapping $P(D): \mathcal{A}(\Omega)/\mathcal{A}(\boldsymbol{R}^n)\to\mathcal{A}(\Omega)/\mathcal{A}(\boldsymbol{R}^n)$. See the proof of Theorem 3.1 below.) One essential difference from the proof of Theorem 2.4 is that we should use in this case the flabbiness of sheaf of the germs of hyperfunctions as well as that of sheaf $\mathcal{C}$.

THEOREM 3.1. *Suppose that the differential operator $P(D)$ satisfies conditions* (1.1) *and* (1.2). *Assume that a relatively compact open set Ω with C^1-boundary satisfies the following geometrical condition* (3.1). *Then* $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ *holds.*

(3.1) *Any bicharacteristic curve $b_{(x,\xi)}$ of $P(D)$ issuing from (x, ξ) intersects Ω in an open interval, where x belongs to $\partial\Omega$ and ξ is a non-zero real cotangent vector satisfying $P_m(\xi)=0$.*

REMARK 1. As is clear from the method of the proof given below, the regularity condition on the boundary of Ω can be relaxed. In fact it is sufficient to assume condition (2.30).

REMARK 2. Taking Remark 1 into account we clearly see that for any relatively compact open convex set Ω in $\boldsymbol{R}^n$ $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ holds if $P(D)$ satisfies conditions (1.1) and (1.2).

PROOF OF THEOREM 3.1. Utilizing the flabbiness of the sheaf of germs of hyperfunctions we first extend $f(x)$ to $\boldsymbol{R}^n$ so that the extension $\tilde{f}(x)$ coincides with $f(x)$ in Ω and has its support in $\bar{\Omega}$. Then it is clear that the following relation (3.2) holds.

$$\text{S.S.}\tilde{f}(x)\subset\{(x,\xi)\in S^*\boldsymbol{R}^n\mid x\in\partial\Omega\}\,. \tag{3.2}$$

In the sequel we denote by M the right hand side of (3.2). We also denote by N the set $\{(x,\xi)\in S^*\boldsymbol{R}^n\mid x\in\partial\Omega,\ P_m(\xi)=0\}$. By the regularity condition on $\partial\Omega$ and assumption (3.1) we can decompose the set N into the union of two closed set N_+ and N_- as in the proof of Lemma 2.2, where $N_+=\{(x,\xi)\in S^*\boldsymbol{R}^n\mid x\in\partial\Omega,\ P_m(\xi)=0$, the positive half side of the bicharacteristic curve $b^+_{(x,\xi)}: x+t\,\mathrm{grad}_\xi P_m(\xi)$ $(t\geqq 0)$ issuing from (x,ξ) does not intersect $\Omega\}$ and $N_-=\{(x,\xi)\in S^*\boldsymbol{R}^n\mid x\in\partial\Omega,\ P_m(\xi)=0$, the negative half side of the bicharacteristic curve $b^-_{(x,\xi)}: x+t\,\mathrm{grad}_\xi P_m(\xi)$ $(t\leqq 0)$ issuing from (x,ξ) does not intersect $\Omega\}$. The regularity condition on $\partial\Omega$ is used to assure the closedness of the sets N_+ and N_-. (Cf. the proof of Lemma 2.2.) By the aid of the flabbiness of sheaf C and the vanishing of $H^1(\boldsymbol{R}^n,\mathcal{A})$ we can decompose $\tilde{f}(x)$ into the sum of two hyperfunctions $\tilde{f}_+(x)+\tilde{f}_-(x)$ as microfunctions so that

$$\text{S.S.}\tilde{f}_+(x),\ \text{S.S.}\tilde{f}_-(x)\subset M\,, \tag{3.3}$$

$$\text{S.S.}\tilde{f}_+(x)\cap N\subset N_+\quad\text{and}\quad \text{S.S.}\tilde{f}_-(x)\cap N\subset N_-\,. \tag{3.4}$$

Consider the microfunction $v(x)$ defined by the following integral:

$$v(x)=\int E_+(x-y)\tilde{f}_+(y)dy+\int E_-(x-y)\tilde{f}_-(y)dy\,. \tag{3.5}$$

Then by (1.3) and the definition of $\tilde{f}_+(x)$ and $\tilde{f}_-(x)$ we have

$$P(D)v(x)=\tilde{f}(x) \tag{3.6}$$

as microfunctions defined on $S^*\boldsymbol{R}^n$. Hence what remains to prove is the regularity property of $v(x)$. It is performed in a similar way to the proof of Lemma 2.2. In fact we easily have also in this case

$$\text{S.S.}\ E_+(x-y)\tilde{f}_+(y)\subset\bigcup_{j=1}^{5}S_{j,+}\,, \tag{3.7}$$

where

(i) $S_{1,+}=\{(x,y;\xi,\eta)\in S^*\boldsymbol{R}^{2n}\mid x=y,\ (\xi,\eta)=(\alpha\zeta,-\alpha\zeta+\beta\theta),\ (y,\theta)\in$ $\text{S.S.}\tilde{f}_+(y)$ where ζ is any non-zero real cotangent vector and $\alpha+\beta=1$ with $\alpha,\beta\geqq 0\}$,

(ii) $S_{2,+}=\{(x, y;\ \xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid x=y,\ y\notin\partial\Omega,\ \xi=-\eta\neq 0\}$,

(iii) $S_{3,+}=\{(x, y;\ \xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid x=y+t\,\mathrm{grad}_\zeta P_m(\zeta)\quad (t\geqq 0),\quad P_m(\zeta)=0,$ $(\xi, \eta)=(\alpha\zeta, -\alpha\zeta+\beta\theta),\ (y, \theta)\in \mathrm{S.S.}\tilde{f}_+(y)$, where ζ is a non-zero real cotangent vector and $\alpha+\beta=1$ with $\alpha, \beta\geqq 0\}$,

(iv) $S_{4,+}=\{(x, y;\ \xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid x=y+t\,\mathrm{grad}_\zeta P_m(\zeta)\quad (t\geqq 0),\quad P_m(\zeta)=0,$ $y\notin\partial\Omega,\ \xi=-\eta=\zeta\neq 0\}$.

and

(v) $S_{5,+}=\{(x, y;\ \xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid \xi=0,\ (y, \eta)\in\mathrm{S.S.}\tilde{f}_+(y),\ x\neq y+t\,\mathrm{grad}_\zeta P_m(\zeta)$ for any $t\geqq 0$ and real non-zero cotangent vector ζ satisfying $P_m(\zeta)=0\}$.

Concerning $\mathrm{S.S.}\,E_-(x-y)\tilde{f}_-(y)$ we have the analogous relation

$$\mathrm{S.S.}\,E_-(x-y)\tilde{f}_-(y)\subset\bigcup_{j=1}^{5}S_{j,-}\,, \tag{3.8}$$

where the sets $S_{j,-}$ are defined by changing the sign of the parameter t and replacing $\mathrm{S.S.}\tilde{f}_+(y)$ by $\mathrm{S.S.}\tilde{f}_-(y)$ in the definition of the sets $S_{j,+}$.

Now we combine Lemma 1.3 with relations (3.7) and (3.8). Since the sets $S_{2,\pm}$, $S_{4,\pm}$ and $S_{5,\pm}$ do not give any contributions to $\mathrm{S.S.}\,v(x)$ by virtue of Lemma 1.3, we conclude by the aid of Lemma 1.3 that

$$\mathrm{S.S.}\,v(x)\subset\bigcup_{j=1}^{4}A_j\,, \tag{3.9}$$

where

(i) $A_1=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid (x, \xi)\in\mathrm{S.S.}\tilde{f}_+(x)\}$,

(ii) $A_2=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid (x, \xi)\in\mathrm{S.S.}\tilde{f}_-(x)\}$,

(iii) $A_3=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x=y+t\,\mathrm{grad}_\xi P_m(\xi)\ (t\geqq 0),\ P_m(\xi)=0,$ $(y, \xi)\in\mathrm{S.S.}\tilde{f}_+(y)\}$,

and

(iv) $A_4=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x=y+t\,\mathrm{grad}_\xi P_m(\xi)\ (t\leqq 0),\ P_m(\xi)=0,$ $(y, \xi)\in\mathrm{S.S.}\tilde{f}_-(y)\}$.

Therefore we have proved that $\mathrm{S.S.}\,v(x)\cap S^*\Omega=\emptyset$ by (3.1) and (3.9). Hence, taking (3.6) into account, we can find a hyperfunction $V(x)$ defined on $\boldsymbol{R}^n$ such that

(3.10) $V(x)$ is real analytic in Ω

and

(3.11) $P(D)V(x)=\tilde{f}(x)+g(x)$, where $g(x)$ belongs to $\mathcal{A}(\boldsymbol{R}^n)$.

These two relations imply that the mapping $P(D): \mathcal{A}(\Omega)/\mathcal{A}(\boldsymbol{R}^n)\to\mathcal{A}(\Omega)/\mathcal{A}(\boldsymbol{R}^n)$ is surjective under condition (3.1). Then we can apply Lemma 2.1 to eliminate $g(x)$ as in the proof of Lemma 2.2 and conclude that the differential operator

$P(D): \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is surjective under condition (3.1). This completes the proof of the theorem.

Since Theorem 3.1 seems to require too much information concerning the global shape of Ω, we modify Theorem 3.1 as follows:

THEOREM 3.2. *Suppose that the differential operator $P(D)$ satisfies conditions* (1.1) *and* (1.2). *Let a relatively compact open set Ω have the form $\{x \in \boldsymbol{R}^n \mid \varphi(x) < 0\}$ for a real valued real analytic function $\varphi(x)$ defined in a neighbourhood of $\bar{\Omega}$ satisfying* $\mathrm{grad}_x \varphi(x) \neq 0$ *on $\partial\Omega$. If the open set Ω satisfies condition* (2.7) *in Lemma* 2.2 *and condition* (3.12) *below, then $P(D)\mathcal{A}(\Omega) = \mathcal{A}(\Omega)$ holds.*

(3.12) *There exists a family of open sets $\{N_j\}_{j=1}^{p}$ which satisfy the following: For any point x in $\partial\Omega$ we can find some j such that N_j is a neighbourhood of x and that for any bicharacteristic curve $b_{(x,\xi)}$ of $P(D)$ issuing from (x, ξ) with $P_m(\xi) = 0$, $b_{(x,\xi)} \cap (\bar{\Omega} \setminus \{x\}) \cap N_j$ is connected.*

REMARK. If Ω is pseudo-convex with respect to $P(D)$ in the sense of Hörmander (Hörmander [1] Definition 8.6.1), then condition (3.12) is clearly satisfied. In fact the notion of pseudo-convexity in the sense of Hörmander concerns only the second order derivatives of $\varphi(x)$ along the bicharacteristics of $P(D)$, though condition (3.12) concerns the higher order derivatives also. The advantages of the way of presentation of condition (3.12) will be better realized when we treat linear differential operators with variable coefficients. (Kawai [7], [8]. See also Theorem 3.3′ and Corollary 3.4 of Kawai [3].)

PROOF OF THEOREM 3.2. We first modify the good elementary solutions of $P(D)$ so that they have smaller singular support. For that purpose we should allow them not to satisfy (1.3) but to satisfy only (3.13) and (3.14) below. By the flabbiness of sheaf $\mathcal{C}$ we can cut off the microfunction $E_+(x)$ to obtain a microfunction $\tilde{E}_+(x)$ such that

(3.13) $\quad P(D)\tilde{E}_+(x) = \delta(x) + \nu_+(x)\,,$

where

$\mathrm{S.S.}\, \tilde{E}_+(x) \subset \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x = 0 \text{ or } x = t\, \mathrm{grad}_\xi P_m(\xi) \ (0 \leqq t \leqq \varepsilon),$
$P_m(\xi) = 0\}$

and

$\mathrm{S.S.}\, \nu_+(x) \subset \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x = \varepsilon\, \mathrm{grad}_\xi P_m(\xi),\ P_m(\xi) = 0\}\,.$

Here ε is a sufficiently small positive constant which will be fixed later.

In the same way we can obtain a microfunction $\tilde{E}_-(x)$ which satisfies the following relation:

(3.14) $\quad P(D)\tilde{E}_-(x) = \delta(x) + \nu_-(x)\,,$

where

$$\text{S.S.}\,\tilde{E}_-(x) \subset \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x=0 \text{ or } x=t\,\mathrm{grad}_\xi P_m(\xi)\ (-\varepsilon \leqq t \leqq 0),\ P_m(\xi)=0\}$$

and

$$\text{S.S.}\,\nu_-(x) \subset \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x=-\varepsilon\,\mathrm{grad}_\xi P_m(\xi),\ P_m(\xi)=0\}\,.$$

By condition (3.12) we can find a positive ε so small that we can decompose the set $N=\{(y, \xi) \in S^*\boldsymbol{R}^n \mid y \in \partial\Omega,\ P_m(\xi)=0\}$ into the union of two closed sets N_+ and N_-, where $N_+=\{(y, \xi) \in S^*\boldsymbol{R}^n \mid y \in \partial\Omega$ and the portion of the bicharacteristic curve $b^{\varepsilon,+}_{(y,\xi)}: x=y+t\,\mathrm{grad}_\xi P_m(\xi)$ $(0 \leqq t \leqq \varepsilon)$ issuing from (y, ξ) with $P_m(\xi)=0$ does not intersect $\Omega\}$ and $N_-=\{(y, \xi) \in S^*\boldsymbol{R}^n \mid y \in \partial\Omega$ and the portion of the bicharacteristic curve $b^{\varepsilon,-}_{(y,\xi)}: x=y+t\,\mathrm{grad}_\xi P_m(\xi)$ $(-\varepsilon \leqq t \leqq 0)$ issuing from (y, ξ) with $P_m(\xi)=0$ does not intersect $\Omega\}$.

Using this decomposition of the set N we decompose $\tilde{f}(x)$ into the sum of two microfunctions $\tilde{f}_+(x)$ and $\tilde{f}_-(x)$ so that

$$\text{S.S.}\,\tilde{f}_+(x),\ \text{S.S.}\,\tilde{f}_-(x) \subset \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x \in \partial\Omega\} \tag{3.15}$$

and

$$\text{S.S.}\,\tilde{f}_+(x) \cap N \subset N_+ \quad \text{and} \quad \text{S.S.}\,\tilde{f}_-(x) \cap N \subset N_-\,. \tag{3.16}$$

Now consider a microfunction $v(x)$ defined by the following integrals along fibers.

$$v(x)=\int \tilde{E}_+(x-y)\tilde{f}_+(y)dy+\int \tilde{E}_-(x-y)\tilde{f}_-(y)dy\,. \tag{3.17}$$

As is usual we want to prove that $\text{S.S.}\,v(x) \cap S^*\Omega=\emptyset$. However, in this case we should first investigate what equation is satisfied by $v(x)$. By relations (3.13) and (3.14) we easily obtain

$$P(D)v(x)=\tilde{f}(x)+\int \nu_+(x-y)\tilde{f}_+(y)dy+\int \nu_-(x-y)\tilde{f}_-(y)dy \tag{3.18}$$

as an equation for microfunctions. We study the singularities of

$$\int \nu_+(x-y)\tilde{f}_+(y)dy \qquad \text{and} \qquad \int \nu_-(x-y)\tilde{f}_-(y)dy\,.$$

We first study that of $\int \nu_+(x-y)\tilde{f}_+(y)dy$. By (3.13) we have

$$\text{S.S.}\,\nu_+(x-y)\tilde{f}_+(y) \subset \bigcup_{j=1}^{3} T_{j,+}\,, \tag{3.19}$$

where

(i) $T_{1,+}=\{(x, y\,;\, \xi, \eta) \in S^*\boldsymbol{R}^{2n} \mid x=y+\varepsilon\,\mathrm{grad}_\zeta P_m(\zeta),\ P_m(\zeta)=0,\ (\xi, \eta)=(\alpha\zeta, -\alpha\zeta+\beta\theta),\ (y, \theta) \in \text{S.S.}\,\tilde{f}_+(y)$, where ζ is a non-zero real cotangent vector and $\alpha+\beta=1$ with $\alpha, \beta \geqq 0\}$,

(ii) $T_{2,+}=\{(x, y\,;\, \xi, \eta) \in S^*\boldsymbol{R}^{2n} \mid x=y+\varepsilon\,\mathrm{grad}_\zeta P_m(\zeta),\ P_m(\zeta)=0,\ y \notin \partial\Omega,\ \xi=-\eta=\zeta \neq 0\}$

and

(iii) $T_{3,+}=\{(x, y; \xi, \eta)\in S^*\boldsymbol{R}^{2n} \mid \xi=0,\ (y, \eta)\in \mathrm{S.S.}\tilde{f}_+(y),\ x\neq y+\varepsilon\,\mathrm{grad}_\zeta P_m(\zeta)$ with a real cotangent vector ζ satisfying $P_m(\zeta)=0\}$.

Therefore applying Lemma 1.3 to $\int \nu_+(x-y)\tilde{f}_+(y)dy$ we conclude by the definition of the set N_+ that

$$\mathrm{S.S.}\int \nu_+(x-y)\tilde{f}_+(y)dy \cap S^*\Omega_\delta=\emptyset, \tag{3.20}$$

where Ω_δ is a sufficiently small open neighbourhood of $\bar{\Omega}$, since the sets $T_{2,+}$ and $T_{3,+}$ do not give any contribution to $\mathrm{S.S.}\int \nu_+(x-y)\tilde{f}_+(y)dy$ and the contribution from the set $T_{1,+}$ to $\mathrm{S.S.}\int \nu_+(x-y)\tilde{f}_+(y)dy$ has a distance δ ($\delta\ll 1$) to $\bar{\Omega}$. Note that we have assumed in (3.12), so to speak, the strict convexity of Ω with respect to the bicharacteristic curve of $P(D)$ and also the analyticity of the boundary $\partial\Omega$. Hence we can choose such a positive constant δ.

The same arguments clearly succeed about the integral $\int \nu_-(x-y)\tilde{f}_-(y)dy$. Hence we have proved that

$$P(D)v(x)=\tilde{f}(x)+h(x), \tag{3.18$'$}$$

where $\mathrm{S.S.}\,h(x)\cap S^*\Omega_\delta=\emptyset$.

Now we investigate the singularities of $v(x)$. The investigation is quite similar to the proof of Theorem 3.1. In fact we have

$$\mathrm{S.S.}\,\tilde{E}_+(x-y)\tilde{f}_+(y)\subset \bigcup_{j=1}^{5} S'_{j,+}, \tag{3.21}$$

where $S'_{1,+}$ and $S'_{2,+}$ are the same as $S_{1,+}$ and $S_{2,+}$ in (3.7) respectively and the definitions of the sets $S'_{3,+}$, $S'_{4,+}$ and $S'_{5,+}$ are given by restricting the parameter t to the closed interval $[0, \varepsilon]$ in the definitions of the sets $S_{3,+}$, $S_{4,+}$ and $S_{5,+}$ in (3.7) respectively, where t has moved in the half line $[0, \infty)$.

Concerning $\mathrm{S.S.}\,E_-(x-y)\tilde{f}_-(y)$ we have the analogous relation by changing the sign of the parameter t and replacing $\mathrm{S.S.}\tilde{f}_+(y)$ by $\mathrm{S.S.}\tilde{f}_-(y)$.

Therefore we have by Lemma 1.3

$$\mathrm{S.S.}\,v(x)\subset \bigcup_{j=1}^{4} A'_j \tag{3.22}$$

where

(i) $A'_1=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid (x, \xi)\in \mathrm{S.S.}\tilde{f}_+(x)\}$

(ii) $A'_2=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid (x, \xi)\in \mathrm{S.S.}\tilde{f}_-(x)\}$

(iii) $A'_3=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x=y+t\,\mathrm{grad}_\xi P_m(\xi)\ \ (0\leqq t\leqq \varepsilon),\ \ P_m(\xi)=0,$ $(y, \xi)\in \mathrm{S.S.}\tilde{f}_+(y)\}$

and

(iv) $A'_4 = \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x = y + t\,\mathrm{grad}_\xi P_m(\xi)\ (-\varepsilon \leqq t \leqq 0), P_m(\xi) = 0,$
$(y, \xi) \in \mathrm{S.S.}\tilde{f}_-(y)\}.$

The precise reasonings of obtaining (3.22) from (3.21) and Lemma 1.3 are just the same as in the proof of Theorem 3.1, hence we do not repeat it here.

By (3.18′) and (3.22) we can find hyperfunctions $V(x)$, $H(x)$ and $G(x)$ for which the following conditions (3.23)~(3.26) are satisfied.

(3.23) $\qquad V(x)$ is real analytic in Ω.

(3.24) $\qquad H(x)$ is real analytic in $\Omega_\delta \supset \bar{\Omega}$.

(3.25) $\qquad G(x)$ belongs to $\mathcal{A}(\boldsymbol{R}^n)$.

and

(3.26) $\qquad P(D)V(x) = \tilde{f}(x) + H(x) + G(x)$ holds on $\boldsymbol{R}^n$.

These relations imply that the mapping $P(D): \mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega}) \to \mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega})$ is surjective under condition (3.12) only. Next we use the assumption (2.7) to eliminate $H(x)+G(x)$ in (3.26). In fact (3.24) and (3.25) immediately imply that $H(x)+G(x)$ belongs to $\mathcal{A}(\bar{\Omega})$, hence we can find a real analytic function $w(x)$ in Ω for which $P(D)w(x) = H(x)+G(x)$ holds in Ω. Therefore subtracting $w(x)$ from $V(x)$ we clearly obtain the required real analytic solution $u(x)$ of the equation $P(D)u(x) = f(x)$ in Ω. This completes the proof of the theorem.

§ 4. Construction of good elementary solutions (II) —the locally hyperbolic case—

In this section we firstly give the definition of locally hyperbolic operators with constant coefficients after Andersson [**1**] and secondly investigate the singularities of good elementary solutions of such an operator. By employing the theory of microfunctions we can clarify the structure of singularities of such elementary solutions so that the global existence theorem of real analytic solutions is obtained by their aid in § 5. We note that the class of linear differential operators satisfying Definition 4.2 is also considered independently by Kawai [**5**]. We also remark that the conditions on lower order terms imposed in Andersson [**1**] are redundant for our purpose, since we consider all problems in the framework of hyperfunctions. Hence the conditions on lower order terms are redundant even in studying the propagation of analyticity of solutions of such equations, which seems to be the main purpose of Andersson [**1**].

To begin with we give the definition of locally hyperbolic operators with constant coefficients after Andersson [**1**].

DEFINITION 4.1. (Cf. Andersson [1] Definition 3.1.) Let $a(\xi)$ be a homogeneous polynomial. For any positive ε, we define $V(a, \varepsilon)$ as the set of all mappings $v(\xi)$ from S^{n-1} to S^{n-1} which is continuous in a neighbourhood of $\{\xi \in S^{n-1} \mid a(\xi)=0\}$ and has the property that to every $\xi_0 \in S^{n-1}$ there are neighbourhoods U and V of ξ_0 and $v(\xi_0)$ respectively such that $a(\xi+s\theta) \neq 0$ if $(\xi, \theta) \in U \times V$, $|s| < \varepsilon$ and $\operatorname{Im} s \neq 0$. We also denote $\bigcup_{\varepsilon>0} V(a, \varepsilon)$ by $V(a)$.

REMARK. Andersson [1] calls the mapping $v(\xi)$ a vector field. The continuity of $v(\xi)$ in a neighbourhood of $\{a(\xi)=0\}$, is added here for the sake of simplicity. By a lemma due to Andersson [1] (Lemma 4.4 in the below) the assumption of continuity of $v(\xi)$ is sometimes superfluous, for example in the proof of Theorem 4.8.

DEFINITION 4.2. A linear differential operator $P(D)$ with constant coefficients of order m is called locally hyperbolic with respect to the mapping $v: S^{n-1} \to S^{n-1}$ if

$$P_m(v(\xi)) \neq 0 \qquad \text{for any} \quad \xi \in S^{n-1} \tag{4.1}$$

and

$$v(\xi) \in V(P_m) \tag{4.2}$$

hold.

The name of local hyperbolicity will be partly understood by the following Lemma 4.4 due to Andersson [1]. We remark that locally hyperbolic operators play the role of hyperbolic operators in the framework of microfunctions. (See for example Kawai [3] Theorems 3.3 and 3.5 and the remarks following them.) Before stating Lemma 4.4 we recall the definition of localization of linear differential operator with constant coefficients, which is due to Atiyah, Bott and Gårding [1]. There is another definition of localization due to Hörmander [2], but the definition of Atiyah, Bott and Gårding is more advantageous than that of Hörmander as far as we are concerned with real analyticity. (Hörmander's definition is more advantageous when one treats infinite differentiability.)

DEFINITION 4.3 (Atiyah, Bott and Gårding [1] Definition 3.26). Let $a(\xi)$ be a homogeneous polynomial of order m. The localization $a_\xi(\zeta)$ of the polynomial a at $\xi \in S^{n-1}$ is by definition the coefficients of the lowest order term in τ in the development of $a(\xi+\tau\zeta)$, namely

$$a(\xi+\tau\zeta) = \tau^p a_\xi(\zeta) + O(\tau^{p+1}), \quad \text{where} \quad a_\xi(\zeta) \not\equiv 0. \tag{4.3}$$

Then Definitions 4.2 and 4.3 clearly give the following

LEMMA 4.4 (Andersson [1] Corollary 3.1). *Let $a(D)$ be a homogeneous locally hyperbolic operator. Then $a_\xi(D)$ is hyperbolic with respect to $v(\xi_0)$ if ξ is sufficiently close to ξ_0.*

If we denote by $a(D)$ the principal part of a locally hyperbolic operator $P(D)$, then we may consider by the above lemma the inner core of the hyperbolic operator $a_{\xi_0}(D)$, i.e., the component of $\{\xi \in \boldsymbol{R}^n \mid a_{\xi_0}(\xi) \neq 0\}$ containing $v(\xi_0)$. We also denote the closed dual cone of the inner core by $K(a_{\xi_0}, v(\xi_0))$ or K_{ξ_0} for short. We call K_{ξ_0} a positive local propagation cone (relative to $v(\xi)$ with vertex at 0) and $-K_{\xi_0}$, i.e., the cone $\{-x \mid x \in K_{\xi_0}\}$, a negative local propagation cone (relative to $v(\xi)$ with vertex at 0). We also denote by $x \pm K_{\xi_0}$ the positive (negative) local propagation cone relative to $v(\xi)$ with vertex at x, i.e., the set $\{y \mid y = x + x',\ x' \in \pm K_{\xi_0}\}$.

Now we define the good elementary solutions of a locally hyperbolic differential operator $P(D)$ with constant coefficients.

DEFINITION 4.5. (Good elementary solutions of a locally hyperbolic operator.) A hyperfunction $E(x)$ satisfying the equation

$$P(D)E(x) = \delta(x) \tag{1.3}$$

is called a good elementary solution of the linear differential operator $P(D)$ with constant coefficients, which is locally hyperbolic with respect to $v(\xi)$, if it satisfies either condition $(4.4)_+$ or condition $(4.4)_-$.

$$(4.4)_+ \qquad \text{S.S. } E(x) \subset \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x = 0 \text{ or } x \in K_\xi \text{ with } P_m(\xi) = 0\}\,.$$

$$(4.4)_- \qquad \text{S.S. } E(x) \subset \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x = 0 \text{ or } x \in -K_\xi \text{ with } P_m(\xi) = 0\}\,.$$

REMARK. When the differential operator $P(D)$ satisfies conditions (1.1) and (1.2), then $P(D)$ becomes locally hyperbolic with respect to $v(\xi)$ satisfying $\langle v(\xi), \operatorname{grad}_\xi P_m(\xi)\rangle \neq 0$ when $P_m(\xi) = 0$ and Definition 4.5 coincides with Definition 1.1. Moreover if the principal part $P_m(D)$ of $P(D)$ is the p-th power $(p \geqq 2)$ of a homogeneous differential operator $Q(D)$ satisfying conditions (1.1) and (1.2), then the positive (negative) local propagation cone of $P(D)$ with vertex at x coincides with the positive (negative) half side of the bicharacteristic curve of $Q(D)$ through x. Hence in this case we can write down all the theorems in this section and in the next section only using the notion of the bicharacteristic curves, though we leave such restatement of the theorems to the reader.

The analysis developed in § 1 gives the following

THEOREM 4.6. *Suppose that the differential operator $P(D)$ is locally hyperbolic with respect to $v(\xi)$. Then we have two good elementary solutions $E_+(x)$ and $E_-(x)$ of $P(D)$ such that $E_+(x)$ satisfies condition $(4.4)_+$ and $E_-(x)$ satisfies condition $(4.4)_-$.*

PROOF. We can proceed just as in the proof of Theorem 1.2. In fact the convergence of the series given in (1.11) is proved there only assuming that $P_m(\zeta) \neq 0$ and that $\langle z, \zeta\rangle \neq 0$. Hence what remains to prove is to clarify the domain where $P_m(\zeta)\neq 0$. For that purpose we need the following lemma due to Andersson [1] on the semi-continuity of the inner core of the localized operator.

LEMMA 4.7. (Andersson [1] Lemma 3.4. See also Atiyah, Bott and Gårding [1] Lemma 5.9.) *Suppose that $P(D)$ is locally hyperbolic with respect to $v(\xi)$. Denote its principal symbol by $a(\xi)$. Assume that M is a compact subset of $\Gamma(a_{\xi_0}, v(\xi_0))$ for $\xi_0\in S^{n-1}$. Then $M\subset\Gamma(a_\xi, v(\xi_0))$ for any $\xi\in S^{n-1}$ sufficiently close to ξ_0.*

For the proof of this lemma we refer to Andersson [1] pp. 284 and 285 and Atiyah, Bott and Gårding [1] p. 151. We note that this lemma implies as its trivial corollary that $\bigcup_{\xi\neq 0} K_\xi$ is a closed subset of $\boldsymbol{R}^n$. By this lemma we can find an open set Ω in $\boldsymbol{C}^{2n}$ which touches the domain W along the real axis, where

(4.5) $W=\{(z, \zeta)\in\boldsymbol{C}^{2n}\mid \operatorname{Im}\langle z, \zeta\rangle>0,\ \operatorname{Im}\zeta\in-\Gamma(a_\xi, v(\xi)),\text{ where }\xi=\operatorname{Re}\zeta\}$.

Then by the inner semi-continuity of $\Gamma(a_\xi, v(\xi))$ or, in other words, by the outer semi-continuity of K_ξ, we conclude by (4.5) and the definition of microfunctions that

(4.6) $\mathrm{S.S.}\,F(x, \xi)\subset\{(x, \xi\,;\zeta_x, \zeta_\xi)\mid\langle x, \xi\rangle=0,\ a(\xi)=0,\ \zeta_x=\xi,\ \zeta_\xi=-\alpha\eta+\beta x,$ where $\eta\in K_\xi,\ \alpha+\beta=1$ and $\alpha, \beta\geqq 0\}\cup\{(x, \xi\,;\zeta_x, \zeta_\xi)\mid\langle x, \xi\rangle=0,$ $a(\xi)\neq 0,\ \zeta_x=\xi,\ \zeta_\xi=x\}\cup\{(x, \xi\,;\zeta_x, \zeta_\xi)\mid\langle x, \xi\rangle\neq 0,\ a(\xi)=0,\ \zeta_x=0,$ $\zeta_\xi=-\eta\in K_\xi$ with $\eta\neq 0\}$, where we have used the same notations as in the proof of Theorem 1.2, namely $F(x, \xi)$ is a hyperfunction in (x, ξ) defined as the boundary value of the holomorphic function defined by the series (1.11) in Ω and (ζ_x, ζ_ξ) denotes the (real) cotangent vector at (x, ξ).

Then we apply Lemma 1.3 to the integral along fiber

(4.7) $$E_+(x)=\frac{1}{(-2\pi\sqrt{-1})^n}\int_{|\xi|=1}F(x, \xi)\omega(\xi)\,,$$

and obtain from (4.6) that

(4.8) $$\mathrm{S.S.}\int_{|\xi|=1}F(x, \xi)\omega(\xi)\subset\{(x, \xi)\in S^*\boldsymbol{R}^n\mid x=0\text{ or }x\in K_\xi\text{ with }a(\xi)=0\}\,.$$

Therefore the hyperfunction $E_+(x)$ defined by (4.7) satisfies condition $(4.4)_+$. On the other hand it is clear by (1.6) and (1.9) that $E_+(x)$ satisfies (1.3), i.e., $P(D)E_+(x)=\delta(x)$. Thus we have constructed a good elementary solution $E_+(x)$ of $P(D)$. In the same way we can construct $E_-(x)$. In fact it is sufficient to vary $\operatorname{Im}\zeta$ in $\Gamma(a(\xi), v(\xi))$ in (4.5). Thus we have completed the proof of the theorem.

REMARK 1. By the aid of the good elementary solutions of a locally hyperbolic operator we have the following precise theorem concerning the propagation of analyticity of solutions of the equation $P(D)u=0$, which Andersson [1] seems not to give in the general form.

THEOREM 4.8. *Suppose that the differential operator $P(D)$ is locally hyperbolic with respect to $v(\xi)$. Then the singularity of the solution of the equation $P(D)u=0$ propagates only along the local propagation cones. More precisely, if there exist a positive constant c and an open set $V\subset S^*\boldsymbol{R}^n$ which contains the set $\{(x, \xi)\in S^*\boldsymbol{R}^n \mid \xi=\eta,\ x\in y-K_\eta$ and $0\leqq\langle y-x, v(\eta)\rangle\leqq c\}$ such that $P(D)u=0$ in V and that (y, η) is not contained in the set $\{(z, \xi)\in S^*\boldsymbol{R}^n \mid P_m(\xi)=0$ and $z\in x+K_\xi$, where $(x, \xi)\in \mathrm{S.S.}\, u(x)\cap\{\langle y-x, v(\xi)\rangle=0\}\}$, then (y, η) does not belong to* S.S. $u(x)$.

PROOF. By the assumption of the theorem we can choose a relatively compact open set $\Omega\subset S^*\boldsymbol{R}^n$ containing (y, η) so that the following condition holds.

(4.9) For any (x, ξ) in $\partial\Omega\cap \mathrm{S.S.}\, u(x)$, the set $\{(z, \xi)\in S^*\boldsymbol{R}^n \mid P_m(\xi)=0$ and $z\in x+K_\xi\}$ does not contain (y, η).

On the other hand since sheaf $\mathcal{C}$ is flabby, we can find a hyperfunction $\tilde{u}(x)$ so that $\tilde{u}(x)=u(x)$ in Ω, S.S. $\tilde{u}(x)$ is contained in $\bar{\Omega}$ and that the set $\{(z, \xi)\in S^*\boldsymbol{R}^n \mid P_m(\xi)=0$ and $z\in x+K_\xi\}$ does not contain (y, η) for any (x, ξ) in S.S. $P(D)\tilde{u}(x)$. Denote $P(D)\tilde{u}(x)$ by $\tilde{f}(x)$ and by $E_+(x)$ the good elementary solution of ${}^tP(D)$, the formal adjoint operator of $P(D)$, which satisfies condition $(4.4)_+$. Then the following integrals along fibers in (4.10) make sense and we can apply Lemma 1.3 to obtain the following equalities in (4.10) by the choice of $\tilde{u}(x)$.

$$
\begin{aligned}
(4.10)\qquad 0&=\int \tilde{f}(z)E_+(x-z)dz\\
&=\int P(D_z)\tilde{u}(z)E_+(x-z)dz\\
&=\int \tilde{u}(z)\,{}^tP(D_z)E_+(x-z)dz\\
&=\int \tilde{u}(z)\delta(x-z)dz=\tilde{u}(x)\quad \text{holds near}\quad (y, \eta).
\end{aligned}
$$

This proves the theorem.

REMARK 2. If the linear differential operator $P(D)$ with constant coefficients is hyperbolic with respect to the direction N, then Theorem 4.6 implies important results of Atiyah, Bott and Gårding [1], namely one can find an elementary solution $E(x)$ of $P(D)$ whose analytic singular support is contained in the wave front set, i.e., $\bigcup_{\xi\neq 0} K_\xi$, where we take the constant vector N as $v(\xi)$. (Atiyah, Bott and Gårding [1] Theorems 7.16 and 7.24.)

We emphasize again the fact that we need no conditions on lower order terms to obtain such an elementary solution.

One may have a question whether the elementary solution which we have constructed in Theorem 4.6 has its support in a proper convex cone. But we have constructed an elementary solution $F(x)$ of $P(D)$ with this property in Kawai [2] Theorems 6.1.3 and 6.3.1. Hence considering $F(x) - E_+(x)$ we easily see by Theorem 4.8 that $f(x) = F(x) - E_+(x)$ is real analytic on $\boldsymbol{R}^n$. Therefore $E_+(x) + f(x)$ has its support in a proper convex cone and its analytic singular support is contained in the wave front set. Thus some results of Atiyah, Bott and Gårding [1] follow trivially from Theorem 4.6.

§5. Global existence of real analytic solutions for locally hyperbolic operators.

In this section we present some theorems concerning the global existence of real analytic solutions of the equation $P(D)u = f$ assuming that the differential operator $P(D)$ is locally hyperbolic. The good elementary solutions of $P(D)$ constructed in Theorem 4.6 play an essential role in the proof. The method of the proof is quite similar to that employed in § 2 and § 3, hence we sometimes omit the details of the proof.

To begin with, we consider the case when $f(x)$ belongs to $\mathcal{A}(K)$ for a compact set K in $\boldsymbol{R}^n$.

LEMMA 5.1. *Suppose that the differential operator $P(D)$ is locally hyperbolic with respect to $v(\xi)$. Assume that the compact set K is the closure of the open set $\Omega = \{x \in \boldsymbol{R}^n \mid \varphi(x) < 0\}$, where $\varphi(x)$ is real valued real analytic function defined in a neighbourhood of K satisfying* $\operatorname{grad}_x \varphi(x) \neq 0$ *on $\partial\Omega$. If the compact set K satisfies the following geometrical condition* (5.1), *then for any $f(x)$ in $\mathcal{A}(K)$ we can find $u(x)$ in $\mathcal{A}(\Omega)$ such that $P(D)u(x) = f(x)$ holds in Ω.*

(5.1) *For any x_0 in $\partial\Omega$ where $P_m(\operatorname{grad}_x \varphi|_{x=x_0}) = 0$ holds, the positive local propagation cones $x_0 + K_{\pm \operatorname{grad}_x \varphi(x)|_{x=x_0}}$ with vertex at x_0 do not intersect Ω.*

PROOF. Consider a hyperfunction $\tilde{u}(x)$ defined by the following integral along fibers.

(5.2) $$\tilde{u}(x) = \int E_+(x-y) f(y) \theta(-\varphi(y)) dy,$$

where $E_+(x)$ is a good elementary solution constructed in Theorem 4.6 and $\theta(t)$ denotes the 1-dimensional Heaviside function.

Now we can proceed just in the same way as in the proof of Lemma 2.1. In fact it is sufficient to replace the term "$x = y + t \operatorname{grad}_\zeta P_m(\zeta)$ $(t \geqq 0)$" by "$x = y + z$ where z belongs to K_ζ" in the definition of the sets S_j, which show where the singularity of the integrand of (5.2) lies. Then we can apply Lemma 1.3 to the integral (5.2) and obtain the following:

(5.3) $\quad$ S.S. $\tilde{u}(x) \subset \{(x, \xi) \in S^*\boldsymbol{R}^n \mid$ there exists y in $\partial\Omega$ such that $\xi = \mathrm{grad}_y\, \varphi(y)$ or $-\mathrm{grad}_y\, \varphi(y)$ and $x \in y + K_\xi\}$.

Therefore the hyperfunction $\tilde{u}(x)$ is real analytic in Ω. As it is clear by (1.3) that $P(D)\tilde{u}(x) = f(x)\theta(-\varphi(x))$ holds, we obtain the required real analytic function $u(x)$ by restricting $\tilde{u}(x)$ to Ω. This completes the proof of the lemma.

As in Lemma 2.2 we can improve Lemma 5.1 a little by the aid of *both* good elementary solutions $E_+(x)$ and $E_-(x)$ of $P(D)$.

LEMMA 5.2. *Condition* (5.1) *in the previous lemma can be replaced by the following two conditions:*

(5.4) $\quad$ *For any characteristic boundary point* x_0, *i.e., the point in* $\partial\Omega$ *where* $P_m(\mathrm{grad}_x\, \varphi(x)|_{x=x_0}) = 0$ *holds, one of the local propagation cones with vertex at* x_0, $x_0 + K_\xi$ *or* $x_0 - K_\xi$ *does not intersect* Ω, *where* $\xi = \mathrm{grad}_x\, \varphi(x)|_{x=x_0}$ *or* $-\mathrm{grad}_x\, \varphi(x)|_{x=x_0}$.

(5.5) $\quad$ *There exists a neighbourhood* V *of* $\partial\Omega$ *for which the followings hold:*

(i) *For any characteristic boundary point* x_0, *if* $(x_0 + K_\xi) \cap \Omega \neq \emptyset$ *then* $V \cap (x_0 + K_\xi) \subset \Omega \cup \{x_0\}$. *Here* $\xi = \mathrm{grad}_x\, \varphi(x)|_{x=x_0}$ *or* $-\mathrm{grad}_x\, \varphi(x)|_{x=x_0}$.

(ii) *For any characteristic boundary point* x_0, *if* $(x_0 - K_\xi) \cap \Omega \neq \emptyset$ *then* $V \cap (x_0 - K_\xi) \subset \Omega \cup \{x_0\}$. *Here* $\xi = \mathrm{grad}_x\, \varphi(x)|_{x=x_0}$ *or* $-\mathrm{grad}_x\, \varphi(x)|_{x=x_0}$.

PROOF. We go on just in the same way as in the proof of Lemma 2.2. To begin with, we decompose the set $N = \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x \in \partial\Omega,\ P_m(\xi) = 0,\ \xi = \pm \mathrm{grad}_x\, \varphi(x)\}$ into the form $N_+ \cup N_-$, where $N_+ = \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x \in \partial\Omega,\ P_m(\xi) = 0,\ \xi = \pm\mathrm{grad}_x\, \varphi(x)$ and the positive local propagation cone $x + K_\xi$ with vertex at x does not intersect $\Omega\}$ and $N_- = \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x \in \partial\Omega,\ P_m(\xi) = 0,\ \xi = \pm\mathrm{grad}_x\, \varphi(x)$ and negative local propagation cone $x - K_\xi$ with vertex at x does not intersect $\Omega\}$. Then condition (5.5) combined with the outer semi-continuity of K_ξ (Lemma 4.7) implies that the sets N_+ and N_- are closed. Now we decompose the hyperfunction $\tilde{f}(x) = f(x)\theta(-\varphi(x))$ by the aid of flabbiness of sheaf $\mathcal{C}$ and the vanishing of $H^1(\boldsymbol{R}^n, \mathcal{A})$ so that

(5.6) $$\text{S.S.}\,(\tilde{f}(x) - \tilde{f}_+(x) - \tilde{f}_-(x)) = \emptyset\,,$$

(5.7) $$\text{S.S.}\,\tilde{f}_+(x) \cap N \subset N_+ \quad \text{and} \quad \text{S.S.}\,\tilde{f}_-(x) \cap N \subset N_-$$

and

(5.8) $$\text{S.S.}\,\tilde{f}_+(x),\ \text{S.S.}\,\tilde{f}_-(x) \subset S^*_{\partial\Omega}\boldsymbol{R}^n$$

hold.

Using these two hyperfunctions $\tilde{f}_+(x)$ and $\tilde{f}_-(x)$ we define a microfunction

$v(x)$ by

(5.9) $$v(x)=\int E_+(x-y)\tilde{f}_+(y)dy+\int E_-(x-y)\tilde{f}_-(y)dy\,.$$

The investigation of S.S. $v(x)$ is performed in a similar way to the proof of Lemma 2.2. In fact in order to investigate S.S. $E_+(x-y)\tilde{f}_+(y)$ and S.S. $E_-(x-y)\tilde{f}_-(y)$ it is sufficient to replace in (2.12) " $x=y+t\,\mathrm{grad}_\zeta P_m(\zeta)$ $(t\geqq 0)$ " by " $x\in y+K_\zeta$ " and to replace in (2.13) " $x=y+t\,\mathrm{grad}_\zeta P_m(\zeta)$ $(t\leqq 0)$ " by " $x\in y-K_\zeta$ ". Next we apply Lemma 1.3 to the integral along fiber (5.9), we easily see by (5.4) that

(5.10) $$\mathrm{S.S.}\, v(x)\cap S^*\Omega=\emptyset\,.$$

On the other hand by (1.3) we have $P(D)v(x)=\tilde{f}(x)$ as an equation for microfunctions, hence we can find a hyperfunction $V(x)$ which satisfies

(5.11) $$P(D)V(x)=\tilde{f}(x)+g(x)$$

where $V(x)\in \mathcal{A}(\Omega)$ and $g(x)\in\mathcal{A}(\boldsymbol{R}^n)$. Then we apply Lemma 5.1 to eliminate $g(x)$ from (5.11) and obtain the required real analytic solution $u(x)$ of the equation $P(D)u(x)=f(x)$ in Ω. This ends the proof of the lemma.

It is easy to modify Lemma 5.2 so that it gives the existence theorem in the space $\mathcal{A}(K)$ as we have done in Theorem 2.3, hence we leave the modification to the reader. See also Theorem 6.2.

We also have the following analogue of Theorem 2.4.

THEOREM 5.3. *Suppose that the differential operator $P(D)$ is locally hyperbolic with respect to $v(\xi)$. Assume that a compact set $K\subset\boldsymbol{R}^n$ is the closure of an open set $\Omega=\bigcap_{j=1}^{p}\Omega_j$, where each Ω_j satisfies the following regularity conditions* (5.12) *and* (5.13). *Moreover we assume that the compact set K satisfies the following geometrical conditions* (5.14) *and* (5.15). *Then for any $f(x)$ in $\mathcal{A}(K)$ we can find $u(x)$ in $\mathcal{A}(\Omega)$ such that $P(D)u(x)=f(x)$ holds in Ω.*

(5.12) *Ω_j is represented as $\{x\in\boldsymbol{R}^n\mid\varphi_j(x)<0\}$ by a real valued real analytic function $\varphi_j(x)$ defined in a neighbourhood of $\bar{\Omega}_j$.*

(5.13) *$\{\mathrm{grad}_x\,\varphi_{j_q}(x)\}_{q=1}^{k}$ are linearly independent as far as $\varphi_{j_1}(x)=\cdots=\varphi_{j_k}(x)=0$.*

(5.14) *If x_0 satisfies $\varphi_{j_1}(x)=\cdots=\varphi_{j_k}(x)=0$ then, for any non-zero ϑ that is a linear combination of $\mathrm{grad}_x\,\varphi_{j_1}(x)|_{x=x_0},\cdots,\mathrm{grad}_x\,\varphi_{j_k}(x)|_{x=x_0}$ and that satisfies $P_m(\vartheta)=0$, either $(x_0+K_\vartheta)\cap\Omega=\emptyset$ or $(x_0-K_\vartheta)\cap\Omega=\emptyset$ holds.*

(5.15) *There exists a neighbourhood V of $\partial\Omega$ for which the followings hold:*
(i) *For any (x_0,ϑ) in* (5.14), *if $(x_0+K_\vartheta)\cap\Omega\neq\emptyset$, then $V\cap(x_0+K_\vartheta)\subset\Omega\cup\{x_0\}$.*
(ii) *For any (x_0,ϑ) in* (5.14), *if $(x_0-K_\vartheta)\cap\Omega\neq\emptyset$, then $V\cap(x_0-K_\vartheta)\subset\Omega\cup\{x_0\}$.*

The proof of this theorem is quite similar to that of Theorem 2.4, hence we omit the details.

Now we present two theorems which assures the global existence of real analytic solution $u(x)$ of the equation $P(D)u(x)=f(x)$ where $f(x)$ belongs to $\mathcal{A}(\Omega)$ for an open set Ω.

THEOREM 5.4. *Suppose that the differential operator $P(D)$ is locally hyperbolic with respect to $v(\xi)$. Assume that a relatively compact open set Ω with C^1-boundary satisfies the following geometrical conditions* (5.16) *and* (5.17). *Then $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ holds.*

(5.16) *Let (x, ξ) belong to $N=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x\in\partial\Omega,\ P_m(\xi)=0\}$. Then either $(x+K_\xi)\cap\Omega=\emptyset$ or $(x-K_\xi)\cap\Omega=\emptyset$ holds.*

(5.17) *There exists a neighbourhood V of $\partial\Omega$ for which the following hold:*
(i) *For any (x, ξ) in N, if $(x+K_\xi)\cap\Omega\neq\emptyset$, then $V\cap(x+K_\xi)\subset\Omega\cup\{x\}$.*
(ii) *For any (x, ξ) in N, if $(x-K_\xi)\cap\Omega\neq\emptyset$, then $V\cap(x-K_\xi)\subset\Omega\cup\{x\}$.*

PROOF. To begin with, we extend $f(x)$ to $\boldsymbol{R}^n$ by the aid of the flabbiness of the sheaf of germs of hyperfunctions so that the extension $\tilde{f}(x)$ coincides with $f(x)$ in Ω and has its support in $\bar{\Omega}$. Then it is clear that $\mathrm{S.S.}\tilde{f}(x)\subset\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x\in\partial\Omega\}$. We decompose the set N into the union of N_+ and N_-, where $N_+=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x\in\partial\Omega,\ P_m(\xi)=0$ and $(x+K_\xi)\cap\Omega=\emptyset\}$ and $N_-=\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x\in\partial\Omega,\ P_m(\xi)=0$ and $(x-K_\xi)\cap\Omega=\emptyset\}$. By condition (5.17) and the outer semi-continuity of K_ξ (Lemma 4.7) we easily see that N_+ and N_- are closed. Then we use the flabbiness of sheaf C to decompose $\tilde{f}(x)$ into the sum of two hyperfunctions $\tilde{f}_+(x)+\tilde{f}_-(x)$ as microfunctions so that

$$\mathrm{S.S.}\tilde{f}_+(x),\ \mathrm{S.S.}\tilde{f}_-(x)\subset\{(x, \xi)\in S^*\boldsymbol{R}^n \mid x\in\partial\Omega\}\,, \tag{5.18}$$

$$\mathrm{S.S.}\tilde{f}_+(x)\cap N\subset N_+ \quad\text{and}\quad \mathrm{S.S.}\tilde{f}_-(x)\cap N\subset N_-\,. \tag{5.19}$$

Now we define a microfunction $v(x)$ by the following integral:

$$v(x)=\int E_+(x-y)\tilde{f}_+(y)dy+\int E_-(x-y)\tilde{f}_-(y)dy\,. \tag{5.20}$$

As is usual, what is important is to investigate $\mathrm{S.S.}E_+(x-y)\tilde{f}_+(y)$ and $\mathrm{S.S.}E_-(x-y)\tilde{f}_-(y)$. It is easy to see, for example, that

(5.21) $\mathrm{S.S.}E_+(x-y)\tilde{f}_+(y)\subset\bigcup_{j=1}^{5}S_{j,+}$, where $S_{j,+}$ is defined in analogous way to that given in (3.7). We have only to replace "$x=y+t\operatorname{grad}_\zeta P_m(\zeta)$ $(t\geqq 0)$" by "$x\in y+K_\zeta$" in the definition.

An analogous statement also holds for $\mathrm{S.S.}E_-(x-y)\tilde{f}_-(y)$. Therefore Lemma 1.3 shows that

$$\mathrm{S.S.}v(x)\cap S^*\Omega=\emptyset\,. \tag{5.22}$$

Up to now $v(x)$ satisfies the equation $P(D)v(x)=\tilde{f}(x)$ only as microfunctions. But we can find a real analytic function $u(x)$ which satisfies the equation $P(D)u(x)=f(x)$ in Ω by Lemma 5.2. Since the procedure is the same as in the proof of Theorem 3.1, we omit the details. This ends the proof of the theorem.

As in Theorem 3.2 we modify Theorem 5.4 in order to localize condition (5.16) as follows

THEOREM 5.5. *Suppose that the differential operator $P(D)$ is locally hyperbolic with respect to $v(\xi)$. Let a relatively compact open set Ω have the form $\{x\in \boldsymbol{R}^n \mid \varphi(x)<0\}$ for a real valued real analytic function $\varphi(x)$ defined in a neighbourhood of $\bar{\Omega}$ satisfying $\mathrm{grad}_x\, \varphi(x)\neq 0$ on $\partial\Omega$. If the open set Ω satisfies conditions* (5.4) *and* (5.17) *and condition* (5.23) *below, then $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ holds.*

(5.23) *There exists a family of open sets $\{N_j\}_{j=1}^p$ which satisfy the following: For any point x in $\partial\Omega$ we can find some neighbourhood N_j of x such that either $(x+K_\xi)\cap(\bar{\Omega}\setminus\{x\})\cap N_j=\emptyset$ or $(x-K_\xi)\cap(\bar{\Omega}\setminus\{x\})\cap N_j=\emptyset$ holds for any non-zero real cotangent vector satisfying $P_m(\xi)=0$.*

Erratum in Kawai [7]. In Theorem 1 of Kawai [7], we have missed out condition (5.17). We add condition (5.17) to Theorem 1 of Kawai [7].

PROOF. We proceed in the same way as in the proof of Theorem 3.2. To begin with, we modify good elementary solutions $E_+(x)$ and $E_-(x)$ in the following way.

(5.24) $P(D)\tilde{E}_+(x)=\delta(x)+\nu_+(x)$, where $\mathrm{S.S.}\,\tilde{E}_+(x)\subset\{(x,\xi)\in S^*\boldsymbol{R}^n \mid x=0$ or $x\in K_\xi\cap\{|x|\leqq\varepsilon\}$, where $P_m(\xi)=0\}$

and

$\mathrm{S.S.}\,\nu_+(x)\subset\{(x,\xi)\in S^*\boldsymbol{R}^n \mid x\in\{|x|=\varepsilon\}\cap K_\xi$, where $P_m(\xi)=0\}$.
Here ε is a sufficiently small positive number.

(5.25) $P(D)\tilde{E}_-(x)=\delta(x)+\nu_-(x)$, where $\mathrm{S.S.}\,\tilde{E}_-(x)\subset\{(x,\xi)\in S^*\boldsymbol{R}^n \mid x=0$ or $x\in -K_\xi\cap\{|x|\leqq\varepsilon\}$, where $P_m(\xi)=0\}$

and

$\mathrm{S.S.}\,\nu_-(x)\subset\{(x,\xi)\in S^*\boldsymbol{R}^n \mid x\in\{|x|=\varepsilon\}\cap(-K_\xi)$, where $P_m(\xi)=0\}$.

These microfunctions $\tilde{E}_+(x)$ and $\tilde{E}_-(x)$ are obtained by the aid of the flabbiness of sheaf $\mathcal{C}$. Then we define a microfunction $v(x)$ by the following integral:

$$v(x)=\int\tilde{E}_+(x-y)\tilde{f}_+(y)dy+\int\tilde{E}_-(x-y)\tilde{f}_-(y)dy\,. \tag{5.26}$$

The microfunction $v(x)$ satisfies

$$P(D)v(x)=\tilde{f}(x)+h(x)\,, \tag{5.27}$$

where

$$\mathrm{S.S.}\, h(x) \cap \{(x, \xi) \in S^*\boldsymbol{R}^n \mid x \in \bar{\Omega}\} = \emptyset .$$

The relation (5.27) is obtained by conditions (5.23), (5.24) and (5.25) in a similar way to obtaining (3.18′).

On the other hand S.S. $\tilde{E}_+(x-y)\tilde{f}_+(y)$ and S.S. $\tilde{E}_-(x-y)\tilde{f}_-(y)$, hence S.S. $v(x)$, can be studied in a similar way to obtaining (3.21) and (3.22). There is nothing new in the reasoning, so we omit the details. As a conclusion we can prove surjectivity of the mapping

$$P(D): \mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega}) \longrightarrow \mathcal{A}(\Omega)/\mathcal{A}(\bar{\Omega}) \tag{5.28}$$

under conditions (5.17) and (5.23). Then we may apply Lemma 5.2 by conditions (5.4) and (5.17) to deduce the surjectivity of the differential operator $P(D)$ from $\mathcal{A}(\Omega)$ to $\mathcal{A}(\Omega)$ from the surjectivity of the mapping given in (5.28). Thus we have completed the proof of the theorem.

We end this section by the following remarks.

REMARK 1. When the space dimension $n=2$, then any linear differential operator with constant coefficients becomes locally hyperbolic operator. Moreover in this case following three notions, i.e., bicharacteristic curve, the union of positive and negative local propagation cones and the characteristic line, coincide. Hence we have the following seemingly strong result as a corollary of Theorem 5.4.

THEOREM 5.7. *Assume that the space dimension n is equal to* 2. *Let a relatively compact domain Ω in $\boldsymbol{R}^2$ satisfy the following condition:*

(5.29) *Any characteristic line of $P(D)$ intersects Ω in an open interval.*

Then $P(D)\mathcal{A}(\Omega)=\mathcal{A}(\Omega)$ holds.

We remark that condition (5.29) was found by Hörmander [1] (and Ehrenpreis) in the discussion of global existence of C^∞-solutions. We also note that if $\Omega=\boldsymbol{R}^2$ and the differential operator $P(D)$ is homogeneous, i.e., $P(D) = P_m(D)$, then $P(D)\mathcal{A}(\boldsymbol{R}^2)=\mathcal{A}(\boldsymbol{R}^2)$ follows trivially from Theorem α in § 0. In fact if the differential operator $P(D)$ is hyperbolic (in $\boldsymbol{R}^n$) then $P(D)\mathcal{A}(\boldsymbol{R}^n) = \mathcal{A}(\boldsymbol{R}^n)$ follows from the uniqueness of Cauchy problem (see Kawai [7] Theorem 6, where only strictly hyperbolic operators (with variable coefficients) are treated. But Remark 2 after Theorem 4.6 can be used to generalize the above quoted theorem to this case), hence at least the case $P(D)=P_m(D)$ can be easily treated if $n=2$.

REMARK 2. If the differential operator $P(D)$ is elliptic, then $P(D)$ is clearly a locally hyperbolic operator by the definition. Moreover it is clear that $K_\xi=\{0\}$ for any $\xi\neq 0$. Therefore we can prove Theorem α in § 0 under the assumption that Ω is relatively compact by the method employed in this section.

§ 6. Remarks on the global existence theorems —duality method—

In this section we restrict ourselves to the consideration of surjectivity of the differential operator

$$P(D): \mathcal{A}(K) \longrightarrow \mathcal{A}(K),$$

where K is a compact set in $\boldsymbol{R}^n$.

Since the linear topological space $\mathcal{A}(K)$ becomes a DFS-space endowed with its topological structure by $\varinjlim_{V \supset K} \mathcal{O}(V)$ (see §0), we can apply Serre's duality theorem. (About Serre's duality theorem we refer to Komatsu [1] for example.) When the differential operator $P(D)$ has complex coefficients, the results given in this section are at present better than those given by the use of elementary solutions constructed in Kawai [4], which we do not discuss here. (See also Kawai [6] Theorem 2.)

Throughout this section we denote by $\mathrm{Ch}\, K$ the convex hull of a compact set K, which is again compact as is well known.

THEOREM 6.1. *Suppose that the differential operator $P(D)$ satisfies conditions* (1.1) *and* (1.2). *Assume that a compact set K in $\boldsymbol{R}^n$ satisfies the following condition* (6.1). *Then $P(D)\mathcal{A}(K) = \mathcal{A}(K)$ holds.*

(6.1) *For any (x, ξ) in $S^*\boldsymbol{R}^n$ such that x belongs to $\mathrm{Ch}\, K$ but not to K and such that ξ satisfies $P_m(\xi) = 0$, there is a point y outside $\mathrm{Ch}\, K$ for which the segment $\overline{xy}$ does not intersect K and is contained in the bicharacteristic curve of $P(D)$ issuing from (x, ξ).*

PROOF. Since $\mathcal{A}(K)$ is a DFS-space and the space $\mathcal{B}_K$, the space of hyperfunctions support in K, is the dual space of $\mathcal{A}(K)$, it is sufficient to prove that the operator ${}^tP(D): \mathcal{B}_K \to \mathcal{B}_K$ has the closed range and is injective, where ${}^tP(D)$ is the formal adjoint operator of $P(D)$.

Since the space $\mathcal{B}_K$ is an FS-space, it is sufficient to prove that ${}^tP(D)\mathcal{B}_K$ is sequentially closed. Assume that μ_n belongs to $\mathcal{B}_K$ and that ${}^tP(D)\mu_n = \nu_n$ converges to ν in $\mathcal{B}_K$. As $\mathcal{B}_K$ is naturally imbedded into $\mathcal{B}_{\mathrm{Ch}\, K}$, we regard μ_n, ν_n and ν as elements in $\mathcal{B}_{\mathrm{Ch}\, K}$. Then by the definition ν_n converges to ν in the topology of $\mathcal{B}_{\mathrm{Ch}\, K}$. Since $\mathrm{Ch}\, K$ is convex, we can prove by the usual Fourier transformation techniques (see e. g. Kawai [2] Theorem 4.2.4, especially its method of the proof) that ${}^tP(D)\mathcal{B}_{\mathrm{Ch}\, K}$ is closed in $\mathcal{B}_{\mathrm{Ch}\, K}$. Therefore we can find a hyperfunction μ with its support in $\mathrm{Ch}\, K$, which satisfies the equation ${}^tP(D)\mu = \nu$. By the definition, $\mathrm{supp}\, \nu \subset K$. What we want to prove is that $\mathrm{supp}\, \mu$ is also contained in K. On the other hand we have a complete result concerning the propagation of analyticity of hyperfunction solutions in Kawai [3] Theorem 3.3′. This theorem combined with condition (6.1) immediately implies that the point (x, ξ) with x in $\mathrm{Ch}\, K \backslash K$ does not belong to

S.S. $\mu(x)$, since (y, ξ) does not belong to S.S. $\mu(x)$ and $\nu(x)$ vanishes in $\operatorname{Ch} K\backslash K$. Therefore $\mu(x)$ is real analytic in $\operatorname{Ch} K\backslash K$. Since $\mu(x)$ is zero outside $\operatorname{Ch} K$ by the definition and any point in $\operatorname{Ch} K\backslash K$ can be connected by a line outside K to a point outside $\operatorname{Ch} K$ by condition (6.1), we conclude that $\mu(x)$ is zero outside K. This implies that supp μ is contained in K. Therefore we have proved that ${}^tP(D)\mathscr{B}_K$ is closed in $\mathscr{B}_K$. Then by Serre's duality theorem we have proved $P(D)\mathscr{A}(K)$ is closed in $\mathscr{A}(K)$. On the other hand the injectivity of the operator ${}^tP(D): \mathscr{B}_K \to \mathscr{B}_K$ is clear, we have proved that $P(D)\mathscr{A}(K) = \mathscr{A}(K)$. This completes the proof of the theorem.

REMARK. If we can connect the sets K and $\operatorname{Ch} K$ by a family of real analytic surfaces, namely, if we can find a real valued real analytic function $\varphi(x)$ defined in a neighbourhood of $\operatorname{Ch} K$ satisfying

$$\{x \mid \varphi(x) \leqq 1\} = K \quad \text{and} \quad \{x \mid \varphi(x) \leqq 2\} \supset \operatorname{Ch} K \tag{6.2}$$

and

$$\operatorname{grad}_x \varphi(x) \neq 0 \quad \text{in} \quad \operatorname{Ch} K\backslash K, \tag{6.3}$$

then we may find that the cotangent vector ξ which plays an essential role in the proof is only $\pm\operatorname{grad}_x \varphi(x)$ satisfying $P_m(\operatorname{grad}_x \varphi(x)) = 0$. This fact is easily seen by Theorem 3.3′ in Kawai [3] and the abstract form of Holmgren's uniqueness theorem for hyperfunctions depending real analytically on a parameter. (About the abstract form of Holmgren's uniqueness theorem we refer to Kawai [1] Corollary of Theorem 5.1.1.) We will deal with these subjects more precisely also for the overdetermined systems of linear differential equations (with constant coefficients) in our forthcoming papers and omit the details here. See also Kawai [8] § 3.

By Theorem 4.8, the following theorem is proved in a similar way to the above proof.

THEOREM 6.2. *Suppose that the differential operator $P(D)$ is locally hyperbolic with respect to $v(\xi)$. Assume that a compact set K in $\boldsymbol{R}^n$ satisfies the following condition* (6.4). *Then $P(D)\mathscr{A}(K) = \mathscr{A}(K)$ holds.*

(6.4) *For any (x, ξ) in $S^*\boldsymbol{R}^n$ such that x belongs to* $\operatorname{Ch} K$ *but not to K and such that ξ satisfies $P_m(\xi) = 0$, either $(x + K_\xi) \cap K = \emptyset$ or $(x - K_\xi) \cap K = \emptyset$ holds. Here $x \pm K_\xi$ denotes the positive* (*negative*) *local propagation cone relative to $v(\xi)$ with vertex at x.*

For the proof of this theorem we only need to apply Theorem 4.8 instead of Theorem 3.3′ of Kawai [3]. Hence we leave it to the reader.

Now we consider the case where the principal symbol of $P(D)$ has complex coefficients. In this case we use the theorem of propagation of analyticity for hyperfunction solutions which is proved by the contact transformation. (See Sato, Kawai and Kashiwara [1].) We also note that an analogous result for propagation of analyticity of *distribution* solutions, which is unsatisfactory for our purpose, is proved by Andersson [2].

THEOREM 6.3. *Suppose that the differential operator $P(D)$ satisfies condition (6.5) below. Assume that a compact set K in $\boldsymbol{R}^n$ satisfies the following condition (6.6). Then $P(D)\mathcal{A}(K)=\mathcal{A}(K)$ holds.*

(6.5) *When we denote the principal symbol $P_m(\xi)$ of $P(D)$ by $A_m(\xi)+\sqrt{-1}\,B_m(\xi)$, where $A_m(\xi)$ and $B_m(\xi)$ are real valued for real ξ, $\mathrm{grad}_\xi A_m(\xi)$ and $\mathrm{grad}_\xi B_m(\xi)$ are linearly independent whenever $P_m(\xi)=0$, $\xi\neq 0$.*

(6.6) *Let us denote by $\Lambda_{(x_0,\xi^0)}$ the bicharacteristic plane of $P(D)$ through (x_0, ξ^0), where $P_m(\xi^0)=0$ is satisfied, i.e., the 2-dimensional linear variety passing through x_0 which is spanned by $\mathrm{grad}_\xi A_m(\xi)|_{\xi=\xi^0}$ and $\mathrm{grad}_\xi B_m(\xi)|_{\xi=\xi^0}$. Then for any bicharacteristic plane Λ of $P(D)$, $\Lambda\cap(\mathrm{Ch}\,K\backslash K)$ has no relatively compact component.*

PROOF. We proceed in a similar way to the proof of Theorem 6.1 by the aid of the duality between $\mathcal{A}(K)$ and $\mathcal{B}_K$. We want to prove that the assumption that ${}^tP(D)\mu_n=\nu_n$ converges to ν in $\mathcal{B}_K$, where μ_n belongs to $\mathcal{B}_K$, implies the existence of $\mu\in\mathcal{B}_K$ for which ${}^tP(D)\mu=\nu$ holds. By the Fourier transformation techniques we can find a hyperfunction μ in $\mathcal{B}_{\mathrm{Ch}\,K}$ for which ${}^tP(D)\mu=\nu$ holds. Then the result concerning the propagation of analyticity of hyperfunction solutions, i.e., the fact that singularities propagate along the bicharacteristic plane, obviously implies the analyticity of μ in $\mathrm{Ch}\,K\backslash K$ by condition (6.6) since μ belongs to $\mathcal{B}_{\mathrm{Ch}\,K}$ and ν belongs to $\mathcal{B}_K$. Therefore, using the fact that μ belongs to $\mathcal{B}_{\mathrm{Ch}\,K}$ again, we immediately see that μ vanishes in $\mathrm{Ch}\,K\backslash K$, namely we have proved that μ belongs to $\mathcal{B}_K$. Therefore we have proved the surjectivity of the operator $P(D):\mathcal{A}(K)\to\mathcal{A}(K)$ by Serre's duality theorem. This ends the proof of the theorem.

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