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On some dissipative boundary value problems for the Laplacian

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§1. Introduction.

Let Ω be a bounded domain in $\boldsymbol{R}^{n+1}$ with boundary Γ of class C^∞. $\bar{\Omega}=\Omega\cup\Gamma$ is a C^∞-manifold with boundary. For a function u in $C^\infty(\bar{\Omega})$ and $s\in\boldsymbol{R}$, $\|u\|_s$ denotes Sobolev norm of u of order s.

We consider Laplacian $\Delta=\frac{\partial^2}{\partial x_1{}^2}+\cdots+\frac{\partial^2}{\partial x_{n+1}{}^2}$ in Ω together with the homogeneous boundary condition

$$\mathscr{B}u\Big|_\Gamma=\frac{\partial u}{\partial\nu}+(a+ib)u+(a_0+ib_0)u\Big|_\Gamma=0\,,$$

where ν is the unit exterior normal to Γ, a and b are real C^∞-vector fields on Γ and a_0 and b_0 are real C^∞-functions on Γ.

The following problem is still open: "Characterize those couples $(\Omega, \mathscr{B})$ for which there exists a constant C such that the estimate

$$-\mathrm{Re}\,(\Delta u, u)+C\|u\|_0{}^2\geqq 0 \tag{1.1}$$

holds for any u in $C^2(\bar{\Omega})$ satisfying $\mathscr{B}u|_\Gamma=0$.

A well-known necessary condition for the estimate (1.1) to hold is that

$$|b(x)|\leqq 1\,, \tag{1.2}$$

where $|b(x)|$ is the length of the vector $b(x)$, $x\in\Gamma$ ([6]). On the other hand if $|b(x)|<1$ at every point $x\in\Gamma$, then there exist constants $C_0>0$ and C_1 such that the estimate

$$-\mathrm{Re}\,(\Delta u, u)+C_1\|u\|_0{}^2\geqq C_0\|u\|_1{}^2 \tag{1.3}$$

holds for any u in $C^2(\bar{\Omega})$ satisfying $\mathscr{B}u|_\Gamma=0$. (J. L. Lions [8], see also [1], [6], [10].)

In this note assuming (1.2), we are concerned with the following estimate:

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$$-\mathrm{Re}\,(\Delta u, u)+C_1\|u\|_0^2 \geqq C_0\|u\|_{\frac{1}{2}}^2, \qquad C_0>0, \quad C_1=\text{Const.} \tag{1.4}$$

for any u in $C^2(\bar{\Omega})$ satisfying $\mathcal{B}u|_{\Gamma}=0$.

In the previous paper [3] in more general situation one of us proved a necessary and sufficient condition for the estimate (1.4) to hold. Combining this with the recent result of Anders Melin [9], we can write down a necessary and sufficient condition (Theorem 1). Since this condition is rather implicit, here we shall also give a necessary condition and a sufficient one which are more explicit.

Our estimate (1.4) can be localized. Lions' result implies that difficulty occurs at every point x_0 on Γ where $|b(x_0)|=1$. If we set $l(x)=1-|b(x)|^2$, the assumption (1.2) means that $l(x)\geqq 0$ and $l(x)$ vanishes at x_0. We shall consider the Hessian $L(x_0)$ of $l(x)$ at x_0. Identifying the tangent space $T_{x_0}(\Gamma)$ of Γ at x_0 with its dual by natural metric, we consider $L(x_0)$ as a linear transformation in $T_{x_0}(\Gamma)$.

The vector field $a(x)$ on Γ induces a linear map $\nabla_* a$; $T_{x_0}(\Gamma)\to T_{x_0}(\Gamma)$ defined by the covariant differentiation $\xi\to\nabla_\xi a$ (cf. [7]). $\omega_{x_0}(\xi,\eta)$, $\xi, \eta\in T_{x_0}(\Gamma)$, will denote the second fundamental form at x_0 of the hypersurface $\Gamma\subset \boldsymbol{R}^{n+1}$. $M(x_0)$ will denote the first mean curvature at x_0 of Γ. Let X be a tangent vector to $T^*(\Gamma)$ at the point $(x_0, b(x_0))$, where $b(x_0)$ is identified with a cotangent vector in $T^*_{x_0}(\Gamma)$. X can be decomposed into the sum of its horizontal component ξ and vertical component η. Since $T_{(x_0,b(x_0))}(T^*(\Gamma))$ has its natural symplectic structure σ, the vertical component η can be identified with a cotangent vector to Γ, which will again be denoted as η. The horizontal component ξ can be identified with a tangent vector $\in T_{x_0}(\Gamma)$. Under this identification the following quadratic form has an intrinsic meaning;

$$X=(\xi,\eta)\longrightarrow \frac{1}{2}(|\eta|^2-\langle b(x_0),\eta\rangle^2)-\langle \nabla_\xi b,\eta\rangle+\frac{1}{4}\langle \xi, L(x_0)\xi\rangle+\frac{1}{2}|\nabla_\xi b|^2.$$

Let A be the matrix expression of this form with respect to the symplectic structure σ. Eigenvalues of iA are real and $-\mu$ is an eigenvalue of iA if μ is. As is shown in § 4, iA has at most $n-1$ positive eigenvalues, which we denote by $\mu_1(x_0)\cdots\mu_{n-1}(x_0)$. Then Anders Melin's result combined with our previous result gives the following

THEOREM 1. *The estimate* (1.4) *holds for any function u in $C^2(\bar{\Omega})$ satisfying $\mathcal{B}u|_{\Gamma}=0$ if and only if the following conditions hold:*

(i) *At every point x on Γ, $|b(x)|\leqq 1$.*

(ii) *At every point x_0 on Γ where $|b(x_0)|=1$, the following inequality holds;*

$$\mu_1(x_0)+\cdots+\mu_{n-1}(x_0)+\omega_{x_0}(b(x_0), b(x_0))-nM(x_0)+2a_0(x_0)-\mathrm{Tr}\,\nabla_* a>0. \tag{1.5}$$

Estimating the sum $\mu_1(x_0)+\cdots+\mu_{n-1}(x_0)$, we have the following theorems.

THEOREM 2. *If the estimate* (1.4) *holds, then the following two conditions hold:*

(i) $|b(x)|\leqq 1$ *at every point* x *on* Γ.

(ii) *At every point* x_0 *on* Γ *where* $|b(x_0)|=1$ *the following inequality holds;*

$$\sqrt{n-1}\Big(\frac{1}{2}\operatorname{Tr}(L(x_0))+\operatorname{Tr}{}^t(\nabla_* b)(\nabla_* b)-\operatorname{Tr}(\nabla_* b)^2 - \frac{1}{2}\langle b(x_0), L(x_0)b(x_0)\rangle-|\nabla_{b(x_0)}b|^2\Big)^{\frac{1}{2}} + \omega_{x_0}(b(x_0), b(x_0))-nM(x_0)+2a_0(x_0)-\operatorname{Tr}\nabla_* a>0. \tag{1.6}$$

THEOREM 3. *The estimate* (1.4) *holds for any* u *in* $C^2(\bar{\Omega})$ *with* $\mathscr{B}u|_\Gamma=0$ *if the following conditions* (a) *and* (b) *hold:*

(a) $|b(x)|\leqq 1$ *at every point* x *on* Γ.

(b) *At every point* x_0 *where* $|b(x_0)|=1$ *the following inequality holds;*

$$\Big(\frac{1}{2}\operatorname{Tr} L(x_0)+\operatorname{Tr}{}^t(\nabla_* b)(\nabla_* b)-\operatorname{Tr}(\nabla_* b)^2 - \frac{1}{2}\langle b(x_0), L(x_0)b(x_0)\rangle-|\nabla_{b(x_0)}b|^2\Big)^{\frac{1}{2}} + \omega_{x_0}(b(x_0), b(x_0))-nM(x_0)+2a_0(x_0)-\operatorname{Tr}\nabla_* a>0. \tag{1.7}$$

COROLLARY 4. *In the case* $n=2$, *the estimate* (1.4) *holds if and only if the conditions* (i) *and* (ii) *of Theorem* 2 *hold.*

§ 2. Green's formula.

Let S be the unit circle, whose generic point will be denoted by s. For any $C^\infty(\bar{\Omega}\times S)$-function $u(x, s)$, its restriction $\varphi(x, s)$ to $\Gamma\times S$ is a $C^\infty(\Gamma\times S)$-function. We can uniquely solve the Dirichlet problem:

$$\Big(\Delta+\frac{\partial^2}{\partial s_1{}^2}\Big)w(x, s)=0 \quad \text{in} \quad \Omega\times S \tag{2.1}$$
$$w(x, s)|_{\Gamma\times S}=\varphi(x, s) \quad \text{on} \quad \Gamma\times S.$$

We shall denote by $\mathscr{P}$ the Poisson operator $\varphi\to w$. Setting $v=u-w$, we have decomposition of any $C^\infty(\bar{\Omega}\times S)$ function u:

$$u=v+w. \tag{2.2}$$

If u satisfies the boundary condition

$$\mathscr{B}u(x, s)|_{\Gamma\times S}=0, \quad x\in\Gamma, \quad s\in S,$$

then Green's formula implies that

$$(2.3)\qquad -\mathrm{Re}\iint_{\Omega\times S}\Big(\Delta+\frac{\partial^2}{\partial s^2}\Big)u\bar{u}\,dx\,ds$$
$$=-\mathrm{Re}\iint_{\Omega\times S}\Big(\Delta+\frac{\partial^2}{\partial s^2}\Big)v\bar{v}\,dx\,ds+\mathrm{Re}\iint_{\Gamma\times S}T\varphi(x,s)\overline{\varphi(x,s)}d\gamma\,ds\,,$$

where $d\gamma$ is the hypersurface element of Γ. The operator T is a pseudo-differential operator of order 1 on $\Gamma\times S$ defined by

$$(2.4)\qquad T\varphi=\frac{\partial P\varphi}{\partial\nu}\Big|_{\Gamma\times S}+(a+ib)\varphi+(a_0+ib_0)\varphi\,.$$

In [4] it is proved that the estimate (1.4) holds if and only if the following estimate

$$(2.5)\qquad \mathrm{Re}\,(T\varphi,\varphi)+C_1\|\varphi\|_{-\frac{1}{2}}{}^2\geqq C_0\|\varphi\|_0{}^2\,,\qquad C_0>0\,,\quad C_1=\mathrm{Const.}$$

holds for any φ in $C^\infty(\Gamma\times S)$.

In the next section we shall calculate the symbol of $\mathrm{Re}\,T$ near an arbitrary point x_0 on Γ.

§ 3. Symbol of T.

Poisson operators can be described, modulo C^∞-operators, by their symbols (cf. [2], [3], [5], [10]). First we shall calculate the symbol of P in our case and next that of the operator T. We will take the following coordinate system: We fix an arbitrary point x_0 on Γ. We make y_{n+1}-axis coincide with the direction of the interior normal and the hyperplane $y_{n+1}=0$ coincide with the tangent hyperplane of Γ at x_0.

Then Ω is given by

$$(3.1)\qquad y_{n+1}-\varphi(y')>0\,,$$

where $\varphi(y')$ is a C^∞-function of variables $y'=(y_1,\cdots,y_n)$. We may assume that the Taylor expansion of $\varphi(y')$ is given by

$$(3.2)\qquad \varphi(y')=\sum_j\omega_jy_j{}^2+\sum_{ijk}\omega_{ijk}y_iy_jy_k+0(|y|^4)$$

where ω_j, ω_{ijk} are constants satisfying

$$(3.3)\qquad \omega_{kij}=\omega_{ijk}=\omega_{jik}\,.$$

Whenever we take summations with respect to indices $i, j, k, \cdots$, these indices range from 1 to n independently. Einstein's convention will not be used.

For any two real vectors $\xi=(\xi_1,\cdots,\xi_n)$, $\eta=(\eta_1,\cdots,\eta_n)$ tangent to Γ at x_0, the bilinear form $\omega_{x_0}(\xi,\eta)=2\sum_j\omega_j\xi_j\eta_j$ is the second fundamental form.

$2/n\sum_j \omega_j$ is the mean curvature $M(x_0)$ of Γ at x_0 (cf. [7]).

Now we choose a new coordinate system $x=(x', x_{n+1})$, $x'=(x_1, \cdots, x_n)$ given by

$$y_i = x_i - \frac{x_{n+1}\dfrac{\partial\varphi}{\partial x_i}}{\sqrt{1+\sum_i\left(\dfrac{\partial\varphi}{\partial x_i}\right)^2}}, \qquad i=1, 2, 3, \cdots, n, \tag{3.4}$$

$$y_{n+1}=\varphi(x')+\frac{x_{n+1}}{\sqrt{1+\sum_i\left(\dfrac{\partial\varphi}{\partial x_i}\right)^2}}.$$

In fact, the Jacobian $\dfrac{D(y_1, \cdots, y_{n+1})}{D(x_1, \cdots, x_{n+1})}$ equals one at the origin and $(x_1, \cdots, x_{n+1})$ can be a coordinate system in some neighbourhood of the origin. We have the Taylor expansion

$$\begin{aligned} x_i &= y_i+2\omega_i y_i y_{n+1}+4\omega_i^2 y_i y_{n+1}^2-2\omega_i y_i(\sum_j \omega_j y_j^2) \\ &\qquad +3y_{n+1}\sum_{jk}\omega_{ijk}y_j y_k+O(|y|^4), \end{aligned} \tag{3.5}$$

$$x_{n+1}=y_{n+1}-\sum_i \omega_i y_i^2-2(\sum_i \omega_i^2 y_i^2)y_{n+1}-\sum_{ijk}\omega_{ijk}y_i y_j y_k+O(|y|^4).$$

The metric is

$$\begin{aligned} ds^2 &= dy_1^2+\cdots+dy_{n+1}^2 \\ &= \sum_i (1-2\omega_i x_{n+1})^2 dx_i^2+4(\sum_i \omega_i x_i dx_i)^2 \\ &\quad -12x_{n+1}\sum_{ijk}\omega_{ijk}x_k dx_i dx_j+dx_{n+1}^2+O(|x|^3|dx|^2). \end{aligned}$$

The symbol of the Laplacian $-\left(\Delta+\dfrac{\partial^2}{\partial s^2}\right)$ on $\Omega\times S$ is given by

$$\begin{aligned} &\sum_j (1+4\omega_j x_{n+1}+12\omega_j^2 x_{n+1}^2+O(|x|^3))\xi_j^2+\xi_{n+1}^2+\sigma^2 \\ &\quad +\sum_{ij}(12x_{n+1}(\sum_k \omega_{ijk}x_k)-4\omega_i\omega_j x_i x_j+O(|x|^3))\xi_i\xi_j \\ &\quad -2i\sum_j(-2(\sum_i \omega_i\omega_j x_j)+3x_{n+1}(\sum_i \omega_{iij})+O(|x|^2))\xi_j \\ &\quad +2i(\sum_j \omega_j+2x_{n+1}\sum_j \omega_j^2+3\sum_{ij}\omega_{iij}x_j+O(|x|^2))\xi_{n+1}. \end{aligned} \tag{3.6}$$

Let $A_2(x, \xi', \xi_{n+1}, \sigma)$ denote the principal symbol of $-\left(\Delta+\dfrac{\partial^2}{\partial s^2}\right)$. This is a polynomial of ξ_{n+1} of degree 2. τ^+ (τ^-) denotes the root of $A_2(x', 0, \xi', \xi_{n+1}, \sigma)$ with positive (negative, respectively) imaginary part. $\tau^\pm$ has the Taylor expansion

$$\tau^\pm=\pm i(\rho_1-2\rho_1^{-1}\sum_{ij}\omega_i\omega_j x_i x_j\xi_i\xi_j+O(|x|^3)) \tag{3.7}$$

where

$$\rho_1=(|\xi'|^2+\sigma^2)^{\frac{1}{2}}, \quad \xi'=(\xi_1, \cdots, \xi_n).$$

Let $f(x, \xi, \sigma)=f_{-2}(x, \xi, \sigma)+f_{-3}(x, \xi, \sigma)+\cdots$ be the symbol of the fundamental solution $\mathcal{F}$ of $-\Delta-\frac{\partial^2}{\partial s^2}$ and its asymptotic expansion. Then the principal symbol is

$$\begin{aligned}(3.8)\qquad f_{-2}(x, \xi, \sigma)=\rho^{-2}&-4\rho^{-4}x_{n+1}(\sum_j \omega_j\xi_j^2)\\ &+16\rho^{-6}x_{n+1}^2(\sum_j \omega_j\xi_j^2)^2-12\rho^{-4}x_{n+1}^2(\sum_j \omega_j^2\xi_j^2)\\ &-\rho^{-4}\sum_{ij}(12x_{n+1}(\sum_k \omega_{ijk}x_k)-4\omega_i\omega_j x_i x_j)\xi_i\xi_j\\ &+O(|x|^3)\rho^{-2},\end{aligned}$$

where $\rho=\sqrt{|\xi|^2+\sigma^2}$. The second symbol is

$$(3.9)\qquad f_{-3}(x, \xi, \sigma)=-2i\xi_{n+1}(4\rho^{-6}\sum_j \omega_j\xi_j^2+\rho^{-4}\sum_j \omega_j)+O(|x|)\rho^{-3}.$$

Now we shall denote by T^+ a pseudo-differential operator on $\Gamma\times S$ with the symbol τ^+. We consider the mapping $Q: C_0^\infty(\Gamma\times S)\to Q\varphi\in C^\infty(\bar{\Omega}\times S)$ defined by

$$(3.10)\qquad Q\varphi=\mathcal{F}\Big(i\frac{\partial\delta(\Gamma\times S)}{\partial\nu}\otimes\varphi-\delta(\Gamma\times S)\otimes T^+\varphi\Big),$$

where $\delta(\Gamma\times S)\otimes\varphi$ is the distribution defined by

$$\mathcal{D}(\boldsymbol{R}^{n+1}\times S)\ni\phi\longrightarrow\int_{\Gamma\times S}\phi|_{\Gamma\times S}\varphi\, d\gamma\, ds.$$

Since the mapping Q is defined by (3.10), Q is a pseudo-Poisson operator in the sense of Boutet de Monvel (cf. [2], [3], [5], [10]). Its symbol has an asymptotic expansion with respect to homogeneous degree of (x_{n+1}^{-1}, ξ'). Following Theorem A.8 in appendix of [3], we shall calculate a few terms of it. The symbol of Q is given by the formula:

$$\begin{aligned}&ie^{ix_{n+1}\tau^+}+(2\pi)^{-1}\int_{-\infty}^{\infty}x_{n+1}\frac{\partial}{\partial x_{n+1}}f_{-2}(x', 0, \xi', \xi_{n+1}, \sigma)(\xi_{n+1}-\tau^-)e^{ix_{n+1}\xi_{n+1}}d\xi_{n+1}\\ &\quad-(2\pi)^{-1}\int_{-\infty}^{\infty}\sum_j\frac{\partial}{\partial\xi_j}f_{-2}(x', 0, \xi', \xi_{n+1}, \sigma)\Big(-i\frac{\partial}{\partial x_j}\tau^-\Big)e^{ix_{n+1}\xi_{n+1}}d\xi_{n+1}\\ &\quad+(2\pi)^{-1}\int_{-\infty}^{\infty}f_{-3}(x', 0, \xi', \xi_{n+1}, \sigma)(\xi_{n+1}-\tau^-)e^{ix_{n+1}\xi_{n+1}}d\xi_{n+1}\\ &\quad+O((x_{n+1}\rho_1^{-1})^2)\\ &=ie^{-x_{n+1}(\rho_1-2\rho_1^{-1}\sum_{ij}\omega_i\omega_j x_i x_j\xi_i\xi_j+O(|x'|^3))}\\ &\quad-ix_{n+1}(\sum_i\omega_j\xi_j^2+3\sum_{ijk}\omega_{ijk}x_k\xi_i\xi_j+O(|x'|^2))(2x_{n+1}\rho_1^{-1}+\rho_1^{-2})e^{-x_{n+1}\rho_1}\end{aligned}$$

$$+(\sum_{jk}\omega_j\omega_k\xi_k x_k\xi_j^2\rho_1^{-1})(2x_{n+1}\rho_1^{-2}+2\rho_1^{-3}+O(|x'|))e^{-x_{n+1}\rho_1}$$

$$+i\Big((\sum_j\omega_j\xi_j^2)\Big(\frac{x_{n+1}^2}{\rho_1}-\frac{1}{2\rho_1^3}\Big)+i(\sum_i\omega_i)(x_{n+1}-(2\rho_1)^{-1})\Big)e^{-\rho_1 x_{n+1}}$$

where $\rho_1=\sqrt{|\xi'|^2+\sigma^2}$.

This implies that the mapping $K: \varphi\to Q\varphi|_{\Gamma\times S}$ is an elliptic pseudo-differential operator of order 0. Its symbol is

$$i\Big(1-\frac{1}{2}(\sum_j\omega_j\xi_j^2)\rho_1^{-3}-\frac{1}{2}(\sum_j\omega_j)\rho_1^{-1}+\cdots\Big).$$

Let $K^{(-1)}$ be its parametrix. Then as is proved in [**3**], [**5**] and [**10**], the operator

$$QK^{(-1)}\varphi=\mathcal{F}\Big(i\frac{\partial\delta(\Gamma\times S)}{\partial\nu}\otimes K^{(-1)}\varphi-\delta(\Gamma\times S)\otimes T^{+}K^{(-1)}\varphi\Big)$$

coincides with the operator $\mathcal{P}$ modulo smooth operators. The symbol of the operator $\mathcal{P}$ is

$$e^{-x_{n+1}(\rho_1-2\rho_1^{-1}(\sum_j\omega_j x_j\xi_j)^2+O(|x'|^2))}\tag{3.12}$$
$$-x_{n+1}((\sum_j\omega_j\xi_j^2)\rho_1^{-2}+O(|x'|))e^{-x_{n+1}\rho_1}$$
$$+x_{n+1}(\sum_j\omega_j+O(|x'|))e^{-x_{n+1}\rho_1}+O((x_{n+1}\rho_1^{-1})^2).$$

So the mapping $\varphi\to\left.\frac{\partial\mathcal{P}\varphi}{\partial\nu}\right|_{\Gamma\times S}$ is a pseudo-differential operator with its principal symbol

$$\rho_1-2\rho_1^{-1}(\sum_j\omega_j x_j\xi_j)^2+O(|x'|^2).\tag{3.13}$$

And its second symbol is

$$(\sum_j\omega_j\xi_j^2)\rho_1^{-2}-\sum_j\omega_j+O(|x'|).\tag{3.14}$$

Now we assume that the real vector fields a and b are expressed as follows:

$$a(x)=\sum_j(\alpha_j+\sum_k\alpha_{jk}x_k+O(|x'|^2))\frac{\partial}{\partial x_j},\tag{3.15}$$

$$b(x)=\sum_j(\beta_j+\sum_k\beta_{jk}x_k+\sum_{kl}\beta_{jkl}x_k x_l+O(|x'|^3))\frac{\partial}{\partial x_j}.\tag{3.16}$$

Using these, we can write down the Taylor expansion of the symbol of the operator $\mathrm{Re}\,T$. Its principal symbol $t_1(x, s, \xi, \sigma)$ is

$$t_1(x, s, \xi, \sigma)=\rho_1-2\rho_1^{-1}(\sum_j\omega_j x_j\xi_j)^2\tag{3.17}$$
$$-\sum_j(\beta_j+\sum_k\beta_{jk}x_k+\sum_{kl}\beta_{jkl}x_k x_l)\xi_j$$
$$+O(|x'|^3)\rho_1.$$

And the second symbol $t_0(x, s, \xi, \sigma)$ is

$$t_0(x, s, \xi, \sigma)=\sum_j \omega_j \xi_j^2 \rho_1^{-2}-\sum_j \omega_j+a_0(x_0)-\frac{1}{2}\sum_j \alpha_{jj}+O(|x'|)\,.$$

We have the following coordinate free expression:

$$t_0(x, s, \xi, \sigma)=\frac{1}{2}\rho_1^{-2}\omega_{x_0}(\xi, \xi)-\frac{n}{2}M(x_0)+a_0(x_0)-\frac{1}{2}\operatorname{Tr} \nabla_* a\,. \tag{3.18}$$

§ 4. Proof of Theorems.

Assume that $l(x_0)=0$ at x_0 on Γ. This means that

$$\sum_j \beta_j^2=1 \tag{4.1}$$

if we make use of the coordinate expression. The condition $l(x)\geqq 0$ implies that

$$\sum_j \beta_j \beta_{jk}=0 \tag{4.2}$$

and that its Hessian $L(x_0)$ at x_0 is a non-negative symmetric matrix. Using coordinates expressions (3.15) and (3.16), we obtain

$$\frac{1}{2}\langle x, L(x_0)x\rangle=-4(\sum_j \omega_j x_j \beta_j)^2-\sum_j(\sum_k \beta_{jk}x_k)^2-2\sum_{jkl}\beta_j\beta_{jkl}x_k x_l\,, \tag{4.3}$$

where $\langle\ ,\ \rangle$ is the inner product in the tangent space $T_{x_0}(\Gamma)$. Since $|\nabla_\eta b|^2=\sum_j(\sum_k \beta_{jk}\eta_k)^2$ for any $\eta=(\eta_1, \cdots, \eta_n)$ in $T_{x_0}(\Gamma)$, we have

$$-4(\sum_j \omega_j \eta_j \beta_j)^2-2\sum_{jkl}\beta_j\beta_{jkl}\eta_k\eta_l=\frac{1}{2}\langle \eta, L(x_0)\eta\rangle+|\nabla_\eta b|^2\,. \tag{4.4}$$

The principal symbol t_1 vanishes at the point where $x=0$, $\xi=\beta$, $\sigma=0$. We can calculate its Hessian $H(x_0)$ there. We have

$$\begin{aligned}\frac{1}{2}\ll X, HX\gg=\frac{1}{2}(\sum_j \eta_j^2+\sigma^2)-\frac{1}{2}(\sum_j \beta_j\eta_j)^2-\sum_{pk}\beta_{pk}x_k\eta_p\\ -2(\sum_j \omega_j\beta_j x_j)^2-\sum_{jkl}\beta_j\beta_{jkl}x_k x_l\,,\end{aligned} \tag{4.5}$$

where X is a column vector $(x_1, \cdots, x_n, \eta_1, \cdots, \eta_n, s, \sigma)$ of $(2n+2)$ components. $\ll X, Y\gg$ is the inner product which is the polarization of the quadratic form $X\to\sum_j x_j^2+\sum_j \eta_j^2+s^2+\sigma^2$. We introduce $n\times n$ matrices:

$$A=\begin{pmatrix}\omega_1^2\beta_1^2, \omega_1\omega_2\beta_1\beta_2 \cdots\cdots & \omega_1\omega_n\beta_1\beta_n\\ \omega_2\omega_1\beta_1\beta_2 \cdots\cdots\cdots\cdots & \vdots\\ \vdots & \\ \omega_n\omega_1\beta_1\beta_n \cdots\cdots\cdots\cdots & \omega_n^2\beta_n^2\end{pmatrix}\qquad B=\begin{pmatrix}\sum_j \beta_j\beta_{j11} \cdots & \sum_j \beta_j\beta_{j1n}\\ \sum_j \beta_j\beta_{j21} \cdots & \vdots\\ \vdots & \\ \sum_j \beta_j\beta_{jn1} \cdots & \sum_j \beta_j\beta_{jnn}\end{pmatrix}$$

$$C=\begin{pmatrix} \beta_{11} & \cdots\cdots & \beta_{1n} \\ \vdots & & \vdots \\ \beta_{n1} & \cdots\cdots & \beta_{nn} \end{pmatrix} \qquad D=\begin{pmatrix} \beta_1^2 & \cdots\cdots & \beta_1\beta_n \\ \vdots & & \vdots \\ \beta_n\beta_1 & \cdots\cdots & \beta_n^2 \end{pmatrix}$$

$$I=\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

We have

$$H=\begin{pmatrix} -4A-2B, & -{}^tC, & 0, & 0 \\ -C\ , & I-D, & 0, & 0 \\ 0\ , & 0\ , & 0, & 0 \\ 0\ , & 0\ , & 0, & 1 \end{pmatrix}.$$

Finally we introduce $2n+2$ square matrix $J=\begin{pmatrix} J_0, & 0, & 0 \\ 0, & 0, & -1 \\ 0, & 1, & 0 \end{pmatrix}$ where J_0 is the $2n\times 2n$ matrix $J_0=\begin{pmatrix} 0, & -I \\ I, & 0 \end{pmatrix}$. Since H is given as the Hessian of the principal symbol t_1 which is non-negative if we assume that $|b(x)|\leqq 1$, H is a non-negative matrix. This implies that eigenvalues of JH are pure imaginary. If λ is its eigenvalue, then its complex conjugate is also its eigenvalue. The matrix $\begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}\begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix}$ is nilpotent. The positive eigenvalues of iJH coincide with those of iJ_0H_0, H_0 being the matrix $H_0=\begin{pmatrix} -4A-2B, & -{}^tC \\ -C, & I-D \end{pmatrix}$. Since the principal symbol of the operator $\operatorname{Re} T$ is homogeneous in (ξ, σ), the rank of H_0 is at most $2n-1$. This implies that the number of positive eigenvalues of iJ_0H_0 is at most $n-1$. Let $\mu_1(x_0), \cdots, \mu_{n-1}(x_0)$ denote non-negative eigenvalues of iJ_0H_0. Anders Melin's theorem leads us to

THEOREM 1. *The estimate* (1.4) *holds for any function u in $C^2(\bar{\Omega})$ satisfying $\mathscr{B}u|_\Gamma=0$ if and only if the following conditions hold:*

1) *At every point x on Γ, $|b(x)|\leqq 1$.*

2) *At every point x_0 on Γ where $|b(x_0)|=1$, the following inequality holds:*

$$\frac{1}{2}(\mu_1(x_0)+\cdots+\mu_{n-1}(x_0))+\operatorname{Re} t_0>0, \tag{4.6}$$

where

$$\operatorname{Re} t_0=\frac{1}{2}\omega_{x_0}(b(x_0), b(x_0))-\frac{n}{2}M(x_0)+a_0(x_0)-\frac{1}{2}\operatorname{Tr} \nabla_* a\,.$$

Since

$$(\mu_1^2+\cdots+\mu_{n-1}^2)^{\frac{1}{2}}\leqq\mu_1+\cdots+\mu_{n-1}\leqq\sqrt{n-1}\,(\mu_1^2+\cdots+\mu_{n-1}^2)^{\frac{1}{2}},$$

we have

$$\left(-\frac{1}{2}\operatorname{Tr}(J_0H_0)^2\right)^{\frac{1}{2}}\leqq\mu_1(x_0)+\cdots+\mu_{n-1}(x_0)\leqq\sqrt{n-1}\left(-\frac{1}{2}\operatorname{Tr}(J_0H_0)^2\right)^{\frac{1}{2}}. \tag{4.7}$$

Using coordinate expression, we have

$$\begin{aligned}\operatorname{Tr}(J_0H_0)^2&=2(\operatorname{Tr}C^2-\operatorname{Tr}(I-D)(-4A-2B))\\&=2\Big(\sum_{pk}\beta_{pk}\beta_{kp}+4\sum_j\omega_j^2\beta_j^2+2\sum_{ij}\beta_i\beta_{ijj}\\&\qquad-4\sum_{jk}\omega_j\omega_k\beta_j^2\beta_k^2-2\sum_{ijk}\beta_i\beta_k\beta_j\beta_{ijk}\Big).\end{aligned} \tag{4.8}$$

Using (4.4) we obtained that

$$\begin{aligned}\operatorname{Tr}(J_0H_0)^2=2\Big(&-\frac{1}{2}\operatorname{Tr}L(x_0)-\operatorname{Tr}{}^t(\nabla_*b)(\nabla_*b)+\operatorname{Tr}(\nabla_*b)^2\\&+\frac{1}{2}\langle b(x_0),L(x_0)b(x_0)\rangle+|\nabla_{b(x_0)}b|^2\Big).\end{aligned} \tag{4.9}$$

Combining these, we have proved

THEOREM 2. *If the estimate* (1.4) *holds for any u in $C^2(\bar{\Omega})$ satisfying $\mathcal{B}u|_\Gamma=0$, then the following conditions hold:*

i) *At every point x on Γ, $|b(x)|\leqq1$.*

ii) *At every point x_0 on Γ where $|b(x_0)|=1$ the following inequality holds:*

$$\begin{aligned}\sqrt{n-1}\Big(&\frac{1}{2}\operatorname{Tr}L(x_0)+\operatorname{Tr}{}^t(\nabla_*b)(\nabla_*b)-\operatorname{Tr}(\nabla_*b)^2\\&-\frac{1}{2}\langle b(x_0),L(x_0)b(x_0)\rangle-|\nabla_{b(x_0)}b|^2\Big)^{\frac{1}{2}}\\&+\omega_{x_0}(b(x_0),b(x_0))-nM(x_0)+2a_0(x_0)-\operatorname{Tr}\nabla_*a>0.\end{aligned} \tag{4.10}$$

And we also have

THEOREM 3. *The estimate* (1.4) *holds for any u in $C^2(\bar{\Omega})$ satisfying $\mathcal{B}u|_\Gamma=0$ if the following two conditions hold:*

(a) *At every point $x\in\Gamma$, $|b(x)|\leqq1$.*

(b) *At every point x_0 where $|b(x_0)|=1$, the inequality* (4.11) *holds;*

$$\begin{aligned}\Big(&\frac{1}{2}\operatorname{Tr}L(x_0)+\operatorname{Tr}{}^t(\nabla_*b)(\nabla_*b)-\operatorname{Tr}(\nabla_*b)^2\\&-\frac{1}{2}\langle b(x_0),L(x_0)b(x_0)\rangle-|\nabla_{b(x_0)}b|^2\Big)^{\frac{1}{2}}\\&+\omega_{x_0}(b(x_0),b(x_0))-nM(x_0)+2a_0(x_0)-\operatorname{Tr}\nabla_*a>0.\end{aligned} \tag{4.11}$$

Corollary 4 is a trivial consequence of these theorems.

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