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Geometrical operations of Whitehead groups

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§ 1. Introduction.

In the paper, we study PL manifolds which are related by h-cobordisms.

Suppose that we are given a PL manifold M of dimension $n \geqq 5$ with $\pi_1(M) = G$. An element τ of $\mathcal{Wh}(G)$ operates on M in such a way that the result $M \cdot \tau$ of the operation is the right end of an h-cobordism U from the left end M with $\tau(U, M) = \tau$. This operation is called an interior operation. A result of Milnor ([**15**], Theorem 11.5) was concerned with the inertia group of this interior operation. If M is located on the boundary of a PL manifold W of dimension $n+1$, then we obtain a new PL manifold pair $(W \cup U, M \circ \tau)$, of which we may think as to be obtained from (W, M) by an operation of τ. This operation is called a boundary operation. A study of the boundary operation gives us a rough information about PL homeomorphism classes of compact PL manifolds whose interiors are PL homeomorphic.

In order to make rigorous definitions of these operations (especially the boundary operation) we need Corollary 2.3 which is deduced from the existence and uniqueness Theorem of embedded h-cobordisms (Theorems 2.1 and 2.2). These are slight modifications of results for abstract h-cobordisms due to Stallings ([**18**], p. 250) and Milnor ([**15**], Theorem 11.3) and may be well-known.

In § 3, we give the precise definition of the interior and boundary operations and obtain an extension of Milnor's result for boundary operations, see Theorem 3.4. In particular, for a compact PL manifold W of dimension $n = \text{odd} \geqq 7$, it is proved that there are finitely many distinct PL homeomorphism classes (respectively h-cobordism classes) of compact PL manifolds whose interiors are PL homeomorphic to $Int\ W$, provided that $\pi_1(bW)$ is finite and that $\pi_1(W) = 1$ (resp. $\pi_1(bW) \cong \pi_1(W)$), see Corollary 3.5.

Suppose that we are given an abstract regular neighborhood N of a polyhedron P. For each element τ of $\mathcal{Wh}(\pi_1(bN))$, we have a new neighborhood $N \cup U$ of P as the result of a boundary operation of τ on (N, bN), where U

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is an h-cobordism from bN with $\tau(U, bN)=\tau$. This neighborhood may not be, in general, a regular neighborhood of P. Thus we extend the notion of regular neighborhoods to the notion of homotopy neighborhoods (h-neighborhoods) in § 4. In virtue of Theorems 2.1 and 2.2, we can establish a theory of h-neighborhoods, which may be regarded as extension and improvement of Mazur's simple neighborhood theory [**13**]. A characterization of regular neighborhoods is obtained from the viewpoint of simple homotopy theory.

In § 5, we investigate operations of Whitehead torsions on higher dimensional knots and their cone ball pairs. A striking consequence is that given a higher dimensional knot such that the Whitehead group of the knot group is non-trivial, then we can produce a counter example of either of collapsing and singularity, see Corollary 5.4.

It follows from results by Siebenmann and Sondow ([**16**] and [**17**]) that there are counter examples for both topological invariance problems, see also [**8**].

Finally, suppose that we are given a homotopy equivalence $f: Q\rightarrow P$. Roughly speaking, we may consider the simple homotopy type of P as to be obtained from an operation of $\tau(f)$ on the simple homotopy type of Q. Further, if N is a regular neighborhood of P in R^n and if $n\geqq 2\cdot\dim Q+2$, then N turns out to be a homotopy neighborhood of Q in R^n such that $(N, \partial N)=(N', \partial N')\circ\tau(f)$ for a regular neighborhood N' of Q in R^n, see Lemma 6.1. Thus the operation of a Whitehead torsion on a polyhedron can be covered by the boundary operation of its regular neighborhood. From this viewpoint, we prove two consequences in § 6. One is that two polyhedra are homotopy equivalent (resp. simple homotopy equivalent) if and only if their Thom complexes are homeomorphic (resp. PL homeomorphic), see Theorem 6.2. The other is a reduction of topological invariance of simple homotopy types to ε-push invariance of regular neighborhoods of polyhedra in sufficiently high dimensional euclidean space, which should be compared with our preceding result [8].

Added in proof. Waldhausen has recently announced that the Whitehead torsion is topologically invariant. Thus abstract regular neighborhoods of a polyhedron with codimension $\geqq 3$ is topologically invariant, provided their dimension $\geqq 6$, see Corollary 4.4 and [8].

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§ 2. Embedded h-cobordisms.

CONVENTIONS. All polyhedra are to be compact, manifolds are to be (orientable and) oriented and homeomorphisms of manifolds are to be orientation preserving. For the notion of Whitehead torsion ([**19**] and [**15**]), we shall follow mainly Milnor [**15**]. However, for our purpose, the Whitehead torsion $\tau(f)$ of a homotopy equivalence $f: P \to Q$ between polyhedra P and Q is defined to be an element

$$(\tau(f|P_1), \cdots, \tau(f|P_q)) \quad \text{in} \quad \mathcal{W}h(\pi_1(P_1)) \times \cdots \times \mathcal{W}h(\pi_1(P_q)),$$

if P has the connected components $P_1, \cdots, P_q$. We extend naturally the notations for Whitehead torsions to those for sequences of Whitehead torsions. For example, if $\tau = (\tau_1, \cdots, \tau_q)$, then $\bar{\tau} = (\bar{\tau}_1, \cdots, \bar{\tau}_q)$. Further, for notational convenience, by $\mathcal{W}h(\pi_1(P))$ we denote the group $\mathcal{W}h(\pi_1(P_1)) \times \cdots \times \mathcal{W}h(\pi_1(P_q))$.

DEFINITIONS. Let W be a manifold of dimension n. By bW and Int W, we denote the boundary and the interior of W, respectively. Let M be a submanifold of dimension $(n-1)$ in bW. We call the pair (W, M) a *manifold couple* of dimension n. By a *manifold triad* of dimension n we mean a triple $(U; M, L)$ consisting from a manifold U of dimension n, submanifolds M and L of ∂U such that $bU = M \cup L$ and $M \cap L = bM = bL$ (possibly $bM = \emptyset$). A manifold triad $(U; M, L)$ is an *h-cobordism*, if both M and L are deformation retracts of U. For a manifold couple (W, M) an h-cobordism $(U; M, L)$ is called an *embedded h-cobordism* from M in W, if U is a submanifold of W such that $bW \cap U = M$. Two manifold couples (W_1, M_1) and (W_2, M_2) are *boundary h-cobordant*, if there is an h-cobordism $(U; M_2, L_2)$ such that (W_1, M_1) is homeomorphic to $(W_2 \cup U; L_2)$. These definitions are valid for the piecewise linear (PL) or smooth categories. In particular, in the smooth case, U should be required to be a smooth manifold with corner, if $bM \neq \emptyset$.

In the following we restrict ourselves in the PL category. The desired results for the smooth case are also obtained by Hirsch's method ([**5**] and [**4**]).

First of all we state the uniqueness and existence Theorems for PL embedded h-cobordisms.

THEOREM 2.1 (Existence Theorem). *Let (W, M) be a PL manifold couple of dimension $n \geqq 6$. Then for any element τ of $\mathcal{W}h(\pi_1(M))$ and for any regular neighborhood N mod $(\overline{bW-M})$ of M in W, there exists a PL embedded h-cobordism $(U; M, L)$ such that $\tau(U, M) = \tau$.* (Refer ([18], p. 250) and for relative regular neighborhoods, see [**6**] and [**7**]).

PROOF. From the existence of a collar neighborhood of bW in W and the uniqueness of relative regular neighborhoods mod $(\overline{bW-M})$ of M in W, we can take a PL homeomorphism $h: (M \times I, M \times 0) \to (N, M)$ such that $h(x, 0)$

$=x$ for all $x \in M$. On the other hand, according to Stallings ([**18**], p. 250), there is a PL h-cobordism $(U; M, L)$ such that $\tau(U, M)=\tau$. In order to complete the proof, it suffices to embed U into $M\times I$ so that $U\cap b(M\times I)=M(\equiv(M\times 0))$. For this, again form a PL h-cobordism $(V; L, K)$ such that $\tau(V, L)=-j_*\tau$, where $j_*: \mathcal{W}h(\pi_1(M))\rightarrow\mathcal{W}h(\pi_1(L))$ denotes the isomorphism induced from a composition $j: M\rightarrow L$ of the inclusion map $M\subset U$ and a homotopy inverse of the inclusion map $L\subset U$. Then a triad $(U\cup V; M, K)$ is a PL h-cobordism with $\tau(U\cup V, M)=0$. Since $n\geqq 6$, it follows from the PL s-cobordism Theorem that there is a PL homeomorphism $g: U\cup V\rightarrow M\times I$ so that $g(x)=(x, 0)$ for all $x\in M$, (for the PL s-cobordism Theorem, refer to Kervaire [**10**] and Zeeman [**20**]). Now $g|U: U\rightarrow M\times I$ is the required embedding, completing the proof.

THEOREM 2.2 (Uniqueness Theorem). *Let (W, M) be a PL manifold couple of dimension $n\geqq 6$. Let $(U_i; M, L_i)$, $i=1, 2$, be two PL embedded h-cobordisms from M in W such that $\tau(U_1, M)=\tau(U_2, M)$ $(=\tau)$. Then for any regular neighborhood N mod $(\overline{bW-M})$ of $U_1\cup U_2$ in W, there exists a PL ambient isotopy $h_t: W\rightarrow W$, $t\in I=[0, 1]$, such that $h_0=id.$, $h_t|W-\mathrm{Int}\, N=id.$ for all $t\in I$ and $h_1(U_1)=U_2$.* (Refer to [**15**], Theorem 11.3.)

PROOF. Let N_0 be a regular neighborhood mod $(\overline{bW-M})$ of M in W such that $N_0\subset(\mathrm{Int}\, U_1\cap\mathrm{Int}\, U_2)\cup M$. By Theorem 2.1, there is a PL embedded h-cobordism $(U; M, L)$ from M in N_0 such that $\tau(U, M)=\tau$. Putting $N_i=\overline{U_i-U}$, $i=1, 2$, we observe that $(N_i; L, L_i)$ are PL embedded h-cobordisms from L in $\overline{W-U}$. Since $n\geqq 6$ and $\tau(N_i, L)=0$ for each $i=1, 2$, it follows from the PL s-cobordism Theorem that (U_i, L) are PL homeomorphic to $(L\times I, L\times 0)$, and hence U_1 and U_2 are regular neighborhoods mod $(\overline{bW-M})$ of L in $\overline{W-U}$. Therefore, by the uniqueness of relative regular neighborhoods, we have a PL ambient isotopy $h_t: W\rightarrow W$, $t\in I$, such that $h_0=id.$, $h_t|U\cup\overline{W-N}=id.$ for all $t\in I$, and $h_1(U_1)=U_2$, completing the proof.

Theorems 2.1 and 2.2 imply the following extension Theorem on which the definition of operations of Whitehead groups on manifolds will be based.

COROLLARY 2.3 (Extension Theorem). *Let $(U_i; M_i, L_i)$, $i=1, 2$, be PL h-cobordisms of dimension $n\geqq 6$. Then a PL homeomorphism $f: M_1\rightarrow M_2$ extends to a PL homeomorphism $F: U_1\rightarrow U_2$ if and only if $f_*\tau(U_1, M_1)=\tau(U_2, M_2)$.*

PROOF. The necessity follows from the combinatorial invariance of Whitehead torsions, (see [**19**] or [**15**]). To prove the sufficiency, we may assume by Theorem 2.1 that $(U_i; M_i, L_i)$, $i=1, 2$, are PL embedded h-cobordisms from $M_i\times 0(\equiv M_i)$ in $M_i\times I$. Define a PL homeomorphism $g: M_1\times I\rightarrow M_2\times I$ by $g(x, t)=(f(x), t)$ for all $(x, t)\in M_1\times I$. Then $g(U_1)$ is a PL embedded h-cobordism from M_2 in $M_2\times I$. Since from the assumption $\tau(U_2, M_2)=f_*\tau(U_1, M_1)$ and from the combinatorial invariance of Whitehead torsions $f_*\tau(U_1, M_2)=$

$\tau(g(U_1), M_2)$, it follows that by Theorem 2.2 there is a PL homeomorphism $h: M_2\times I\to M_2\times I$ such that $h|M_2\times 0=id.$ and $h\circ g(U_1)=U_2$. Thus the composition $F=h\circ g|U_1: U_1\to U_2$ is the required extension of f, completing the proof.

The following may be well-known, (for example, see [**2**]). However, the author does not know the complete proof published and the argument of the proof will be used later.

THEOREM 2.4. *Let* (W_i, M_i), $i=1, 2$, *be PL manifold couples. Then* W_1-M_1 *and* W_2-M_2 *are PL homeomorphic if and only if* (W_1, M_1) *and* (W_2, M_2) *are boundary h-cobordant by an invertible PL h-cobordism.* (For invertible h-cobordisms, see [**18**], p. 249.)

PROOF. To see the necessity, suppose that there is an invertible PL h-cobordism $(U; M_2, L_2)$ from M_2 such that $(W_2\cup U, L_2)$ and (W_1, M_1) are PL homeomorphic. By Stallings ([**18**], Theorem 2), $U-L_2$ is PL homeomorphic to $M_2\times[0, \infty)$ and hence $(W_2\cup U)-L_2$ is PL homeomorphic to W_2-L_2. It follows from the assumption that W_2-L_2 and W_1-L_1 are PL homeomorphic.

It remains to prove the sufficiency. For this, form an open PL manifold $W_i\cup M_i\times[0, \infty)$ and a compact polyhedron $W_i\cup M_i*\infty$ from W_i by attaching an open collar $M_i\times[0, \infty)$ and a cone $M_i*\infty$ naturally. Now we may assume by the assumption that there is a PL homeomorphism $h: W_1\cup M_1\times[0, \infty)\to W_2\cup M_2\times[0, \infty)$. Since $M_i*\infty$ is homeomorphic to a single point compactification of $M_i\times[0, \infty)$, the PL homeomorphism h extends to a (topological) homeomorphism $H: W_1\cup M_1*\infty\to W_2\cup M_2*\infty$. If we put $C_i(m)=M_i\times[m, \infty)$ and $B_i(m)=bM_i\times[m, \infty)$ for each integer $m\geqq 0$, then we have sequences $\{C_i(m)\cup\{\infty\}\}_{m=0,1,2,\cdots}$ and $\{B_i(m)\cup\{\infty\}\}_{m=0,1,2,\cdots}$ of neighborhoods of ∞ in $M_i*\infty$ and $bM_i*\infty$, respectively. By choosing a suitable parameter $t\in[0, \infty)$, if necessary, we may assume that

$$\begin{aligned}&\mathrm{Int}\, C_1(0)\cup\mathrm{Int}\, B_1(0)\supset h^{-1}(C_2(1))\,,\\ &\mathrm{Int}\, C_2(1)\cup\mathrm{Int}\, B_2(1)\supset h(C_1(1)) \qquad \text{and}\\ &\mathrm{Int}\, C_1(1)\cup\mathrm{Int}\, B_1(1)\supset h^{-1}(C_2(2))\,.\end{aligned}$$

Putting $U=\overline{h^{-1}(C_2(1))}-C_1(1)$, $M=h^{-1}(M_2\times 1)$ and $L=M_1\times 1$, we will show that both M and L are deformation retracts of U. Since $M_i\times 1$ is a deformation retract of $C_i(1)$ for each $i=1, 2$, both M and U are deformation retracts of $h^{-1}(C_2(1))=U\cup C_1(1)$. Hence M is a deformation retract of U. In order to show that L is a deformation retract of U, we prove that $C_1(1)$ is a deformation retract of $h^{-1}(C_2(1))$. To do this, we observe the following sequence of homotopy groups:

$$\pi_p(h^{-1}(C_2(2)), x_0)\xrightarrow{i}\pi_p(C_1(1), x_0)\xrightarrow{j}\pi_p(h^{-1}(C_2(1)), x_0)\xrightarrow{k}\pi_p(C_1(0), x_0)\,,$$

where x_0 is a base point of $h^{-1}(C_2(2))$, and i, j, k stand for the homomorphisms induced from the inclusion maps. Since compositions $j \circ i$ and $k \circ j$ are isomorphisms, so is the homomorphism j for each $p \geqq 0$. Hence, by the Whitehead Theorem, $C_1(1)$ is a deformation retract of $h^{-1}(C_2(1))$. Since L and U are deformation retracts of $C_1(1)$ and $h^{-1}(C_2(1))$, respectively, it follows that L is a deformation retract of U. In the same way, we may prove that both $bM_1 \times 1$ and $h^{-1}(bM_2 \times 1)$ are deformation retracts of $\overline{h^{-1}(B_2(1)-B_1(1))}$. Therefore, a PL manifold triad $(U; M, \overline{bU-M})$ is a PL h-cobordism. The invertibility follows from the construction of the h-cobordism. Consequently, putting $h(U)=U_2$ and $h(\overline{bU-M})=L_2$, we have obtained an invertible PL h-cobordism $(U_2; M_2 \times 1, L_2)$ such that $h(W_1 \cup M_1 \times [0, 1]) = W_2 \cup M_2 \times [0, 1] \cup U_2$ and $h(M \times 1) = L_2$. Therefore, (W_1, M_1) and (W_2, M_2) are boundary h-cobordant, completing the proof.

§ 3. Operations of Whitehead groups on manifolds.

In the section, we define operations of Whitehead groups on manifolds and manifold couples, called *interior* and *boundary* operations, respectively. The precise definition is rather dull. However, we will give it as a pattern of geometrical operations of Whitehead groups involved in the paper.

Before giving the definition of those operations, we define some notations. All manifolds considered here are to be connected.

Let M be a PL manifold of dimension n and let $\varphi: G \to \pi_1(M)$ be a homomorphism from a group G to the fundamental group of M. Then the pair (M, φ) is called a *G-manifold of dimension n*. A second G-manifold (L, ψ) is *isomorphic* to (M, φ), if there is a PL homeomorphism $h: M \to L$, called an *isomorphism*, making a commutative diagram:

$$\begin{array}{ccc} \mathcal{W}h(\pi_1(M)) & \xrightarrow{h_*} & \mathcal{W}h(\pi_1(L)) \\ & {\scriptstyle\varphi_*}\nwarrow \quad \nearrow{\scriptstyle\psi_*} & \\ & \mathcal{W}h(G) & \end{array}$$

The isomorphism between G-manifolds is clearly an equivalence relation, and the isomorphism class of (M, φ) is again denoted by (M, φ). By a *G-manifold couple of dimension n* we mean a triple (W, M, φ) obtained from a PL manifold couple (W, M) of dimension n and a G-manifold (M, φ). A second G-manifold couple (V, L, ψ) is *isomorphic* to (W, M, φ), if there is a PL homeomorphism $h: (W, M) \to (V, L)$, called an *isomorphism*, such that $h|M: (M, \varphi) \to (L, \psi)$ is an isomorphism. The isomorphism between G-manifold couples is clearly an equivalence relation, and the isomorphism class of (W, M, φ) is again denoted by (W, M, φ).

Suppose that we are given a G-manifold couple of (W, M, φ) of dimension $n+1$ (respectively a G-manifold (M, φ) of dimension n), and that $n \geqq 5$. Then for each element τ of $\mathcal{W}h(G)$, by taking a PL h-cobordism $(U; M, M')$ such that $\tau(U, M)=\varphi_*\tau$, we obtain a new G-manifold couple $(W\cup U, M', j\circ\varphi)$ (resp. $(M', j\circ\varphi)$), written $(W, M, \varphi)\circ\tau$ (resp. $(M, \varphi)\circ\tau$), where $j: \pi_1(M)\to\pi_1(M')$ is the natural isomorphism $\pi_1(M)\cong\pi_1(U)\cong\pi_1(M')$. Corollary 2.3 guarantees us that the isomorphism class of $(W, M, \varphi)\circ\tau$ (resp. $(M, \varphi)\circ\tau$) depends only upon the class of (W, M, φ) (resp. (M, φ)) and τ. A G-manifold couple (W, M, φ) (resp. G-manifold (M, φ)) is *boundary* h-*cobordant* (resp. *interior* h-*cobordant*) to (V, L, ϕ) (resp. (L, ϕ)), if (V, L, ϕ) (resp. (L, ϕ)) is isomorphic to $(W, M, \varphi)\circ\tau$ (resp. $(L, \phi)\circ\tau$) for some element τ of $\mathcal{W}h(G)$. By $[W, M, \varphi]$ (resp. $[M, \varphi]$) we denote the set of all isomorphism classes of G-manifold couples (resp. G-manifolds) which are boundary h-cobordant to (W, M, φ) (resp. interior h-cobordant to (M, φ)). We define the *boundary operation* (resp. *interior operation*) of $\mathcal{W}h(G)$ on $[W, M, \varphi]$ (resp. $[M, \varphi]$)

$$[W, M, \varphi]\times\mathcal{W}h(G)\longrightarrow[W, M, \varphi]$$

$$(\text{resp. } [M, \varphi]\times\mathcal{W}h(G)\longrightarrow[M, \varphi]) \text{ by}$$

$$((V, L, \phi), \tau)\longmapsto(V, L, \phi)\circ\tau$$

$$(\text{resp. } ((L, \phi), \tau)\longmapsto(L, \phi)\circ\tau)$$

for all (V, L, ϕ) in $[W, M, \varphi]$ (resp. (L, ϕ) in $[M, \varphi]$) and for all τ in $\mathcal{W}h(G)$.

It is not hard to see that these maps are actually transitive operations.

The *inertia groups* $I[W, M, \varphi]$ and $I[M, \varphi]$ of these operations are subgroups of $\mathcal{W}h(G)$ consisting of elements τ of $\mathcal{W}h(G)$ such that

$$(W, M, \varphi)\circ\tau=(W, M, \varphi) \quad\text{and}$$

$$(M, \varphi)\circ\tau=(M, \varphi), \quad\text{respectively.}$$

Since $\mathcal{W}h(G)$ is abelian and operates on $[W, M, \varphi]$ (resp. $[M, \varphi]$) transitively, if τ belongs to $I[W, M, \varphi]$ (resp. $I[M, \varphi]$), then $(V, L, \phi)\circ\tau=(V, L, \phi)$ (resp. $(L, \phi)\circ\tau=(L, \phi)$) for any (V, L, ϕ) in $[W, M, \varphi]$ (resp. (L, ϕ) in $[M, \varphi]$). Thus those operations make the sets $[W, M, \varphi]$ and $[M, \varphi]$ into abelian groups which are isomorphic to $\mathcal{W}h(G)/I[W, M, \varphi]$ and $\mathcal{W}h(G)/I[M, \varphi]$, respectively. By $[W, M]$ and $[M]$ we denote the sets of PL homeomorphism classes of PL manifold couples which are boundary h-cobordant to (W, M) and PL manifolds which are PL h-cobordant to M. Then we have a natural map $[W, M, \varphi]\to[W, M]$ and $[M, \varphi]\to[M]$, which are clearly surjections, if $G=\pi_1(M)$ and $\varphi=id.$. Thus by investigating the groups $[W, M, \varphi]$ and $[M, \varphi]$ we can obtain rough information about the sets $[W, M]$ and $[M]$.

We turn to investigate those groups. First of all, we define an endomorphism $d_n: \mathcal{Wh}(G) \to \mathcal{Wh}(G)$ by $d_n(\tau)=\tau+(-1)^{n-1}\bar{\tau}$ for all τ in $\mathcal{Wh}(G)$. We list here several remarkable properties of the subgroup $d_n \mathcal{Wh}(G)$ for a finite group G.

PROPOSITION 3.1. *Suppose that G is a finite group (resp. finite abelian group).*

(1) *In case $n=$odd, $d_n \mathcal{Wh}(G)$ is isomorphic to $\mathcal{Wh}(G)$ modulo finite groups (resp. 2-torsions).*

(2) *In case $n=$even, $d_n \mathcal{Wh}(G)$ is finite (resp. trivial).*

(3) *Given a G-manifold (M, φ) of dimension $n \geqq 5$, then $d_{n+1}\mathcal{Wh}(G)$ is a subgroup of $I[M, \varphi]$. More precisely, if $(U; M, L)$ is a PL h-cobordism such that $\tau(U, M)=d_{n+1}(\tau)$ for some element τ in $\mathcal{Wh}(\pi_1(M))$, then there is a PL homeomorphism $h: M \to L$ such that $h|bM=id.$.*

PROOF. By ([15], Corollary 6.10 (resp. Lemma 6.7)), $\tau \equiv \bar{\tau}$ (modulo elements of finite order) (resp. $\tau=\bar{\tau}$) for all τ in $\mathcal{Wh}(G)$. Since G is finite and hence $\mathcal{Wh}(G)$ is finitely generated, the statements (1) and (2) follow immediately. To show (3), taking a PL h-cobordism V from M with $\tau(V, M)=\tau$, we put $(U'; M, M')=(\overline{b(V\times I)-M\times I}; M\times 0, bM\times I \cup M\times 1)$. Then it is easily seen that this is a PL h-cobordism with $\tau(U', M)=d_{n+1}(\tau)$ and that there is a PL homeomorphism $g: M \to M'$ such that $g|bM=id.$. Since (U', M) and (U, M) are PL homeomorphic, we can pull back g to the required one, completing the proof.

The following is an immediate consequence of (3) and (1) in Proposition 3.1.

COROLLARY 3.2 (Milnor). *Let (M, φ) be a G-manifold of dimension $n=$even $\geqq 6$. Suppose that G is finite (resp. finite abelian).*

Then the group $[M, \varphi]=\mathcal{Wh}(G)/I[M, \varphi]$ is finite (resp. consists of elements of order 2). In particular, there are only finitely many distinct PL homeomorphism classes of PL manifolds which are h-cobordant to M.

Further, we can get some informations about the group $[W, M, \varphi]=\mathcal{Wh}(G)/I[W, M, \varphi]$ as follows.

Let (W, M, φ) be a G-manifold couple of dimension $n \geqq 6$. In the rest of the section, we fix the symbol $i: \pi_1(M) \to \pi_1(W)$ to denote a homomorphism induced from the inclusion map $M \subset W$. Then we have a G-manifold $(W, i\circ\varphi)$, which is called an associated G-manifold to (W, M, φ) and written $|W, M, \varphi|$. The isomorphism class of $|W, M, \varphi|$ clearly depends only on the class of (W, M, φ).

Now the endomorphism $d_n: \mathcal{Wh}(G) \to \mathcal{Wh}(G)$ plays a role to connect the boundary operation on $[W, M, \varphi]$ and the interior operation of $[|W, M, \varphi|]$.

LEMMA 3.3. *Let (W, M, φ) be a G-manifold couple of dimension $n \geqq 6$. Then we have an identity: $|(W, M, \varphi)\circ d_n(\tau)|=(W, i\circ\varphi)\circ\tau$ for all τ in $\mathcal{Wh}(G)$.*

In particular, the following holds.

(1) *In case that $i_* \circ \varphi_* \mathcal{Wh}(G)$ is contained in $d_{n+1}\mathcal{Wh}(\pi_1(W))$, $d_n\mathcal{Wh}(G)$ is a*

subgroup of $I[W, M, \varphi]$.

(2) *In case* $bW=M$ *and* $\ker i_*=0$,

$$d_n I[W, i\circ\varphi]=I[W, M, \varphi]\cap d_n \mathcal{W}h(G).$$

PROOF. By taking a *PL* h-cobordism $(U; M, L)$ such that $\tau(U, M)=\varphi_*\tau$, we put $V=(W\cup U)\times I$ and $\overline{bV-W}=W'$. Since a triad $(V; W, W')$ is a *PL* h-cobordism such that $\tau(V, W)=i_*\circ\varphi_*\tau$, we have $(W', i'\circ\varphi)=(W, i\circ\varphi)\circ\tau$, where $i': \pi_1(M)\to\pi_1(W')$ is a homomorphism induced from the inclusion map $M\equiv M\times 0\subset W'$. On the other hand, since $W'=(\overline{bW-M})\times I\cup W\times 1\cup U\times 0 \cup L\times I\cup U\times 1$, it follows from Duality Theorem ([**15**], p. 394) that $(W', M, \varphi) =(W\times 1, M\times 1, \varphi\times 1)\circ d_n(\tau)=(W, i\circ\varphi)\circ\tau$. In particular, if $i_*\circ\varphi_*\mathcal{W}h(G)$ is contained in $d_{n+1}(\mathcal{W}h(\pi_1(W)))$, then for each element τ in $\mathcal{W}h(G)$ there is an element τ' in $\mathcal{W}h(\pi_1(W))$ such that $i_*\circ\varphi_*\tau=d_{n+1}(\tau')$. From Proposition 3.1, (3), we can take a *PL* homeomorphism $h: W\to W'$ such that $h|bW=id.$, which turns out to be an isomorphism between $(W', M, \varphi)=(W, M, \varphi)\circ d_n(\tau)$ and (W, M, φ). This implies that $d_n\mathcal{W}h(G)$ is a subgroup of $I[W, M, \varphi]$, proving (1).

In case $bW=M$ and $\operatorname{Ker} i_*=0$, an isomorphism $h:(W, i\circ\varphi)\to(W', i'\circ\varphi')$ turns out always to be an isomorphism $(W, M, \varphi)\to(W', M, \varphi)$, since $h(bW) =bW'$ and $h_*\circ i_*\circ\varphi_*=i'_*\circ\varphi_*$ implies $(h|M)_*\circ\varphi_*=\varphi_*$. Therefore $d_n I[W, i\circ\varphi]$ is a subgroup of $I[W, M, \varphi]$. It follows that $d_n I[W, i\circ\varphi]=I[W, M, \varphi] \cap d_n\mathcal{W}h(G)$, completing the proof.

Consequently, we have the following.

THEOREM 3.4. *Let* (W, M, φ) *be a G-manifold couple of dimension* $n\geqq 6$.

(1) *In case that* $i_*\circ\varphi_*\mathcal{W}h(G)$ *is contained in* $d_{n+1}\mathcal{W}h(\pi_1(W))$, *there is a natural epimorphism*

$$\mathcal{W}h(G)/d_n\mathcal{W}h(G)\longrightarrow[W, M, \varphi]\longrightarrow 0.$$

(2) *In case that* $bW=M$ *and* $\operatorname{Ker} i_*=0$, *the endomorphism* $d_n: \mathcal{W}h(G)\to \mathcal{W}h(G)$ *induces a monomorphism* $\bar{d}_n: [W, i\circ\varphi]\to[W, M, \varphi]$ *which gives rise to an exact sequence:*

$$0\to[W, i\circ\varphi]\xrightarrow{\bar{d}_n}[W, M, \varphi]\to\mathcal{W}h(G)/I[W, M, \varphi]+d_n\mathcal{W}h(G)\to 0,$$

where $I[W, M, \varphi]+d_n\mathcal{W}h(G)$ *is the smallest subgroup of* $\mathcal{W}h(G)$ *containing* $I[W, M, \varphi]$ *and* $d_n\mathcal{W}h(G)$.

(3) *In case that G is finite abelian and n is even, then the operation of* $\mathcal{W}h(G)$ *on* $[W, i\circ\varphi]$ *is trivial*

$$I[W, i\circ\varphi]=\mathcal{W}h(G).$$

PROOF. The statements (1) and (2) are immediate consequences of Lemma 3.3. Since if G is finite abelian, then each element of $\mathcal{W}h(G)$ is self conjugate, it follows that if $n=$ even, then $d_n(\tau)=\tau+(-1)^{n-1}\bar{\tau}=0$. Hence (3) also follows from Lemma 3.3, completing the proof.

In case that G is finite, we have the following consequence.

COROLLARY 3.5. [I] *Let* (W, M) *be a PL manifold couple of dimension* $n=\text{odd}\geqq 7$. *Suppose that* $\pi_1(M)$ *is finite.*

(1) *In case* $\mathcal{W}h(\pi_1(W))=0$, *there are finitely many distinct PL homeomorphism classes of PL manifolds whose interiors are PL homeomorphic to* Int W.

(2) *In case* $bW=M$ *and* $i_*:\mathcal{W}h(\pi_1(M))\cong\mathcal{W}h(\pi_1(W))$, *there are finitely many distinct PL h-cobordism classes of PL manifolds whose interiors are PL homeomorphic to* Int W.

[II] *Let* W *be a PL manifold of dimension* $n=\text{even}\geqq 6$. *Suppose that* $i_*:\mathcal{W}h(\pi_1(bW))\cong\mathcal{W}h(\pi_1(W))$ *and* $\pi_1(W)$ *is finite abelian. Then a PL manifold which is h-cobordant to* W *is PL homeomorphic to* W.

PROOF. Notice that if a PL manifold couple (V, L) (resp. PL manifold L) is boundary (resp. interior) h-cobordant to (W, M) (resp. M), then we may take an isomorphism $\varphi:\pi_1(M)\to\pi_1(L)$ so that $\pi_1(M)$-manifold couples $(W, M, id.)$ and (V, L, φ) (resp. $\pi_1(M)$-manifolds $(M, id.)$ and (L, φ)) are isomorphic. By Theorem 2.4, this reduces the proof of [I] and [II] to computing the orders of groups $[W, M, id.]$, $[W, M, id.]/[W, i\circ id.]$ and $[W, id.]$. Hence [II] follows from Theorem 3.4, (3), since $i_*:\mathcal{W}h(\pi_1(bW))\cong\mathcal{W}h(\pi_1(W))$, and [I] follows from Theorem 3.4, (1) and (2) together with Proposition 3.1, (1), completing the proof.

§ 4. Homotopy neighborhoods.

Let W be a PL manifold of dimension n and let P be a subpolyhedron of dimension p in W.

A *homotopy neighborhood* (abbreviated by *h-neighborhood*) of P in W is defined to be a PL submanifold N of dimension n in W satisfying the following conditions (1) and (2):

(1) $P\subset \text{Int}\, N$, (necessarily $P\subset \text{Int}\, W$), and

(2) for any derived neighborhood D of P in N, $(\overline{N-D}; bD, bN)$ is an h-cobordism. (For derived neighborhoods, see [**20**].)

REMARK. In virtue of the uniqueness of derived neighborhoods, the condition (2) may be replaced by a condition

(2′) for *some* derived neighborhood D of P in N, $(\overline{N-D}; bD, bN)$ is an h-cobordism.

For example, if $P\subset \text{Int}\, W$, then by the regular neighborhood annulus Theorem, regular neighborhoods are h-neighborhoods, (see [**6**]).

By an abstract h-neighborhood (resp. abstract regular neighborhood) of dimension (n, p), we mean a polyhedral pair (N, P) such that N is a PL manifold of dimension n which is an h-neighborhood (resp. regular neighborhood) of a polyhedron P of dimension p in N. A characterization of an abstract h-neighborhood is as follows:

THEOREM 4.1. *Let (N, P) be a pair consisting of a PL manifold N and a subpolyhedron P of N.*

Then (N, P) is an abstract h-neighborhood if and only if

(1) $P \subset \operatorname{Int} N$,

(2.1) *P is a deformation retract of N and*

(2.2) *for some derived neighborhood D of P in N, the inclusion maps $bN \subset N-P$ and $bD \subset N-P$ induce isomorphisms $\pi_1(bN) \cong \pi_1(N-P)$ and $\pi_1(bD) \cong \pi_1(N-P)$.*

PROOF. Under the common condition (1) $P \subset \operatorname{Int} N$, we will show that the condition (2) in the definition of the h-neighborhood is equivalent to the conditions (2.1) and (2.2) in Theorem 4.1.

For this taking a derived neighborhood D of P in N, we put $A = \overline{N-D}$. Recall a well-known fact that bD is a deformation retract of $D-P$. Hence A is a deformation retract of $N-P$. Thus (2) implies (2.1) and (2.2). Conversely, (2.2) implies that the inclusion map $bD \subset A$ and $bN \subset A$ induce isomorphisms $\pi_1(bD) \cong \pi_1(A) \cong \pi_1(bN)$. Following Milnor ([**14**], Lemma 2), we may deduce that $(A; bD, bA)$ is an h-cobordism as follows:

Let $\hat{A}$ be the universal covering space of A. Since $\pi_1(bN) \cong \pi_1(A) \cong \pi_1(bD)$, the restrictions $\widehat{bN}$ and $\widehat{bD}$ over bN and bD of $\hat{A}$ are also universal covering spaces of bN and bD, respectively. Notice that $H_k(\hat{A}, \widehat{bN})$ and $H_k(\hat{A}, \widehat{bD})$ can be identified with $H_k(A, bN; \mathcal{G})$ and $H_k(A, bD; \mathcal{G})$, where $\mathcal{G}$ is the integral group ring over $\pi_1(A)$. Then by the excision, we have

$$H_k(\hat{A}, \widehat{bD}) \cong H_k(A, bD; \mathcal{G}) \cong H_k(N, D; \mathcal{G}) = 0$$

for all k, since by (2.1) D is a deformation retract of N. Therefore, by the Whitehead Theorem bD is a deformation retract of A. By Poincaré duality, we have

$$H_k(\hat{A}, \widehat{bN}) \cong H_k(A, bN; \mathcal{G}) \cong H^{n-k}(A, bD; \mathcal{G}) = 0\,,$$

since bD is a deformation retract of A. Again by the Whitehead Theorem, bN is a deformation retract of A. It follows that $(A; bD, bN)$ is an h-cobordism, completing the proof.

An implication of Theorem 4.1 is the topological invariance of the abstract h-neighborhood;

COROLLARY 4.2. *Let (N, P) be a pair consisting from a PL manifold and a subpolyhedron P of N.*

If (N, P) is homeomorphic to an abstract h-neighborhood, then (N, P) is *itself an abstract h-neighborhood.*

PROOF. Let (N', P') be an abstract h-neighborhood and let $h:(N, P)\to(N', P')$ be a homeomorphism.

It is clear that P is a deformation retract of N and $\pi_1(bN)\cong\pi_1(N-P)$, since $h|N-P$ is also a homeomorphism. To complete the proof, it remains to prove that $\pi_1(bD)\cong\pi_1(N-P)$ for some derived neighborhood D of P in N. For this, by making use of the fact that h is a homeomorphism, we take derived neighborhoods D and D_1 of P in N, D' and D_1' of P' in N', respectively so that $h(D)\subset \mathrm{Int}\, D'$, $D_1'\subset \mathrm{Int}\, h(D)$ and $h(D_1)\subset \mathrm{Int}\, D_1'$. By the regular neighborhood annulus Theorem, $\overline{D-D_1}$ and $\overline{D'-D_1'}$ are PL homeomorphic to $bD\times I$ and $bD'\times I$, respectively. Hence by the same argument as in the proof of Theorem 2.4, we have $\pi_1(h(bD_1))\cong\pi_1(\overline{D'-D_1'})$. Since $\overline{D'-D_1'}$ is a deformation retract of $N'-P'$ and $h|N-P$ is a homeomorphism, it follows that $\pi_1(bD)\cong(\pi_1(N-P))$. Now Theorem 4.1 completes the proof of Corollary 4.2.

We will establish the uniqueness and existence Theorem of h-neighborhoods of P in W. For this, we define the Whitehead group $\mathcal{W}h(P, W)$ of P in W as follows: Suppose that $P\subset \mathrm{Int}\, W$. Let D and D' be derived neighborhoods of P in W. By the uniqueness of regular neighborhoods, for any derived neighborhood D_0 of P in $\mathrm{Int}\, D\cap\mathrm{Int}\, D'$, there is a PL homeomorphism $h: W\to W$ so that $h(D)=D'$ and $h|D_0=id.$. This gives a canonical isomorphism

$$(h|bD)_*:\mathcal{W}h(\pi_1(bD))\cong\mathcal{W}h(\pi_1(bD')).$$

Thus the Whitehead group $\mathcal{W}h(P, W)$ is defined as the projective limit of $\mathcal{W}h(\pi_1(bD))$ for any derived neighborhood D of P in W.

The *torsion* $\tau[N, P]$ of an h-neighborhood N of P in W is defined as an element of $\mathcal{W}h(P, W)$ corresponding to a Whitehead torsion $\tau(A, bD)$ for some derived neighborhood D of P in N.

The uniqueness of regular neighborhoods and the combinatorial invariance of Whitehead torsions guarantee us that the torsion $\tau[N, P]$ is well-defined in $\mathcal{W}h(P, W)$.

The uniqueness and existence Theorem of h-neighborhoods are stated as follows:

THEOREM 4.3. *Let W be a PL manifold of dimension $n\geqq 6$ and let P be a subpolyhedron of* Int W.

EXISTENCE THEOREM. *For any element τ of $\mathcal{W}h(P, W)$ and for any open neighborhood U of P in W, there exists an h-neighborhood N of P in W such that $\tau[N, P]=\tau$ and $N\subset U$.*

UNIQUENESS THEOREM. *Let N and N' be h-neighborhoods of P in W such that $\tau[N, P]=\tau[N', P]$. Then for any derived neighborhood D of P in* Int $N\cap$ Int N', *there exists a PL homeomorphism $h: N\to N'$ such that $h|D=id.$.*

Further, if $N \cup N' \subset \operatorname{Int} W$, *then for any open neighborhood* U *of* $N \cup N'$ *in* W, *we can take a* PL *ambient isotopy* $h_t: W \to W'$, $t \in I$, *so that* $h_0 = id.$, $h_t | D \cup W - U = id.$ *and* $h_1(N) = N'$.

PROOF. This follows immediately from Theorems 2.2 and 2.3.

A characterization of the regular neighborhood is obtained from Theorem 4.3, (Uniqueness Theorem).

COROLLARY 4.4. *Let* W *be a* PL *manifold of dimension* $n \geqq 6$ *and let* P *be a subpolyhedron of* $\operatorname{Int} W$.

Then a PL *submanifold* N *of dimension* n *in* W *is a regular neighborhood of* P *in* W *if and only if* N *is an h-neighborhood of* P *in* W *with* $\tau[N, P] = 0$.

This together with Corollary 4.2 gives a topological invariance theorem of abstract regular neighborhoods in a special case:

COROLLARY 4.5. *Let* (N, P) *be a pair consisting of a* PL *manifold* N *of dimension* $n \geqq 6$ *and a subpolyhedron* P *of* $\operatorname{Int} N$. *Suppose that* $\mathcal{W}h(P, N) = 0$.

If (N, P) *is homeomorphic to an abstract regular neighborhood, then* (N, P) *is itself an abstract regular neighborhood.*

We consider of the special case $\dim W - \dim P \geqq 3$. Let D be a derived neighborhood of P in W. Then by the general position argument, we have $\pi_1(D-P) \cong \pi_1(D)$, and hence $\pi_1(bD) \cong \pi_1(P)$. Thus we have a canonical isomorphism $\mathcal{W}h(\pi_1(P)) \cong \mathcal{W}h(P, W)$.

Suppose that N is a PL submanifold of W such that $P \subset \operatorname{Int} N$ and $\dim N = \dim W$. Then again by the general position argument, we have $\pi_1(N-P) \cong \pi_1(N)$. Further, if $D \subset \operatorname{Int} N$ and P is a deformation retract of N, then we have $\pi_1(bD) \cong \pi_1(N-P)$. Therefore, in Theorem 4.1, the condition (2.2) can be replaced by a single condition (2.2′) $\pi_1(bN) \cong \pi_1(N)$. On the other hand, D collapses P and hence $\tau(D, P) = 0$. It follows that the torsion $\tau[N, P]$ corresponds to the Whitehead torsion $\tau(N, P)$ by the isomorphism $\mathcal{W}h(\pi_1(P)) \cong \mathcal{W}h(P, W)$.

Consequently, we may conclude the following:

PROPOSITION 4.6. *Let* (N, P) *be a pair consisting of a* PL *manifold and a subpolyhedron* P *of dimension* p *in* N. *Suppose that* $n-p \geqq 3$. *Then* (N, P) *is an abstract h-neighborhood if and only if*

(1) $P \subset \operatorname{Int} N$

(2.1) *P is a deformation retract of N and*

(2.2′) $\pi_1(bN) \cong \pi_1(N)$.

Further, $\tau(N, P)$ *corresponds to the torsion* $\tau[N, P]$ *by the isomorphism* $\mathcal{W}h(\pi_1(P)) \cong \mathcal{W}h(P, W)$.

The following examples show that the condition (2.2) can be weakened no more in case $\dim N - \dim P \leqq 2$.

EXAMPLE 4.7. *For each integer* $n \geqq 6$, *there exist a* PL *embedded* $(n-2)$-

sphere Σ in S^n and a PL submanifold N of dimension n in S^n such that

(1) $\Sigma \subset \operatorname{Int} N$,

(2.1) *Σ is a deformation retract of N and*

(2.2″) *bN is a deformation retract of $N-\Sigma$, but N is not an h-neighborhood of Σ in S^n.*

PROOF. In [9], we have constructed a *PL* embedded $(n-2)$-sphere Σ in S^n such that the complement of an open derived neighborhood *Int D* of Σ in S^n is *PL* homeomorphic to a product space $S^1 \times W$ of the circle S^1 and a contractible *PL* manifold W such that $\pi_1(bW)$ is non-trivial.

Taking a *PL* $(n-1)$-ball B in Int W, we put $N=\overline{S^n-S^1\times B}$ and $A=\overline{N-D}(=S^1\times(\overline{W-B}))$. Since bB is a deformation retract of $\overline{W-B}$ and A is a deformation retract of $N-\Sigma$, it follows that bN is a deformation retract of $N-\Sigma$, and Σ is a deformation retract of N. However, $\pi_1(bD)\cong\pi_1(S^1\times bW)$ $\cong Z\times\pi_1(bW)$ is not isomorphic to $\pi_1(bN)\cong\pi_1(S^1\times bB)\cong Z$. Therefore, $(A\,;bD, bN)$ can not be an h-cobordism, completing the proof.

REMARK. By attaching a feeler to Σ or by thickening Σ in S^n, we can obtain such an example for each case of codimension $\leqq 1$. These also give counter examples of Uniqueness Theorem of simple neighborhoods in [**13**], (also see [**11**] and [**12**]). As for this, we have already obtained the first counter examples in [8], which are, however, h-neighborhoods.

§5. Operations of Whitehead torsions on higher dimensional knots.

In the preceding section, we have studied a h-neighborhood which is a union of a derived neighborhood and a *PL* h-cobordism from its boundary. This notion may be relativilized. In the section, however, we involve only higher dimensional knots and their cone ball pairs.

DEFINITION. By a *knot of dimension n*, we mean a *PL* $(n+2, n)$-sphere pair $\kappa_n=(S, \Sigma)$ such that Σ has a product neighborhood $(\Sigma\times B^2)$ in S^{n+2} so that $(\Sigma\times 0)=\Sigma$, where B^2 stands for the standard 2-ball $[-1, 1]\times[-1, 1]$. From a knot κ_n we have a ball pair $a*\kappa_n=(a*S, a*\Sigma)$, called a *cone ball pair.* By a *G-knot* we mean a pair (κ_n, φ) consisting of a knot $\kappa_n=(S, \Sigma)$ and a homomorphism $\varphi: G\to\pi_1(S-\Sigma)$ inducing a monomorphism $\varphi_*: \mathcal{W}h(G)\to \mathcal{W}h(\pi_1(S-\Sigma))$. A second G-knot (κ_n', φ') is *isomorphic* to (κ_n, φ), if there is a *PL* homeomorphism $h:\kappa_n\to\kappa_n'$ such that $(h|S-\Sigma)_*\circ\varphi_*=\varphi_*'$. The isomorphism between G-knots is an equivalence relation and the isomorphism class of a G-knot (κ_n, φ) is called a G-knot type and written again (κ_n, φ).

For a proper manifold pair (W, M), by $E=E(W, M)$ we denote the closure of the complement of a derived neighborhood of M in W. We identify $\pi_1(E(W, M))$ with $\pi_1(W-M)$ by the natural isomorphism induced from the inclusion map $E\subset W-M$ which is a homotopy equivalence.

Now suppose that we are given a higher dimensional G-knot (κ_n, φ) of dimension $n\geqq 3$, where $\kappa_n=(S, \Sigma)$. For each element τ of $\mathcal{W}h(G)$, by taking a PL h-cobordism $(U; E, E')$ such that $\tau(U, E)=\varphi_*\tau$ and by applying PL Smale Theorem, we have a new knot $((\Sigma\times B^2)\cup E', \Sigma)$, written $\kappa_n\circ\tau$, and a ball pair $((a*S)\cup U, a*\Sigma)$, written $(a*\kappa_n)\circ\tau$, where $E=\overline{S-(\Sigma\times B^2)}$. Then the PL homeomorphism classes of $\kappa_n\circ\tau$ and $(a*\kappa_n)\circ\tau$ depend only on the isomorphism class of (κ_n, φ), and we have the following:

LEMMA 5.1. *If $(a*\kappa_n)\circ\tau$ is PL homeomorphic to $a*\kappa_n$, then $\tau=0$; that is to say, τ operates freely on $a*\kappa_n$.*

PROOF. Notice that there are PL homeomorphisms $(E(a*\kappa_n), E(\kappa_n))\rightarrow(E\times I, E\times 0)$ and $(E((a*\kappa_n)\circ\tau), E(\kappa_n\circ\tau))\rightarrow(U, E)$.

If there is a PL homeomorphism $a*\kappa_n\rightarrow(a*\kappa_n)\circ\tau$, then by PL invariance of derived neighborhoods there is a PL homeomorphism

$$(E(a*\kappa_n), E(\kappa_n))\rightarrow(E((a*\kappa_n)\circ\tau), E(\kappa_n\circ\tau)).$$

Hence (U, E) and $(E\times I, E\times 0)$ are PL homeomorphic. Therefore, $\tau=\tau(U, E)=0$, completing the proof.

COROLLARY 5.2. *If $\tau\neq 0$, then $(a*\kappa_n)\circ\tau$ can not be an abstract regular neighborhood; that is to say, if $(a*\kappa_n)\circ\tau=((a*S)\cup U, a*\Sigma)$, then $(a*S)\cup U$ never collapses $a*\Sigma$.*

PROOF. By the local flatness of Σ in S, the ball pair $(a*\kappa_n)\circ\tau$ satisfies the condition for the uniqueness of relative regular neighborhoods ([**6**] and [**7**]). Since $a*S$ is a relative regular neighborhood of $a*\Sigma$ in $(a*S)\cup U$, or $a*\kappa_n$ is an abstract relative regular neighborhood, it follows that if $(a*\kappa_n)\circ\tau$ is an abstract regular neighborhood, then, by the uniqueness, $(a*\kappa_n)\circ\tau$ and $a*\kappa_n$ are PL homeomorphic, contradicting Lemma 5.1, and completing the proof.

On the contrary, in the topological category, we have the following commutativity.

LEMMA 5.3. *For any element τ of $\mathcal{W}h(G)$, $(a*\kappa_n)\circ\tau$ and $a*(\kappa_n\circ\tau)$ are homeomorphic.*

PROOF. By the argument in the proof of ([**16**], Lemma 3.1) $(a*\kappa_n)\circ\tau-(a, a)$ and $(\kappa_n\circ\tau)\times[0, 1)$ are PL homeomorphic. Hence those single point compactifications $(a*\kappa_n)\circ\tau$ and $a*(\kappa_n\circ\tau)$ are homeomorphic, completing the proof.

By the cone extension argument, we conclude as follows.

COROLLARY 5.4. *Let (κ_n, φ) be a G-knot of dimension $n\geqq 3$ such that $\mathcal{W}h(G)$ is non-trivial. Then for each non-zero element τ of $\mathcal{W}h(G)$, either of the following two statements holds.*

(1) *If κ_n and $\kappa_n \circ \tau$ are homeomorphic, then $a * \kappa_n$ and $(a * \kappa_n) \circ \tau$ are homeomorphic. That is to say, if $\kappa_n = (S, \Sigma)$, then the collapsing $a * s \searrow a * \Sigma$ is not topologically invariant.*

(2) *If κ_n and $\kappa_n \circ \tau$ are not homeomorphic, then the point a has two topologically distinct cone neighborhoods $a * \kappa_n$ and $a * (\kappa_n \circ \tau)$ in $(a * \kappa_n) \circ \tau$. That is to say, there exist two distinct PL structures $(a * \kappa_n) - \kappa_n$ and $a * (\kappa_n \circ \tau) - (\kappa_n \circ \tau)$ on an open $(n+3, n+1)$-ball pair $(a * \kappa_n) \circ \tau - (\kappa_n \circ \tau)$ such that in these PL structures the point a has topologically distinct link pairs κ_n and $\kappa_n \circ \tau$.*

In virtue of analysis of strong h-cobordisms of knots due to Siebenmann and Sondow [**16**] and [**17**] for each $n = \text{even} \geqq 4$, we have an example of the knot in (1) and for each $n = \text{odd} \geqq 3$, we have an example of the knot in (2). In fact, for example, by [**21**] we can take a PL knot $\kappa_n = (S, \Sigma)$ of dimension $n \geqq 2$ such that $\pi_1(S - \Sigma)$ is a direct product $G \times J$ of the binary icosahedral group G of order 120 and infinite cyclic group J. Let $\varphi : Z_5 \to G \times J (\equiv \pi_1(S - \Sigma))$ be an embedding of Z_5 to a 5-Sylow group of $G \subset G \times J$. Siebenmann and Sondow have proved that φ induces a monomorphism $\varphi_* : \mathcal{W}h(Z_5)(\cong Z) \to \mathcal{W}h(G \times J)$ into a direct summand $\mathcal{W}h(G)$ of $\mathcal{W}h(G \times J)$, [**16**] and also see ([**15**], p. 421). Thus we have a Z_5-knot (κ_n, φ) such that $\mathcal{W}h(Z_5) \cong Z$.

If $n = \text{even} \geqq 4$, then by applying Proposition 3.1, (3), we may conclude that κ_n and $\kappa_n \circ 2\tau$ are PL homeomorphic for each τ of $\mathcal{W}h(Z_5) \cong Z$. Thus we have obtained an example of the knot in the statement (1) of Corollary 5.4. As for (2), we have an example of the knot of dimension $n = \text{odd} \geqq 3$ from ([**17**], p. 741).

In the sequel, we have the following.

COROLLARY 5.5. *Generally, the collapsing and the singularity are not topologically invariant.*

§ 6. Simple homotopy types of polyhedra.

All polyhedra considered here are to be compact and connected, and Whitehead torsions of homotopy equivalences are to be identified by appropriate isomorphisms between fundamental groups.

The following is a generalization of ([**11**], (11.5)).

LEMMA 6.1. *Let N be a regular neighborhood of a subpolyhedron P of dimension p in R^n and let N' be a PL submanifold of dimension n of R^n.*

Suppose that $n \geqq 6$, $n \geqq 2p+2$ and $\pi_1(bN') \cong \pi_1(N')$.

For any homotopy equivalence $f : N \to N'$, there exists a PL h-cobordism U from bN such that $\tau(U, bN) = \tau(f)$ and a PL homeomorphism $h : N \cup U \to N'$ such that $h|N$ is homotopic to f.

PROOF. If we put $f' = f|P : P \to N'$, then f' is a homotopy equivalence with $\tau(f')=\tau(f)$, since $\tau(N, P)=0$. By the general position argument, we may approximate f' by a PL embedding $g: P\to \operatorname{Int} N'$. By Gugenheim's Theorem [3], there is a PL homeomorphism $H: R^n\to R^n$ such that $H\circ g: P\to R^n$ is the inclusion map $P\subset R^n$. Since $n\geqq 6$, $n-p\geqq p+2$, $\pi_1(bN')\cong \pi_1(N')$ and P is a deformation retract of $H(N')$, it follows that $H(N')$ is an h-neighborhood of P in R^n with $\tau(H(N'), P)=\tau(g)=\tau(f)$. Taking a PL h-cobordism U from bN such that $\tau(f)=\tau(U, bN)$, we may assume that $H(N')=N\cup U$. Then $H^{-1}|N: N\to N'$ is homotopic to f, since P is a deformation retract of N. Thus $H^{-1}|N\cup U$ is the required PL homeomorphism, completing the proof.

Let P be a polyhedron of dimension p. By a *Thom complex* $T^n(P)$ of P of dimension $n(\geqq 2p+1)$ we mean a polyhedron $N\cup(bN*\infty)$ which is a union of a regular neighborhood N of P in R^n and a cone $(bN*\infty)$ for a point ∞.

THEOREM 6.2. *Let P and Q be polyhedra of dimensions p and q, respectively. Assume $n\geqq \max(2p+2, 2q+1, 6)$.*

Then the following holds.

(1) *P and Q are homotopy equivalent if and only if $T^n(P)$ and $T^n(Q)$ are homeomorphic.*

(2) *P and Q are simple homotopy equivalent if and only if $T^n(P)$ and $T^n(Q)$ are PL homeomorphic.*

PROOF. Let $T^n(P)=M\cup(bM*\infty)$ and $T^n(Q)=N\cup(bN*\infty)$. Suppose that P and Q are homotopy equivalent. By Lemma 6.1 and Theorem 2.4, $\operatorname{Int} M$ and $\operatorname{Int} N$ are PL homeomorphic, and hence their single point compactifications $T^n(P)$ and $T^n(Q)$ are homeomorphic. Further, if P and Q are simple homotopy equivalent, then N and M are PL homeomorphic. Hence by the cone extension argument, $T^n(P)$ and $T^n(Q)$ are PL homeomorphic.

Conversely, suppose that there is a homeomorphism $h: T^n(P)\to T^n(Q)$. If bM is not a homotopy sphere, then by the same local argument as in the proof of Theorem 2.4 we have $h(\infty)=\infty$, since only the link bM of ∞ has $\pi_1(bM)\neq 1$ or $H_*(bM)\not\cong H_*(S^{n-1})$. If bM is a homotopy sphere, then bM is actually a PL $(n-1)$-sphere and hence $T^n(P)$ is a PL manifold. We may take a PL homeomorphism $g: T^n(P)\to T^n(P)$ such that $g\circ h(\infty)=\infty$. Therefore, we may also assume that $h(\infty)=\infty$. Now a homeomorphism $h|T^n(P)-\infty$: $T^n(P)-\infty\to T^n(Q)-\infty$ gives rise to a homotopy equivalence between P and Q. Further, if $T^n(P)$ and $T^n(Q)$ are PL homeomorphic, then we may take a PL homeomorphism $h: T^n(P)\to T^n(Q)$ such that $h(\infty)=\infty$. By the PL invariance of cone neighborhoods guaranteed by pseudo radial projection argument we may assume that $h(bM*\infty)=bN*\infty$. Therefore, M and N are PL homeomorphic. Now a PL homeomorphism between M and N gives rise to a simple homotopy equivalence between P and Q, completing the proof.

An implication of Theorem 6.2 is the following.

COROLLARY 6.3. *Let P and Q be polyhedra of dimensions p and q, respectively. Suppose that for each element τ of $\mathcal{W}h(\pi_1(P))$ there is a homotopy equivalence $\varphi: P\rightarrow P$ such that $\tau(\varphi)=\tau$ and that $n\geqq\max(2p+2, 2q+1, 6)$.*

Then $T^n(P)$ and $T^n(Q)$ are homeomorphic if and only if they are PL homeomorphic.

PROOF. To show this, in virtue of Theorem 6.2, it is sufficient to prove that if P is homotopically equivalent to Q, then P is simple homotopy equivalent to Q. For this, let $\varphi: P\rightarrow Q$ be a homotopy equivalence. By the assumption we have a homotopy equivalence $\psi: P\rightarrow P$ such that $\tau(\varphi)=-\tau(\psi)$. Then $\varphi\circ\psi: P\rightarrow Q$ is a simple homotopy equivalence, completing the proof.

For example, let P be a polyhedron of dimension 2 obtained by attaching a 2-cell onto a circle by a map of degree 5. Then by ([**18**], p. 252); for a generator τ of $\mathcal{W}h(Z_5)=Z$ there is a homotopy equivalence $\psi: P\rightarrow P$ such that $\tau(\psi)=\tau$. It follows that for $n\geqq\max(2q+1, 6)$, $T^n(P)$ and $T^n(Q)$ are homeomorphic if and only if they are PL homeomorphic.

Finally we consider the following two statements [I] and [II].

[I] (*Topological invariance of Whitehead torsions*).

Let (Q, P) be a polyhedral pair such that P is a deformation retract of Q, and let (Q', P') be a polyhedral pair. Suppose that there is a homeomorphism $h:(Q', P')\rightarrow(Q, P)$.

Then $h_*\tau(Q', P')=\tau(Q, P)$.

[II] (*Topological invariance of regular neighborhoods by ε-push*).

Let N be a regular neighborhood of a polyhedron P of dimension p in R^n $(n\geqq 3p+1)$.

Then there is a number $\delta>0$ such that for any number $\varepsilon<\delta$, if $h: N\rightarrow N$ is an ε-push of (N, P) such that $h(P)$ is a subpolyhedron of N, then N is a regular neighborhood of $h(P)$ in R^n. (For ε-push, see [**1**].)

The statement [II] is a special case of the statement [I]. We prove the following.

THEOREM 6.4. *The statements* [I] *and* [II] *are equivalent.*

PROOF. We will prove that [II] implies [I]. For this, we think of Q as a subpolyhedron of R^{2q+1}, where $q=\dim Q$, and define a topological embedding $k: Q'\rightarrow R^n$ as the composition

$$Q'\xrightarrow{h} Q\subset R^{2q+1}\equiv R^{2q+1}\times 0\subset R^{2q+1}\times R^r=R^n,$$

where $r=n-2q-1$. Let N be a regular neighborhood of Q in R^n. Then N is a h-neighborhood of P in R^n with $\tau(N, P)=\tau(Q, P)$. We may take a regular neighborhood M of P in Int N so that $Q\subset\operatorname{Int} M$ and $i_*\tau(Q, P)=\tau(N, M)$, where $i: Q\rightarrow N$ is the inclusion map. On the other hand, if $n\geqq 3q+1$, then from ([**1**], Theorem 1.1) and the statement [II], for sufficiently small number ε, we have an ε-push g of (M, Q) so that $g\circ k: Q'\to M$ is a PL embedding and that M and N are regular neighborhoods of $g(Q)=g\circ k(Q')$ and $g(P)=g\circ k(P')$ in R^n, respectively. Now we have, by the combinatorial invariance of Whitehead torsions, $\tau(N, M)=j_*\tau(g\circ k(Q'), g\circ k(P'))=(j\circ g\circ k)_*\tau(Q', P')$, where $j: g(Q)\to M$ is the inclusion map, and hence $i_*\tau(Q, P)=(j\circ g\circ k)_*\tau(Q', P')$. Since $(j\circ g\circ k)_*=(i\circ h)_*$, it follows that $h_*\tau(Q', P')=\tau(Q, P)$, completing the proof.

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References

[1] R.H. Bing and J.M. Kister, Taming complexes in hyperplanes, Duke Math. J., 31 (1964), 491-511.
[2] W. Browder, J. Levine and G.R. Livesay, Finding a boundary for an open manifold, Amer. J. Math., 87 (1965), 1017-1028.
[3] V.K.A.M. Gugenheim, Piecewise linear isotopy and embedding of elements and spheres I, Proc. London Math. Soc., 3 (1953), 29-53.
[4] M.W. Hirsch, On combinatorial submanifolds of differentiable manifolds, Comment. Math. Helv., 36 (1961), 103-111.
[5] M.W. Hirsch, Smooth regular neighborhoods, Ann. of Math., 76 (1962), 524-530.
[6] J.F.P. Hudson and E.C. Zeeman, On regular neighborhoods, Proc. London Math. Soc., 14 (1964), 719-745.
[7] L.S. Hush, Thesis, Florida State University.
[8] M. Kato, Regular neighborhoods are not topologically invariant, Bull. Amer. Math. Soc., 74 (1968), 988-991.
[9] M. Kato, Higher dimensional PL knots and knot manifolds, J. Math. Soc. Japan, 21 (1969), 458-480.
[10] M. Kervaire, Le théorème de Barden-Mazur-Stallings, Comment. Math. Helv., 40 (1965), 31-42.
[11] B. Mazur, Differential topology from the point of view of simple homotopy theory, Publ. Math. IHES No. 15.
[12] B. Mazur, Corrections to: Differential topology from the point of view of simple homotopy theory, Publ. Math. IHES No. 23.
[13] B. Mazur, Relative neighborhoods and the theorems of Smale, Ann. of Math., 77 (1963), 232-249.
[14] J. Milnor, Two complexes which are homeomorphic but combinatorially distinct, Ann. of Math., 74 (1961), 575-590.
[15] J. Milnor, Whitehead torsion, Bull. Amer. Math. Soc., 72 (1966), 358-426.
[16] L.C. Siebenmann and J. Sondow, Some homeomorphic sphere pairs that are combinatorially distinct, Comment. Math. Helv., 41 (1966-67), 261-272.
[17] L.C. Siebenmann, On the homotopy type of compact topological manifolds, Bull. Amer. Math. Soc., 74 (1968), 738-742.
[18] J. Stallings, On infinite processes leading to differentiability in the complement of a point, Differential and Combinatorial Topology, Princeton Univ. Press, Prince-

ton, N.J., U.S.A., (1965), 245–254.
[19] J.H.C. Whitehead, Simple homotopy types, Amer. J. Math., 72 (1950), 1–57.
[20] E.C. Zeeman, Seminar on combinatorial topology, IHES (1963), (mimeographed notes).
[21] E.C. Zeeman, Twisting spun knots, Trans. Amer. Math. Soc., (3) 115 (1965), 471–495.