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Some results on Γ-extensions of algebraic number fields

Dedicated to Professor Iyanaga on his 60th birthday

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Let l denote a prime number which we fix throughout the present paper. Let F_0 be an algebraic number field of finite degree, and let F/F_0 be a Γ-extension over F_0. Namely F/F_0 is a Galois extension whose Galois group is isomorphic to the additive group of l-adic integers. In the following we shall consider a Γ-module $A(K/F)$, attached to F/F_0, which will be defined analogously to the cyclotomic case considered by Iwasawa [8]. After the preliminaries in § 1 we shall give in § 2 a necessary and sufficient condition for the regularity of $A(K/F)$ (as Γ-module) in terms of characters of idèle groups of intermediate fields of F and F_0 (Theorems 1 and 2). The Γ-module $A(K/F)$ is intimately related to l-adic behaviour of global unit groups of algebraic number fields (Theorem 3 in § 2).

Now let in particular the ground field F_0 be an imaginary quadratic extension of the rational number field. In such a case there exist, in a fixed algebraic closure of F_0, two independent Γ-extensions over F_0 (with respect to our fixed prime number l). Under additional conditions on F_0 the regularity of $A(K/F)$ will be obtained in § 3 (Theorem 4 in § 3).

General notations. We denote by l a prime number which we fix throughout the present paper[1]. $\boldsymbol{Z}$ and $\boldsymbol{Q}$ stand for the ring of rational integers and the rational number field, respectively. We denote by $\boldsymbol{Z}_l$ and $\boldsymbol{Q}_l$ the ring of l-adic integers and the l-adic completion of $\boldsymbol{Q}$, respectively. $\boldsymbol{Z}/(d)\boldsymbol{Z}$ means the additive group of integers modulo d, where $d \in \boldsymbol{Z}$.

§ 1. Preliminaries.

1.1.[2] Now let in general E be a field and K/E a Galois extension. Then the Galois group of K/E equipped with the Krull topology will be denoted by $G(K/E)$. Let F be an intermediate field of K and E which is also a Galois extension over E. Then the Galois group $G(F/E)$ is canonically isomorphic to

1) We reserve the notations p, $\mathfrak{p}$, etc. for general prime numbers or prime divisors.

2) Cf. Iwasawa [8], § 1. The purpose of the descriptions in § 1.1 and § 1.2 is to introduce notations.

the factor group $G(K/E)/G(K/F)$, $G(K/F)$ being of course a closed normal subgroup of $G(K/E)$. Furthermore if K/F is abelian then every inner automorphism $x \to s^{-1}xs$ of $G(K/E)$ induces a topological automorphism of $G(K/F)$ which depends only upon the coset σ of $s \bmod G(K/F)$. $G(K/F)$ is thus made into a $G(F/E)$-group on which $G(F/E)$ acts unitarily (i. e. $1 \cdot x = x$) and continuously. The discrete character group of the compact abelian group $G(K/F)$ will be denoted by $A(K/F)$. The action of $G(F/E)$ on $A(K/F)$ is defined by setting

$$(1) \qquad a^{\sigma}(x) = a(x^{\sigma}) = a(s^{-1}xs), \qquad x \in G(K/F),$$

where $a \in A(K/F)$ and s is any element of $G(F/E)$ such that σ is the coset of $s \bmod G(K/F)$. We notice that $G(F/E)$ acts on $G(K/F)$ from the right, and $G(F/E)$ acts on $A(K/F)$ from the left: e. g. $(a^{\sigma})^{\tau} = a^{\tau\sigma}$.

Let a be an element in $A(K/F)$, and let K_a denote provisionally the fixed field of the kernel of a. If $G(F/E)$ is cyclic or contains a dense cyclic subgroup, then K_a/E is abelian if and only if $a^{\sigma} = a$ for every σ in $G(F/E)$.

1.2. Let Γ denote a multiplicative topological group isomorphic to the additive group of $\boldsymbol{Z}_l$. For an integer $m \geqq 0$ we denote by Γ_m the unique open subgroup with index l^m in Γ. Let F_0 be a finite algebraic extension over $\boldsymbol{Q}$, and let F/F_0 be a Γ-extension. Namely F/F_0 is a Galois extension whose Galois group is isomorphic to Γ[3]. We identify the Galois group $G(F/F_0)$ with Γ, and we denote by F_m the fixed field of Γ_m. F is the union of the increasing sequence of all F_m $(m \geqq 0)$. We denote by S the set of all prime divisors of F which divide the rational prime divisor l. If K' and K'' are two algebraic extensions of F in which no prime divisor of F outside S is ramified, then the same holds good for the composite field $K' \cdot K''$. Thus there exists the unique maximal l-primary abelian extension K over F in which no prime divisor of F outside S is ramified[4]. Furthermore if $\mathfrak{l}$ is an element in S then the prime divisors conjugate to $\mathfrak{l}$ with respect to F/F_0 are also contained in S. Thus K/F_0 is a Galois extension, and we are in the situation described in § 1.1 with $E = F_0$. In particular the Galois group Γ of F/F_0 acts on $A(K/F)$, the dual of the Galois group $G(K/F)$, as described in § 1.1.

A discrete group A is said to be l-primary if A is the direct limit of a family of finite l-groups, and a compact group G is said to be l-primary if G is the inverse limit of a family of finite l-groups. A discrete l-primary (additive) abelian group A is said to be a discrete Γ-module if Γ acts on A unitarily and continuously. Similarly a compact l-primary (additive) abelian group

3) Cf. Iwasawa [**6**], where the fixed prime number is denoted by p.

4) 'l-primary abelian' means here that the Galois group $G(K/F)$ is an inverse limit of a family of finite l-abelian groups. Cf. also Remark at the end of § 2.3.

G is said to be a compact Γ-module if Γ acts on G unitarily and continuously[5].

Thus $G(K/F)$ and $A(K/F)$ mentioned above are compact and discrete Γ-modules, respectively.

1.3. Let in general A be a discrete Γ-module. Then, as usual, we denote by A_m the submodule of A which consists of all the elements a in A such that $a^\sigma = a$ for every $\sigma \in \Gamma_m$. Let m and n be integers such that $m \geqq n \geqq 0$. Then A_m is naturally made into a Γ_n/Γ_m-module. It is known by Iwasawa [7] that a discrete Γ-module A is regular (as Γ-module) if and only if we have

$$H^i(\Gamma_n/\Gamma_m, A_m) \cong (0), \qquad i = 1, 2,$$

for every $m \geqq n \geqq 0$. It is also known that A is regular if only we have

$$H^1(\Gamma_n/\Gamma_m, A_m) \cong (0), \qquad m \geqq n,$$

whenever both m and n are sufficiently large.

We shall make use of the following

LEMMA 1. *Let A be a discrete Γ-module. Then A is regular if we have*

$$H^i(\Gamma/\Gamma_m, A_m) \cong (0), \qquad i = 1, 2,$$

for every sufficiently large integer m.

PROOF. We choose m so large that the assumption in Lemma 1 is satisfied. Since the order of Γ/Γ_m is a power of a prime number l, and since the cohomology groups of Γ/Γ_m in A_m vanish for two consecutive dimensions $i = 1, 2$, we get, by Nakayama's theorem on cohomological triviality[6], $H^i(G, A_m) \cong (0)$ for every dimension i and for every subgroup G of Γ/Γ_m. Thus all the cohomology groups of Γ_n/Γ_m in A_m vanish for every m and n such that $m \geqq n \geqq 0$ and that m is sufficiently large, which together with the above referred facts proves Lemma 1.

1.4. REMARK. For an integer m ($\geqq 0$) let ζ_m denote a primitive l^m-th root of unity, and let $F_m = \boldsymbol{Q}(\zeta_{m+1})$ for $m \geqq 0$. Let F denote the union of the increasing sequence of all F_m ($m \geqq 0$). If l is an odd prime number then F/F_0 is a Γ-extension. In such a case the Γ-module $A(K/F)$ has already been considered by Iwasawa [8], and it is known that $A(K/F)$ is regular if and only if the group of principal units[7] of the local cyclotomic field $\Phi_m = \boldsymbol{Q}_l(\zeta_{m+1})$ contains $l^m(l-1)/2-1$ global units in F_m which are independent over Z_l for every $m \geqq 0$. The regularity of $A(K/F)$ is known to be the case when the class

5) The structure theorems on Γ-modules are given by Iwasawa [6] and [7]. For the definition of the regularity of Γ-modules, see [6], p. 187.

6) Cf. e. g. Serre [13], p. 152.

7) A local unit u in Φ_m is said to be principal if $u \equiv 1$ $(\mathfrak{l}_m)$, where $\mathfrak{l}_m$ stands for the valuation ideal of Φ_m.

number of F_0 is prime to l[8].

§ 2. Formulation in terms of characters of idèles.

2.1. Let F/F_0 be, as in § 1.2, a Γ-extension over an algebraic number field F_0 of finite degree, and let $A(K/F)$, F_m etc. be as in § 1.2. In this section we give a necessary and sufficient condition for the regularity of $A(K/F)$ in terms of characters of idèle group of F_m. We denote the idèle group and the principal idèle group of F_m by I_m and P_m, respectively. Let $C_m = I_m/P_m$, and let D_m denote the connected component of the identity of the idèle class group C_m of F_m. We denote by $\mathcal{D}_m$ the group of all continuous characters of I_m with finite orders which are trivial (i. e. take the value 1) on P_m. Then $\mathcal{D}_m$ may be naturally regarded as the dual of the compact abelian group C_m/D_m. Now let $\mathfrak{P}$ be a prime divisor of F_m, and let χ be an element in $\mathcal{D}_m$. Then the local component $\chi_{\mathfrak{P}}$ of χ at $\mathfrak{P}$ is defined by means of the local component of idèles. χ is said to be unramified at $\mathfrak{P}$ if $\chi_{\mathfrak{P}}$ is trivial on the unit group of the $\mathfrak{P}$-completion of F_m[9], and, if otherwise, said to be ramified at $\mathfrak{P}$. χ is ramified at $\mathfrak{P}$ if and only if $\mathfrak{P}$ is ramified by the cyclic extension over F_m with which χ is associated in the sense of class field theory[10].

We define the action of the Galois group Γ_n/Γ_m of F_m/F_n on $\mathcal{D}_m$ by setting

$$\chi^{\sigma}(\tilde{\alpha}) = \chi(\tilde{\alpha}^{\sigma}), \qquad \sigma \in \Gamma_n/\Gamma_m, \tag{2}$$

where $\tilde{\alpha} \in I_m$ and $\tilde{\alpha}^{\sigma}$ is an idèle conjugate to $\tilde{\alpha}$ by σ[11].

Let $\mathcal{A}_m$ denote the subgroup of $\mathcal{D}_m$ which consists of all the elements in $\mathcal{D}_m$ whose orders are powers of l. We denote by S_m the set of all prime divisors of F_m which divide the rational prime divisor l. Now we define two subgroups of $\mathcal{A}_m$ by setting

$$\mathcal{A}'_m = \{\chi \in \mathcal{A}_m \mid \chi \text{ is unramified at every prime divisor of } F_m \text{ outside } S_m\},$$

and

$$\mathcal{A}^F_m = \{\chi \in \mathcal{A}_m \mid \exists m' \geqq m : \ker \chi = P_m \cdot N'(I_{m'})\},$$

where N' stands here for the norm mapping from $I_{m'}$ to I_m. Namely elements of $\mathcal{A}^F_m$ are associated with sub-extensions of F/F_m in the sense of class field

8) Cf. also Ax [**2**], Iwasawa and Sims [**9**], Jehne [**10**], where other results on this subject are found.

9) In the present paper no sign condition is imposed on real infinite components of idèles.

10) For the class field theory used in the present paper without references, see Chevalley [**3**], Weil [**14**], § 1 and Whaples [**15**], Theorem 3. As is well-known, the differentials of F_m defined by Chevalley loc. cit. are nothing but elements of $\mathcal{D}_m$.

11) The Galois group acts on idèles from the right.

theory, and therefore $\mathcal{A}_m^F$ is isomorphic to the dual of the Galois group $G(F/F_m)$. Now, since no prime divisors of F_m outside S_m are ramified by F/F_m[12], we have $\mathcal{A}_m^F \subset \mathcal{A}_m'$. It is easy to observe that $\mathcal{A}_m^F$ and $\mathcal{A}_m'$ are Γ_n/Γ_m-subgroups of $\mathcal{A}_m$.

2.2. Let M denote the maximal abelian extension over F_m. For a while we put $\mathfrak{g} = G(M/F_m)$ and $A = A(M/F_m)$, the dual of $\mathfrak{g}$. By class field theory the dual of C_m/D_m is canonically isomorphic to A. Namely $\mathcal{D}_m$ is canonically isomorphic to A. Let m and n be integers such that $m \geqq n \geqq 0$. Then, by our convention, the Galois group of F_m/F_n is Γ_n/Γ_m. The above mentioned canonical isomorphism gives a canonical Γ_n/Γ_m-isomorphism of $\mathcal{D}_m$ and A by (2) and by a well-known property of the reciprocity map[13].

Let $\mathcal{B}$ be a Γ_n/Γ_m-subgroup of $\mathcal{D}_m$, and let B denote the canonical image of $\mathcal{B}$ in A. We denote by $\Phi(\mathfrak{g}, B)$ the annihilator of B in $\mathfrak{g}$, which is a Γ_n/Γ_m-subgroup of $\mathfrak{g}$. Then B is the dual of $\mathfrak{g}/\Phi(\mathfrak{g}, B)$, and thus $\mathcal{B}$ is canonically Γ_n/Γ_m-isomorphic to the dual of $\mathfrak{g}/\Phi(\mathfrak{g}, B)$.

2.3. Now let K_m denote the unique maximal l-primary abelian extension over F_m in which no prime divisor of F_m outside S_m is ramified. Then, putting $\mathcal{B} = \mathcal{A}_m'$, we have, by § 2.2,

$$A(K_m/F_m) \cong \mathcal{A}_m', \qquad (\Gamma_n/\Gamma_m\text{-isomorphism}),$$

$$A(F/F_m) \cong \mathcal{A}_m^F, \qquad (\Gamma_n/\Gamma_m\text{-isomorphism}).$$

These isomorphisms may be regarded as Γ-isomorphisms in which the action of Γ_m is trivial. On the other hand we have a canonical Γ-isomorphism $A(K_m/F) \cong A(K_m/F_m)/A(F/F_m)$. Hence we have a canonical Γ-isomorphism $A(K_m/F) \cong \mathcal{A}_m'/\mathcal{A}_m^F$. Since, by the remark at the end of § 1.1, we have $A(K_m/F) = A(K/F)_m$, the following lemma is obtained.

LEMMA 2. *$A(K/F)_m$ is Γ_n/Γ_m-isomorphic to $\mathcal{A}_m'/\mathcal{A}_m^F$ for every $m \geqq n \geqq 0$.*

REMARK. 1. The meaning of the suffix m of $A(K/F)$ is described at the beginning of § 1.3.

2. As far as the extension K/F is concerned, only Lemma 2 and the formula (9) in § 2.7 will be necessary for our later argument. Thus we may rather define K as the union of the increasing sequence of all K_m ($m \geqq 0$).

2.4. The following Proposition will be proved in the next § 2.5.

PROPOSITION. *We have*

$$H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m'/\mathcal{A}_m^F) \cong (0) \tag{3}$$

12) Cf. Iwasawa [**6**], p. 218.

13) Cf. e. g. the last formula in Chap. XI, 3 of Serre [**13**], in which the Galois group G acts on the G-modules (in 'class formation') from the left, contrary to our convention.

if and only if the natural homomorphism

$$H^i(\Gamma_n/\Gamma_m, \mathcal{A}'_m) \to H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m) \tag{4}$$

is injective.

Combining Lemmas 1, 2 and Proposition, we get immediately the following

THEOREM 1. *Let F/F_0 be a Γ-extension over an algebraic number field F_0 of finite degree. Then the Γ-module $A(K/F)$ is regular if and only if the natural homomorphisms*

$$H^i(\Gamma/\Gamma_m, \mathcal{A}'_m) \longrightarrow H^i(\Gamma/\Gamma_m, \mathcal{A}_m), \qquad i=1, 2,$$

are both injective for every sufficiently large integer m, where Γ/Γ_m stands for the Galois group of F_m/F_0.

2.5. For the proof of Proposition we prepare some lemmas.

LEMMA 3. *We have $H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m) \cong \boldsymbol{Z}/(l^{m-n})\boldsymbol{Z}$ and $H^2(\Gamma_n/\Gamma_m, \mathcal{A}_m) \cong (0)$ for every $m \geqq n \geqq 0$.*

PROOF. Let D_m denote the connected component of the identity in the idèle class group C_m of F_m. Then we have[14)]

$$\begin{cases} H^1(\Gamma_n/\Gamma_m, C_m) \cong (0), & H^2(\Gamma_n/\Gamma_m, C_m) \cong \boldsymbol{Z}/(l^{m-n})\boldsymbol{Z}, \\ H^1(\Gamma_n/\Gamma_m, D_m) \cong (0), & H^2(\Gamma_n/\Gamma_m, D_m) \cong (0), \end{cases} \tag{5}$$

the first three of which are of general character, and we have the last, because no infinite prime divisor of F_n is ramified by F_m/F_n. From the exact sequence $(1) \to D_m \to C_m \to C_m/D_m \to (1)$, we get the exact sequence

$$\begin{array}{ccccc} H^1(\Gamma_n/\Gamma_m, D_m) & \longrightarrow & H^1(\Gamma_n/\Gamma_m, C_m) & \longrightarrow & H^1(\Gamma_n/\Gamma_m, C_m/D_m) \\ \uparrow & & & & \downarrow \\ H^2(\Gamma_n/\Gamma_m, C_m/D_m) & \longleftarrow & H^2(\Gamma_n/\Gamma_m, C_m) & \longleftarrow & H^2(\Gamma_n/\Gamma_m, D_m), \end{array}$$

because Γ_n/Γ_m is cyclic. Then we get by (5) $H^1(\Gamma_n/\Gamma_m, C_m/D_m) \cong (0)$ and $H^2(\Gamma_n/\Gamma_m, C_m/D_m) \cong \boldsymbol{Z}/(l^{m-n})\boldsymbol{Z}$. Since $\mathcal{D}_m$ is dual to the compact abelian group C_m/D_m, and since Γ_n/Γ_m is cyclic, $H^1(\Gamma_n/\Gamma_m, \mathcal{D}_m)$ is dual to $H^2(\Gamma_n/\Gamma_m, C_m/D_m)$, and $H^2(\Gamma_n/\Gamma_m, \mathcal{D}_m)$ is dual to $H^1(\Gamma_n/\Gamma_m, C_m/D_m)$. On the other hand we have $H^i(\Gamma_n/\Gamma_m, \mathcal{D}_m) \cong H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m)$, because the order of Γ_n/Γ_m is a power of a prime number l. Now Lemma 3 follows from the above mentioned duality.

Now we prepare some notations. Let the element ν in the group ring $\boldsymbol{Z}[\Gamma_n/\Gamma_m]$ be defined by $\nu = 1+\sigma+\cdots+\sigma l^{m-n-1}$, where σ is a generator of Γ_n/Γ_m. Let in general M be a multiplicative abelian Γ_n/Γ_m-group. Then we put

$$B^1(M) = \{a^{1-\sigma} \mid a \in M\}, \qquad C^1(M) = \{a \in M \mid a^\nu = 1\},$$

$$B^2(M) = \{a^\nu \mid a \in M\}, \qquad C^2(M) = \{a \in M \mid a^\sigma = a\}.$$

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14) Cf. Artin and Tate [**1**], Chevalley [**4**], Hochschild and Nakayama [**5**], Weil [**14**].

These notations will be retained in the following. Moreover we identify $H^i(\Gamma_n/\Gamma_m, M)$ with $C^i(M)/B^i(M)$, where $i=1, 2$.

LEMMA 4. *$H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m^F)$ and $H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m)$ are canonically isomorphic for every $m \geqq n \geqq 0$.*

PROOF. Since F/F_n is abelian, every element in $\mathcal{A}_m^F$ is invariant under the action of the Galois group Γ_n/Γ_m. Let $\chi_{m,2m-n}$ be an element in $\mathcal{A}_m^F$ whose order is equal to l^{m-n}. Then $\chi_{m,2m-n}$ generates $C^1(\mathcal{A}_m^F)$. Put now $\chi_{m,m+1}=(\chi_{m,2m-n})^{l^{m-n-1}}$. Then $\chi_{m,m+1}$ is associated with the class field F_{m+1}/F_m. Namely F_{m+1}/F_m is the class field defined over the kernel of $\chi_{m,m+1}$. Let $\chi_{n,m+1}$ be an element in $\mathcal{A}_n^F$ which is associated with F_{m+1}/F_n. Then by the translation theorem in class field theory we have

$$\ker \chi_{m,m+1} = \{\tilde{\mathfrak{a}} \in I_m \mid N(\tilde{\mathfrak{a}}) \in \ker \chi_{n,m+1}\} ,$$

where N stands here for the norm mapping from I_m to I_n. The factor group $I_n/\ker \chi_{n,m+1}$ is cyclic and of order l^{m-n+1}. We denote by ι the natural injective homomorphism of I_n into I_m, and let $\tilde{\mathfrak{b}}$ be an idèle of F_n which belongs to a generating coset of $I_n/\ker \chi_{n,m+1}$. Then $N(\iota(\tilde{\mathfrak{b}}))=\tilde{\mathfrak{b}}^{l^{m-n}} \notin \ker \chi_{n,m+1}$; namely we have

$$\chi_{m,m+1}(\iota(\tilde{\mathfrak{b}})) \neq 1 . \tag{6}$$

If there exists an element χ in $\mathcal{A}_m$ such that $\chi_{m,m+1}=\chi^{1-\sigma}$, then we have, by (2), $\chi_{m,m+1}(\iota(\tilde{\mathfrak{b}}))=\chi(1)=1$, which contradicts (6). Thus we get $B^1(\mathcal{A}_m) \cap \mathcal{A}_m^F = B^1(\mathcal{A}_m^F)=(1)$, which together with Lemma 3 proves the assertion in Lemma 4 for $i=1$ (and also for odd i by the periodicity of cyclic cohomologies). For $i=2$ it is easily observed that $\mathcal{A}_m^F = C^2(\mathcal{A}_m^F)=B^2(\mathcal{A}_m^F)$, which together with Lemma 3 proves the assertion for $i=2$. Lemma 4 is proved.

We notice in particular that the natural homomorphism

$$H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m^F) \longrightarrow H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m') \tag{7}$$

is injective.

Now we prove Proposition stated in § 2.4. Assume that the natural homomorphism for $i=1$ in Proposition is injective. Then it follows from the injectiveness of (7) and Lemmas 3, 4, that $H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m')$ and $H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m)$ are canonically isomorphic. From the exact sequence

$$\begin{tikzcd} H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m^F) \arrow[r, "i_1"'] & H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m') \arrow[r] & H^1(\Gamma_n/\Gamma_m, \mathcal{A}_m'/\mathcal{A}_m^F) \arrow[d] \\ H^2(\Gamma_n/\Gamma_m, \mathcal{A}_m'/\mathcal{A}_m^F) \arrow[u] & H^2(\Gamma_n/\Gamma_m, \mathcal{A}_m') \arrow[l] & H^2(\Gamma_n/\Gamma_m, \mathcal{A}_m^F) \arrow[l] \end{tikzcd} \tag{8}$$

it then follows that i_1 in the above sequence is a surjective isomorphism. Then the isomorphism (3) in Proposition for $i=1$ follows from the fact that $H^2(\Gamma_n/\Gamma_m, \mathcal{A}_m^F) \cong (0)$. Conversely assume now (3) for $i=1$, then by the above sequence we get $H^1(\Gamma_n/\Gamma_m, \mathcal{A}'_m) \cong \boldsymbol{Z}/(l^{m-n})\boldsymbol{Z}$, which means by Lemmas 3 and 4 that the natural homomorphism for $i=1$ in Proposition is injective. For $i=1$ this completes the proof of Proposition. Our proposition for $i=2$ follows similarly from Lemmas 3 and 4 and the above sequence (8).

By the above and by the fact referred from [7] in § 1.3 we observe also the following

THEOREM 2. *Let the notation be as in Theorem* 1. *Then the natural homomorphisms*

$$H^i(\Gamma_n/\Gamma_m, \mathcal{A}'_m) \longrightarrow H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m), \qquad i=1, 2,$$

are both bijective for every $m \geqq n \geqq 0$ *if and only if the* Γ*-module* $A(K/F)$ *is regular.*

2.6.[15] To introduce the next theorem we first prepare some notations concerning infinite abelian groups. Let $Z(l, \infty)$ denote the group of all the roots of unity whose orders are powers of l. An abelian group M is said to be a torsion l-abelian group if every element of M is of order a power of l. Let $M^{(0)}$ be the subgroup of M which consists of all the elements x of M with $x^l=1$. Then $M^{(0)}$ may be regarded as a vector space over the prime field of characteristic l, of which dimension we shall call the rank of the torsion l-abelian group M. A subgroup N of M is said to be divisible if, for any element x of N and any power l^r of l, there exists an element y in N such that $x=y^{l^r}$. The torsion l-abelian group M contains a unique largest divisible subgroup M_∞, and M_∞ is isomorphic to the direct product of finite or infinite number of $Z(l, \infty)$. If the rank of M is finite, then M is the direct product of M_∞ by a finite subgroup of M. After the terminology of Kubota [12] we shall call the rank of M_∞ the dimension of M, and we denote it by $\dim M$.

Let M and M' be torsion l-abelian groups, and let there be given a homomorphism of M onto M' whose kernel is finite. Then we have $\dim M = \dim M'$[16].

2.7. We next consider the ring $R_m = F_m \otimes \boldsymbol{Q}_l$. Let R_m^* denote the multiplicative group of all the regular elements in R_m. Then R_m^* is canonically identified with the direct product $\prod_{\mathfrak{l} \in S_m} F_{m,\mathfrak{l}}^*$ where $F_{m,\mathfrak{l}}^*$ stands for the multiplicative group of the $\mathfrak{l}$-completion of F_m for $\mathfrak{l} \in S_m$. S_m denotes, as before, the set of all prime divisors of F_m which divide the rational prime divisor l. The elements in R^* which are congruent 1 modulo l form a multiplicative group H_{R_m}, and the power u^α is defined for every $u \in H_{R_m}$ and $\alpha \in \boldsymbol{Z}_l$. The dimension over $\boldsymbol{Z}_l$ of H_{R_m} (modulo the finite torsion subgroup if $l=2$) is equal to the degree d_m of F_m over $\boldsymbol{Q}$, as observed from the well-known structure theo-

15) Cf. Kaplansky [**11**].

16) Notice that any divisible subgroup of a torsion l-abelian group is a direct summand, cf. Kaplansky, loc. cit. p. 8.

rem of the local unit groups. Let $r_l(m)$ denote the dimension of Z_l-subspace of H_{R_m} spanned by units $\varepsilon\ (=\varepsilon\otimes 1)$ of F_m contained in H_{R_m}. Then the equality $\dim \mathcal{A}_m=\dim \mathcal{A}'_m=d_m-r_l(m)$ is known by Kubota [**12**], Theorem 5, where $\dim \mathcal{A}_m$ etc. are defined in § 2.6. Then Lemma 2 entails

$$\dim A(K/F)_m=\dim \mathcal{A}'_m-1=d_m-r_l(m)-1\,, \tag{9}$$

where $d_m=[F_m:\boldsymbol{Q}]$. Furthermore the Γ-module $A(K/F)$ is Γ-finite; namely the rank of $A(K/F)_m$ is finite for every $m\geqq 0$.

THEOREM 3. *Let F/F_0 be a Γ-extension over an algebraic number field F_0 of finite degree. Let $A(K/F)$ be the Γ-module described in* § 1.1, *and let $r_l(m)$ be as above. Assume that the Γ-module $A(K/F)$ is regular. Then we have*

$$r_l(m)=l^m(r_l(0)+1)-1 \tag{10}$$

for every $m\geqq 0$.

REMARK. Let $r_\infty(m)$ denote the usual rank of the unit group of F_m. If we assume moreover $r_l(0)=r_\infty(0)$ in Theorem 3, then (10) implies $r_l(m)=r_\infty(m)$ for every $m\geqq 0$, because no infinite prime divisor is ramified by F/F_0. The proof of Theorem 3 given below shows that the equality (10) follows if we assume only the regularity of the maximal divisible submodule of $A(K/F)$.

Theorem 3 is a direct consequence of (9) and the following

LEMMA 5. *Let A be a discrete Γ-finite Γ-module. If A is regular, then* $\dim A_n=l^n\dim A_0$ *for every* $n\geqq 0$[17].

PROOF. If A is Γ-finite and *regular*, then A is a sum of a divisible regular submodule B' of finite rank and a characteristic submodule C such that $C\cong E(m_1,\cdots,m_s)/D$ for some $0\leqq m_i\leqq\infty$ and for a finite submodule D of $E(m_1,\cdots,m_s)$. The intersection $B'\cap C$ is finite[18]. We have then the surjective homomorphisms f and g such that

$$\bar{A}=B\oplus E(m_1,\cdots,m_s)\xrightarrow{f}B\oplus C\xrightarrow{g}A=B+C$$

(where $\oplus$ stands for the direct sum), and that the kernel $\mathfrak{K}$ of $g\circ f$ is finite. Let σ denote here a generator of Γ/Γ_n, and put $(\bar{A}_n)'=\{a\in A\,|\,(1-\sigma)a\in\mathfrak{K}\}$, the inverse image of A_n by $g\circ f$. Then we have $(\bar{A}_n)'/\mathfrak{K}=A_n$. Since $\mathfrak{K}$ is finite, we have $((\bar{A}_n)':\bar{A}_n)<\infty$. Thus we get $\dim(\bar{A}_n)'=\dim\bar{A}_n$ and $\dim(\bar{A}_n)'=\dim A_n$, and consequently $\dim\bar{A}_n=\dim A_n$. Thus the proof is reduced to the cases where $A=B$ (divisible, regular and of finite rank) or $A=E(m_1,\cdots,m_s)=E(m_1)\oplus\cdots\oplus E(m_s)$. Lemma 5 is then a direct consequence of the fact that

17) In the proof of Lemma 5 notations and terminologies are in accordance with those of Iwasawa [**6**]; cf. in particular Theorems 1 and 2 of [**6**]. Submodule, homomorphism etc. mean Γ-submodule, Γ-homomorphism, etc.

18) Because we have $B\supset B'$, where B is the submodule appearing in loc. cit. Theorem 1, and $B\cap C$ is finite. Moreover in our case we have $B=B'$.

B_n and $E(m_i)_n$ for $m_i < \infty$ are finite modules for every $n \geqq 0$, and that $\dim E(\infty)_n = l^n$ for every n[19].

§ 3. Γ-extensions over imaginary quadratic fields.

3.1. In this section we shall prove the following

THEOREM 4. *Let F_0 be an imaginary quadratic extension over $\boldsymbol{Q}$ in which the fixed prime number l is not fully decomposed: namely S_0 consists of a single element $\mathfrak{l}$. Furthermore we assume that the class number of F_0 is prime to l and that the $\mathfrak{l}$-completion of F_0 contains no primitive l-th root of unity (this last assumption being always the case if $l > 3$). Let F/F_0 be a Γ-extension over F_0*[20]. *Then the Γ-module $A(K/F)$ is regular, and the natural homomorphisms*

$$H^i(\Gamma_n/\Gamma_m, \mathcal{A}'_m) \longrightarrow H^i(\Gamma_n/\Gamma_m, \mathcal{A}_m), \qquad i = 1, 2,$$

are both injective for every $m \geqq n \geqq 0$.

Let F/F_0 be as in Theorem 4. Since $r_l(0) = r_\infty(0) = 0$, we get, by Theorems 3 and 4, $r_l(m) = r_\infty(m)$ for every finite intermediate field F_m of F/F_0[21].

For the proof of Theorem 4 it suffices, by § 2, to show the following Lemmas 6 and 7.

LEMMA 6. *If the ground field F_0 of a Γ-extension F/F_0 is an imaginary quadratic extension over $\boldsymbol{Q}$, then the natural homomorphism*

$$H^1(\Gamma/\Gamma_m, \mathcal{A}'_m) \longrightarrow H^1(\Gamma/\Gamma_m, \mathcal{A}_m)$$

is injective for every $m \geqq 0$.

LEMMA 7. *Under the same assumptions in Theorem 4 the natural homomorphism*

$$H^2(\Gamma/\Gamma_m, \mathcal{A}'_m) \longrightarrow H^2(\Gamma/\Gamma_m, \mathcal{A}_m)$$

is injective for every $m \geqq 0$.

3.2. PROOF OF LEMMA 6. In the proofs of Lemmas 6 and 7 the Galois group Γ/Γ_m of F_m/F_0 is simply denoted by G, and σ stands for a generator of G. Let the element ν in the group ring $\boldsymbol{Z}[G]$ be defined as in § 2.5. S_m(resp. S_0) is, as before, the set of all prime divisors of F_m (resp. F_0) which divide l. Let $\mathfrak{p}$ be any prime divisor of F_0 outside S_0. We put $U_m^{\mathfrak{p}} = \prod_{\mathfrak{P} \mid \mathfrak{p}} U_m^{\mathfrak{P}}$, where $U_m^{\mathfrak{P}}$ stands for the unit group of the $\mathfrak{P}$-completion of F_m for a prime divisor $\mathfrak{P}$ of

19) Cf. loc. cit. in particular Lemma 5.1.

20) There exist two independent Γ-extensions over F_0 (with respect to the fixed prime number l); cf. Kubota [12], Theorem 5. Thus our F/F_0 is not necessarily 'cyclotomic'. Here we note also that our argument in the proof of this theorem is also applicable for Γ-extentions over $\boldsymbol{Q}$.

21) The corresponding fact for Γ-extensions over $\boldsymbol{Q}$ is known by Jehne [**10**] as 'o-th stability' of l.

F_m such that $\mathfrak{P}|\mathfrak{p}$. Since $\mathfrak{p} \notin S_0$, $\mathfrak{p}$ is unramified by F_m/F_0, and we have[22)]

$$H^i(G, U_m^{\mathfrak{p}}) \cong (0), \qquad \text{for } i=1, 2. \tag{11}$$

Now let χ_0 be an element in $\mathcal{A}_m' \cap B^1(\mathcal{A}_m)$, where the notation $B^1(\mathcal{A}_m) = \{\chi^{1-\sigma} | \chi \in \mathcal{A}_m\}$ is defined in § 2.5. Then there exists an element χ_1 in $\mathcal{A}_m$ for which

$$\chi_0 = \chi_1^{1-\sigma}, \qquad \chi_1 \in \mathcal{A}_m. \tag{12}$$

We consider χ_1 on $U_m^{\mathfrak{p}}$, $U_m^{\mathfrak{p}}$ being regarded as imbedded in the idèle group I_m of F_m. Since $\chi_1^{1-\sigma}$ is unramified at $\mathfrak{P}|\mathfrak{p}$, it follows from (2) and (11) for $i=1$ that χ_1 is trivial on $C^1(U_m^{\mathfrak{p}})$. Thus we can define a character $\varphi_{\mathfrak{p}}'$ of $B^2(U_m^{\mathfrak{p}})$ by setting

$$\varphi_{\mathfrak{p}}'(\tilde{\mathfrak{a}}^{\nu}) = \chi_1(\tilde{\mathfrak{a}}), \qquad \tilde{\mathfrak{a}} \in U_m^{\mathfrak{p}}. \tag{13}$$

Then $\varphi_{\mathfrak{p}}'$ is defined on $C^2(U_m^{\mathfrak{p}})$ by virtue of (11) for $i=2$. Let now N denote the norm mapping from I_m to I_0. We put

$$\varphi_{\mathfrak{p}}(N(\tilde{\mathfrak{a}})) = \varphi_{\mathfrak{p}}'(\tilde{\mathfrak{a}}^{\nu}), \qquad \tilde{\mathfrak{a}} \in U_m^{\mathfrak{p}}. \tag{14}$$

Then $\varphi_{\mathfrak{p}}$ is a character defined on the unit group $U_0^{\mathfrak{p}}$ of the $\mathfrak{p}$-completion of F_0 for $\mathfrak{p} \notin S_0$. Since $\varphi_{\mathfrak{p}}$ is of finite order, $\varphi_{\mathfrak{p}}$ is continuous on $U_0^{\mathfrak{p}}$.

For a non-zero element α of F_0 we denote by $\tilde{\alpha}$ the element in the principal idèle group P_0 corresponding to α, and let τ_0 denote the endomorphism of I_0 given by

$$(\tau_0(\tilde{\mathfrak{a}}))_{\mathfrak{l}} = \tilde{\mathfrak{a}}_{\mathfrak{l}}, \qquad \text{for } \mathfrak{l} \in S_0,$$

$$(\tau_0(\tilde{\mathfrak{a}}))_{\mathfrak{p}} = 1, \qquad \text{for } \mathfrak{p} \notin S_0,$$

where $\tilde{\mathfrak{a}} \in I_0$. Let E_0 denote the unit group of F_0. We define a character φ_{S_0} on $\tau_0(\tilde{E}_0)$ by setting

$$\varphi_{S_0}(\tau_0(\tilde{\varepsilon})) = \prod_{\mathfrak{l} \in S_0} \varphi_{\mathfrak{p}}^{-1}((\tilde{\varepsilon}\tau_0(\tilde{\varepsilon})^{-1})_{\mathfrak{p}}), \qquad \varepsilon \in E_0. \tag{15}$$

Since $\tau_0(\tilde{\varepsilon}) \to \tilde{\varepsilon}\tau_0(\tilde{\varepsilon})^{-1}$ is an isomorphism, and since the right hand side of (15) is a character on $\prod_{\mathfrak{p} \notin S_0} U_0^{\mathfrak{p}}$, φ_{S_0} is an (algebraic) character defined on $\tau_0(\tilde{E}_0)$. Moreover, since $\tau_0(\tilde{E}_0)$ is a finite group, φ_{S_0} is continuous on $\tau_0(\tilde{E}_0)$ with respect to the topology induced by that of $U_{S_0} = \prod_{\mathfrak{l} \in S_0} U_0^{\mathfrak{l}}$, where $U_0^{\mathfrak{l}}$ is the unit group of the $\mathfrak{l}$-adic completion of F_0. Since $\tau_0(\tilde{E}_0)$ is closed in U_{S_0}, and since φ_{S_0} is of order a power of l, we can extend φ_{S_0} onto U_{S_0} as a continuous character of order a power of l, which we shall denote by the same notation φ_{S_0}[23)]. We denote by U_0 the unit idèle group of F_0, and we define φ on U_0 by setting

22) Cf. e. g. Chevalley [4], Theorem 12.1.

23) We note that every continuous character on U_{S_0} is of finite order.

$\varphi=\varphi_{S_0}\cdot\prod_{\mathfrak{p}\in S_0}\varphi_{\mathfrak{p}}$. Then φ is a continuous character on U_0 with order a power of l. φ is trivial on $U_0\cap P_0$ because of $U_0\cap P_0=\tilde{E}_0$ and (15). Thus we can extend φ onto $P_0\cdot U_0$ by putting $\varphi(\tilde{\alpha})=1$ for every $\tilde{\alpha}\in P_0$. The continuous character φ thus defined on $P_0\cdot U_0$ extends now onto I_0, preserving the property that the order of φ is a power of l, because the closed subgroup $P_0\cdot U_0$ of I_0 is of finite index. Namely there exists an element φ in $\mathcal{A}_0$ whose $\mathfrak{p}$-component on $U_0^{\mathfrak{p}}$ is given by (14). Then there exists an element $\tilde{\varphi}$ in $\mathcal{A}_m$ such that

$$\tilde{\varphi}(\tilde{\mathfrak{a}})=\varphi(N(\tilde{\mathfrak{a}}))\,,\qquad \text{for } \tilde{\mathfrak{a}}\in I_m\,. \tag{16}$$

By virtue of (13) and (14), $\chi_1\cdot\tilde{\varphi}^{-1}$ is unramified at every prime divisor of F_m outside S_m; namely $\chi_1\cdot\tilde{\varphi}^{-1}\in\mathcal{A}'_m$. Moreover it is observed by (16) that $\tilde{\varphi}$ belongs to $C^2(\mathcal{A}_m)$. We have thus $(\chi_1\cdot\tilde{\varphi}^{-1})^{1-\sigma}=\chi_1^{1-\sigma}=\chi_0$. The existence of such $\chi_1\cdot\tilde{\varphi}^{-1}$ in $\mathcal{A}'_m$ is nothing but the assertion in Lemma 6.

REMARK. In the above proof the assumption that F_0 is an imaginary quadratic field is essentially used only in the form $r_l(0)=r_\infty(0)$.

3.3. PROOF OF LEMMA 7. By the assumption in Theorem 4, F/F_0 contains no non-trivial unramified extension, and it follows further that S_m consists of a single element $\mathfrak{L}: S_m=\{\mathfrak{L}\}$. We denote by Φ_m the $\mathfrak{L}$-completion of F_m and by Φ_0 the $\mathfrak{l}$-completion of F_0, where $S_0=\{\mathfrak{l}\}$. Then in our case the Galois group of Φ_m/Φ_0 can be identified with that of F_m/F_0. Moreover, since the class number of F_0 is assumed to be prime to l, it follows in particular that *no non-principal ideal of F_0 becomes principal in F_m.*

Let U_m denote the unit idèle group of F_m and U'_m the group of unit idèles of F_m whose $\mathfrak{L}$-components are 1. We denote by $\tau_{\mathfrak{L}}$ the endomorphism of I_m given by

$$(\tau_{\mathfrak{L}}(\tilde{\mathfrak{a}}))_{\mathfrak{L}}=\tilde{\mathfrak{a}}_{\mathfrak{L}}\,,\qquad \text{for } \mathfrak{L}\in S_m\,,$$

$$(\tau_{\mathfrak{L}}(\tilde{\mathfrak{a}}))_{\mathfrak{P}}=1\,,\qquad \text{for } \mathfrak{P}\notin S_m\,,$$

where $\tilde{\mathfrak{a}}\in I_m$. We put $U_{\mathfrak{L}}=\tau_{\mathfrak{L}}(U_m)$. We denote by E_m the unit group of F_m.

Now let χ_0 be an element in $\mathcal{A}'_m\cap B^2(\mathcal{A}_m)$. Then there exists an element χ_1 in $\mathcal{A}_m$ for which

$$\chi_0=\chi_1^{\nu}\,,\qquad \chi_1\in\mathcal{A}_m\,. \tag{17}$$

We define a character χ_2 on U'_m by setting

$$\chi_2(\tilde{\mathfrak{a}})=\chi_1^{-1}(\tilde{\mathfrak{a}})\,,\qquad \text{for } \tilde{\mathfrak{a}}\in U'_m\,. \tag{18}$$

Since $P_m\cap U'_m=(1)$, χ_2 extends onto $P_m\cdot U'_m$ by setting

$$\chi_2(\tilde{\alpha})=1\,,\qquad \text{for } \tilde{\alpha}\in P_m\,. \tag{19}$$

Now let $\tilde{\mathfrak{a}}$ be an element in $P_m\cdot U'_m\cap I_m^{\nu}$, and let $\tilde{\mathfrak{a}}=\tilde{\alpha}\tilde{\mathfrak{b}}$, where $\tilde{\alpha}\in P_m$ and

$\tilde{\mathfrak{b}} \in U'_m$. We get $\alpha \in F_0$, because $\tilde{\mathfrak{a}}$, and thus in particular the $\mathfrak{L}$-component of $\tilde{\mathfrak{a}}$, is invariant under the action of the Galois group G. Let $(\tilde{\mathfrak{a}})$ denote the ideal of F_m corresponding to the idèle $\tilde{\mathfrak{a}}$. Then we have $(\tilde{\mathfrak{a}}) = (\alpha)$, $\alpha \in F_0$. The principal ideal (α) is a norm of an ideal of F_m. Since there exists only one prime divisor which is ramified by F_m/F_0, it follows that α is a norm of an element in F_m[24]. Thus $\tilde{\alpha}$ is an element in $B^2(I_m) = I_m^{\nu}$. We get then $\tilde{\mathfrak{b}} \in I_m^{\nu}$, because $\tilde{\mathfrak{a}} = \tilde{\alpha}\tilde{\mathfrak{b}} \in I_m^{\nu}$. From (11) it follows further that $\tilde{\mathfrak{b}} \in U_m'^{\nu}$. Hence $P_m \cdot U'_m \cap I_m^{\nu} = (P_m \cdot U'_m)^{\nu}$. This enables us to extend χ_2 on $P_m \cdot U'_m \cdot I_m^{\nu}$ by setting

$$\chi_2(\tilde{\mathfrak{a}}^{\nu}) = 1, \qquad \text{for } \tilde{\mathfrak{a}} \in I_m, \tag{20}$$

because χ_2 previously defined on $P_m \cdot U'_m$ is trivial on $(P_m \cdot U'_m)^{\nu}$.

We next consider the continuity of χ_2 defined on $P_m \cdot U'_m \cdot I_m^{\nu}$ by (18), (19) and (20). For this purpose it suffices to consider χ_2 only on $P_m \cdot U'_m \cdot I_m^{\nu} \cap U_m$, which is the direct product of $\boldsymbol{D} = P_m \cdot U'_m \cdot I_m^{\nu} \cap U_L$ and U'_m (as topological group). That χ_2 is continuous on U'_m is clear by (18). Thus we have only to consider χ_2 on $\boldsymbol{D}$. Let $\tilde{\mathfrak{a}} \in \boldsymbol{D}$ and $\tilde{\mathfrak{a}} = \tilde{\alpha}\tilde{\mathfrak{b}}\tilde{\mathfrak{c}}^{\nu}$, where $\tilde{\alpha} \in P_m$, $\tilde{\mathfrak{b}} \in U'_m$, $\tilde{\mathfrak{c}} \in I_m$. Then the ideal (α) corresponding to the principal idèle $\tilde{\alpha}$ is an image by ν of an ideal of F_m. Then, by the remark at the beginning of this § 3.3, there exists an element α' in F_0 and a unit ε of F_m for which we have $\alpha = \alpha' \cdot \varepsilon$. Then there exists $\beta \in F_m$ such that $\alpha' = \beta^{\nu}$[24]. Thus we have $\tilde{\mathfrak{a}} = \tilde{\beta}^{\nu} \cdot \tilde{\varepsilon} \cdot \tilde{\mathfrak{b}} \cdot \tilde{\mathfrak{c}}^{\nu}$, where $\beta \in F_m^*$, $\varepsilon \in E_m$, $\mathfrak{b} \in U'_m$ and $\tilde{\mathfrak{c}} \in I_m$. Hence the $\mathfrak{L}$-component of $\tilde{\mathfrak{a}}$ is of the form $\varepsilon \cdot \alpha^{\nu}$, where $\varepsilon \in E_m$ and $\alpha \in U_{\mathfrak{L}}$. Conversely an idèle of F_m whose $\mathfrak{L}$-component is of the form $\varepsilon \cdot \alpha^{\nu}$ ($\in U_{\mathfrak{L}}$) and all other local components are 1 is clearly contained in $\boldsymbol{D}$.

We now consider $U_{\mathfrak{L}}$ as contained in the multiplicative group Φ_m^* of the non-zero elements of Φ_m. The structure of $U_{\mathfrak{L}}$ is as follows. Let V denote the group of all the roots of unity contained in Φ_m whose orders are prime to l. Then the order v of V is equal to the absolute norm of $\mathfrak{L}$ minus 1. Let H denote the subgroup of $U_{\mathfrak{L}}$ which consists of all the elements α in $U_{\mathfrak{L}}$ such that $\alpha \equiv 1 \pmod{\mathfrak{L}}$. As topological group, $U_{\mathfrak{L}}$ is the direct product of the subgroups V and H. Now, by our assumption, Φ_0 contains no primitive l-th root of unity. This immediately implies that Φ_m also contains no primitive l-th root of unity, because $[\Phi_m : \Phi_0] = l^m$ in our case. In such a case H is, as topological group, isomorphic to the direct product of $[\Phi_m : \boldsymbol{Q}_l]$ groups all isomorphic to the additive group of $\boldsymbol{Z}_l$. In particular H is torsion free. If we put $H^{(1)} = C^1(H)$ and $H^{(2)} = C^2(H)$, then we have $H^{(1)} \cap H^{(2)} = (1)$. Moreover it is easily observed by local class field theory that the direct product $H^{(1)} \cdot H^{(2)}$ is an open subgroup of finite index in H. Therefore, if $\alpha_1 \in H^{(1)}$, $\alpha_2 \in H^{(2)}$ and

24) Cf. Iwasawa [8], p. 550.

$\alpha_1 \cdot \alpha_2 \equiv 1 \pmod{\mathfrak{L}^c}$ for a sufficiently large integer c, then there exists an integer d independent of c such that $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod{\mathfrak{L}^{c-d}}$.

Since χ_2 is of order a power of l on $\boldsymbol{D}$, χ_2 is continuous on $\boldsymbol{D}$ if and only if χ_2^ν is continuous on $\boldsymbol{D}$. Then χ_2 is continuous on $\boldsymbol{D}$ if χ_2 is continuous on $\boldsymbol{D}'$, where $\boldsymbol{D}' = \{\varepsilon\alpha^\nu \mid \varepsilon \in \tau_{\mathfrak{L}}(\tilde{E}_m) \cap H,\ \alpha^\nu \in H\}$. Since the assumptions in Theorem 4 implies that l is odd, we have $\varepsilon^\nu = 1$ if $\varepsilon \in H$: namely we have $\tau_{\mathfrak{L}}(\tilde{E}_m) \cap H \subset H^{(1)}$. Thus, if $\varepsilon\alpha^\nu \equiv 1 \pmod{\mathfrak{L}^c}$ for a sufficiently large integer c, where $\varepsilon \in \tau_{\mathfrak{L}}(\tilde{E}_m) \cap H$ and $\alpha^\nu \in H^{(2)}$, then we have $\varepsilon \equiv 1 \pmod{\mathfrak{L}^{c-d}}$. On the other hand we have $\chi_2(\varepsilon\alpha^\nu) = \chi_2(\varepsilon) = \chi_1^{-1}(\varepsilon)$ locally at $\mathfrak{L}$. Since χ_1 is continuous, χ_2 is also continuous on $\boldsymbol{D}'$. Therefore, as noticed above, it follows that χ_2 is continuous on $P_m \cdot U'_m \cdot I_m^\nu$.

Now, as a continuous character, χ_2 extends uniquely onto the closure of $P_m \cdot U'_m \cdot I_m^\nu$. By this procedure the value group of χ_2 remains unchanged, because the order of the original χ_2 is finite. The closure of $P_m \cdot U'_m \cdot I_m^\nu$ is a closed subgroup of the locally compact abelian group I_m, and therefore χ_2 now extends onto the whole group I_m. The restriction of χ_2 thus defind on I_m to the unit idèle group U_m is of finite order. Hence χ_2 thus extended onto I_m is of finite order, because $I_m/P_m \cdot U_m$ is of finite order and $\chi_2(P_m) = 1$. Then we can take χ_2 extended on I_m so as to be of order a power of l. χ_2 is then an element of $\mathcal{A}_m$, and moreover we have, by (18), $\chi_1 \cdot \chi_2 \in \mathcal{A}'_m$. It follows finally from (17), (20) and (2) that $(\chi_1 \cdot \chi_2)^\nu = \chi_1^\nu = \chi_0$. The existence of such $\chi_1 \cdot \chi_2$ in $\mathcal{A}'_m$ completes the proof of Lemma 7.

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References

[1] E. Artin and J. Tate, Class field theory, Princeton notes, 1951, distributed by Harvard University.

[2] J. Ax, On the unit of an algebraic number field, Illinois J. Math., **9** (1965), 584–589.

[3] C. Chevalley, La théorie du corps de classes, Ann. of Math., **41** (1940), 394–418.

[4] C. Chevalley, Class field theory, Nagoya, 1954.

[5] G. Hochschild and T. Nakayama, Cohomology in class field theory, Ann. of Math., **55** (1952), 348–366.

[6] K. Iwasawa, On Γ-extensions of algebraic number fields, Bull. Amer. Math. Soc., **65** (1959), 183–226.

[7] K. Iwasawa, On some properties of Γ-finite modules, Ann. of Math., **70** (1959), 291–312.

[8] K. Iwasawa, On the theory of cyclotomic fields, ibid., 531–561.

[9] K. Iwasawa and C. C. Sims, Computation of invariants in the theory of cyclotomic fields, J. Math. Soc. Japan, **18** (1966), 86–96.

[10] W. Jehne, Zur Verschärfung des F. K. Schmidtschen Einheitensatzes, Math. Z., **77** (1961), 439–452.

[11] I. Kaplansky, Infinite abelian groups, University of Michigan Publ. in Math.. No. 2, 1954.
[12] T. Kubota, Galois group of the maximal abelian extension over an algebraic number fields. Nagoya Math. J., **12** (1957), 177-189.
[13] J.P. Serre, Corps locaux, Paris, 1962.
[14] A. Weil, Sur la théorie du corps de classes, J. Math. Soc. Japan, **3** (1951), 1-35.
[15] G. Whaples, Non-analytic class field theory and Grunwald's theorem, Duke Math. J. **3** (1942), 455-473.

[REMARK added in proof 6 Feb. 1968 at the 'Goethe Institut' in Brannenburg] I have heard in Japan that A. Brumer has proved the p-adic analogue of Dirichlet's unit theorem (cf. § 1.4 and § 2.7) for absolutely abelian fields and that his paper will appear in a forthcoming issue of Mathematika, to which I am not yet accessible.