J. Math. Soc. Japan
Vol. 19, No. 1, 1967

A study of transformation groups on manifolds

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(Received June 25, 1966)

§ 0. Introduction.

It might be interesting to ask to what extent the topological and algebraic structures of the group $H(M)$ of the homeomorphisms of a manifold M reflect the topological structure of M. In spite of its importance, unfortunately, little has been known about it. Though it seems very difficult to determine the structures of $H(M)$, many conjectures or problems have been set up by several authors.

Among them, our main concern in this paper is a problem raised by A. M. Gleason and R. S. Palais [2], which is given as follows:

"*Is the closure of a homomorphic image of a connected Lie group G into $H(M)$ locally compact?*"

In this connection, the author in a previous paper [4] has shown the following:

i) If G is a connected Lie group with compact center and if the image of the adjoint representation $Ad(G)$ of G is closed in the general linear group $GL(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G, then any monomorphic image of G into $H(M)$ is closed and locally compact.

ii) Let φ be a monomorphism from G into $H(M)$. If $\varphi(V)$ has the locally compact closure for any closed vector subgroup V of G, then so does $\varphi(G)$.

iii) If M is connected and one dimensional, then any monomorphic image of any vector group has the locally compact closure.

The manifolds treated in i)—iii) are all assumed to satisfy the second countability axiom and the topology for $H(M)$ is of course the compact open topology. Under the same assumptions, the present author has also shown in [6] the following:

iv) Let φ be a homomorphism from a vector group V into $H(M)$ of a connected manifold M. If every orbit $\varphi(V)(x)$, $x \in M$, is homeomorphic to a circle or a point, then $\varphi(V)$ is closed in $H(M)$ and locally compact.

The object of this paper is to obtain the following theorem and example.

THEOREM A. *Let M be a two-dimensional, metric and connected manifold*

* The author would like to acknowledge a financial support given by the Sakko-kai Foundation during the preperation of this paper.

which is not homeomorphic to a torus. Then any monomorphic image of any vector group into $H(M)$ is closed and locally compact.

EXAMRLE. *Let M be a two-dimensional torus. There is a monomorphism ψ of one-dimensional vector group R into $H(M)$ such that the closure of $\psi(R)$ in $H(M)$ is not locally compact.*

Throughout this paper, the manifolds M are always assumed to be connected and to satisfy the 2nd countability axiom and the topology for $H(M)$ is the compact open topology.

The above example shows that the conjecture of Gleason and Palais does not hold in general. Moreover, by the construction of the monomorphism ψ, $\psi(R)(x)$ is dense in M for any $x \in M$ and $\psi(t)$ is a diffeomorphism of M (class C^1). Since $\psi(R)$ is not closed in $H(M)$, this example seems far from our intuition.

§ 1. One-parameter transformation groups on two-dimensional manifolds.

Let M be a connected and two-dimensional manifold with metric ρ. Let φ be a continuous monomorphism from R into $H(M)$ with compact open topology. In this section, we shall give some propositions which will be used in the next section.

PROPOSITION 1. *Notations being as above, if for any $N > 0$, $\varepsilon > 0$ and $x \in M$, there is a positive number $t = t(x) > N$ such that $\rho(\varphi(\pm t)(x), x) < \varepsilon$, then the subset M'' consisting of the points x such that the orbit $\varphi(R)(x)$ is homeomorphic to a circle is an open subset.*

PROOF. Assume $M'' \neq \emptyset$. Put $M' = \{x \in M;\ \varphi(R)(x) \neq \{x\}\}$. Then we see that M' is an open subset. Let x_0 be an arbitrary point in M'. Considering the family of curves $\{\varphi(R)(x)\}$ in M', there is a subset C containing x_0 which is a local cross-section [8]. That is, on defining the mapping ψ' from $R \times C$ into M' by $\psi'(t, x) = \varphi(t)(x)$, the restriction $\psi' | (-\delta, \delta) \times C$ is a homeomorphism from $(-\delta, \delta) \times C$ onto $\psi'((-\delta, \delta), C)$ for some δ. Since M is two dimensional and metric manifold, C contains a relatively open subset C' containing x_0 which is homeomorphic to R [**9**]. The homeomorphism from R onto C' is denoted by ξ. Assume $\xi(0) = x_0$.

Now, assume that $\xi(0) \in M''$. Since the interval $(-1, 1)$ is homeomorphic to R and there is $K > 0$ such that $\psi' | (-K, K) \times \xi((-1, 1))$ is a homeomorphism from $(-K, K) \times \xi((-1, 1))$ onto $\psi'((-K, K), \xi((-1, 1)))$ which is an open subset of M', we can assume without loss of generality that $\varphi(t)(\xi(R)) \cap \xi(R) = \emptyset$ for any t such that $0 < |t| < K$. Now, under this assumption, since $\varphi(R)(x_0)$ is homeomorphic to a circle, there is $\kappa > 0$ such that $\xi([-\kappa, \kappa]) \cap \varphi(t)(\xi(0)) = \emptyset$ for $0 < t < t_0$, where $t_0 = \min \{t > 0;\ \varphi(t)(\xi(0)) = \xi(0)\}$.

On putting $\eta(u)=\min\{t>0\,;\ \varphi(t)(\xi(u))\in\xi([-\kappa,\kappa])\}$, we have $\eta(0)=t_0$. η is continuous at 0. In fact, let $\{u_n\}$ be a sequence converging to 0. Since $[0, t_0]$ is compact and φ is continuous, we have $\eta(u_n)\leqq t_0+\frac{t_0}{2}$ for large n. There is a subsequence $\{u_{n'}\}$ such that $\eta(u_{n'})$ converges to some η'. Since $\varphi(\eta(u_{n'}))(\xi(u_{n'}))\in\xi([-\kappa,\kappa])$, $\varphi(\eta')(\xi(0))\in\xi([-\kappa,\kappa])$. It follows that $t_0\leqq\eta'\leqq t_0+\frac{t_0}{2}$. Thus, $\varphi(\eta'-t_0)\varphi(t_0)(\xi(0))=\varphi(\eta')(\xi(0))\in\xi([-\kappa,\kappa])$. On the other hand, $\varphi(\eta'-t_0)\varphi(t_0)(\xi(0))=\varphi(\eta'-t_0)(\xi(0))$ and $0\leqq\eta'-t_0\leqq\frac{t_0}{2}$. This gives $\eta'=t_0$. Thus, 0 is a point of continuity. Therefore for any $\varepsilon>0$, there is $\delta'>0$ such that $|\eta(u)-t_0|<\varepsilon$ if $-\delta'<u<\delta'$. Assume $\varepsilon<\frac{K}{4}$. We show that η is continuous on $(-\delta',\delta')$. For a sequence $\{u_n\}\subset(-\delta',\delta')$ converging to $u\in(-\delta',\delta')$, we have $|\eta(u_n)-\eta(u)|<2\varepsilon<\frac{K}{2}$. Thus, there is a subsequence $\{u_{n'}\}$ such that $\eta(u_{n'})$ converges to some η'. It follows that $\varphi(\eta')(\xi(u))\in\xi([-\kappa,\kappa])$. Since $\varphi(\eta(u))(\xi(u))\in\xi([-\kappa,\kappa])$, we see that $\varphi(\eta'-\eta(u))(\xi[-\kappa,\kappa]))\cap\xi([-\kappa,\kappa])\neq\emptyset$. Since $|\eta'-\eta(u)|\leqq\frac{K}{2}$, we have $\eta'=\eta(u)$. Thus, η is continuous on $(-\delta',\delta')$. By the same argument as above, we see that $\hat{\phi}:\xi(u)\to\varphi(\eta(u))(\xi(u))$ is a one-to-one correspondence on $\xi(-\delta',\delta')$. Thus, by defining $\hat{\eta}=\xi^{-1}\hat{\phi}\xi$, $\hat{\eta}$ is a homeomorphism from $(-\delta',\delta')$ onto $\hat{\eta}((-\delta',\delta'))$. Clearly, the origin $0\in(-\delta',\delta')$ is a fixed point of $\hat{\eta}$. If $\hat{\eta}([0,\delta'))\subsetneqq[0,\delta')$, then $\hat{\eta}^n([0,\delta'))$ converges to some $[0,\lambda]$, $0\leqq\lambda<\delta'$ for $n\to\infty$. This means that $\varphi(t)(\xi(u))$, $\lambda<u<\delta$, converges to a limit cycle $\varphi(R)(\xi(\lambda))$ for $t\to\infty$. This contradicts the assumption. Thus, $\hat{\eta}([0,\delta'))=[0,\delta')$ and $\hat{\eta}(u)=u$ for any $u\in[0,\delta')$. If $\hat{\eta}([0,\delta'))\supsetneqq[0,\delta')$, then taking $\hat{\eta}^{-1}$ instead of $\hat{\eta}$, we have the same conclusion. If $\hat{\eta}([0,\delta'))\leqq 0$, then $\hat{\eta}\cdot\hat{\eta}([0,\delta'))\geqq 0$. From the same argument as above, we obtain the same result. Thus we conclude that $\hat{\eta}(u)=u$ for any $u\in(-\delta',\delta')$. It follows that every orbit $\varphi(R)(\xi(u))$ is homeomorphic to a circle if $u\in(-\delta',\delta')$. Thus, M'' is an open subset. q. e. d.

Let x_0 be an arbitrarily fixed point in M'. Consider the family of curves $\{\varphi(R)(x)\,;\ x\in M'\}$. There is a continuous injection ξ from R into M' such that $\xi(0)=x_0$ and $\xi(R)$ is a local cross-section of the family. That is, the mapping $\phi|(-K,K)\times R$ is a homeomorphism from $(-K,K)\times R$ onto $\varphi((-K,K))(\xi(R))$ which is an open subset of M', where ϕ is defined by $\phi(t,u)=\varphi(t)(\xi(u))$.

Now, assume, in addition to the assumption in Proposition 1, that there is a positive continuous function $t_0(u)$ on $[-L,L]$ such that (i) $t_0(u)>K$, (ii) putting $\tilde{\eta}(u)=\phi(t_0(u),u)$, $\tilde{\eta}$ is a continuous injection from $[-L,L]$ into $\phi((-K/2,K/2),R)$ and (iii) $\eta([-L,L])\cap\phi((-K/2,K/2),[-L,L])\neq\emptyset$.

PROPOSITION 2. *Notations and assumptions being as above, if, moreover,*

M is not a two-dimensional torus, then there is a point $x \in \xi([-L, L])$ such that the orbit $\varphi(R)(x)$ is homeomorphic to a circle.

PROOF. Let $\alpha(u)$ denote $t \in (t_0(u)-K/2, t_0(u)+K/2)$ such that $\psi(t, u) \in \xi(R)$, where $u \in [-L, L]$. Since the curve $\psi((t_0(u)-K/2, t_0(u)+K/2), u)$ intersects the curve $\xi(R)$ at one point, $\alpha(u)$ is well-defined and continuous on $[-L, L]$. Let $\beta(u)=\xi^{-1}\psi(\alpha(u), u)$. From the property (iii) above, $[-L, L] \cap \beta([-L, L]) \neq \emptyset$. Since $\bar{\eta}$ and $\psi|(-K, K)\times R$ are injective, β is a homeomorphism from $[-L, L]$ onto $\beta([-L, L])$. Since $[-L, L] \cap \beta([-L, L]) \neq \emptyset$, if β is orientation reversing, then there is a point $u \in [-L, L]$ such that $\beta(u)=u$. This means that $\varphi(R)(\xi(u))$ is homeomorphic to a circle.

Assume that β is orientation preserving and there is no fixed point of β. Since $u \to -u$ is a homeomorphism from R onto R, we can assume without loss of generality that $\beta(u) > u$.

Let $\bar{t}_0=\min\{t>0;\ \psi(t, [-L, L]) \cap \xi([-L, L]) \neq \emptyset\}$ and let $x_1 \in \xi([-L, L]) \cap \psi(\bar{t}_0, [-L, L])$ and $x_1=\xi(\bar{u}_0)=\varphi(\bar{t}_0)(\xi(u_0))$. Obviously, $u_0, \bar{u}_0 \in [-L, L]$. There is a positive number δ_0 such that $\psi(\bar{t}_0, [u_0-\delta_0, u_0+\delta_0])$ is contained in $\psi((-K/2, K/2), R)$. Thus, the function α_0 defined by $\alpha_0(u)=\{t;\ t>0,\ |t-\bar{t}_0| < K/2, \psi(t, u) \in \xi(R)\}$ is continuous on $[-L, L] \cap [u_0-\delta_0, u_0+\delta_0]$ and $\alpha_0(u_0)=\bar{t}_0$. Put $\beta_0(u)=\xi^{-1}(\psi(u), u)$ and we see β_0 is a homeomorphism from $[-L, L] \cap [u_0-\delta_0, u_0+\delta_0]$ into R.

Put $[u_0, \bar{u}_0] \cap [u_0-\delta_0, u_0+\delta_0]=I$. There are following two cases; (i) β_0 is orientation preserving, (ii) β_0 is orientation reversing.

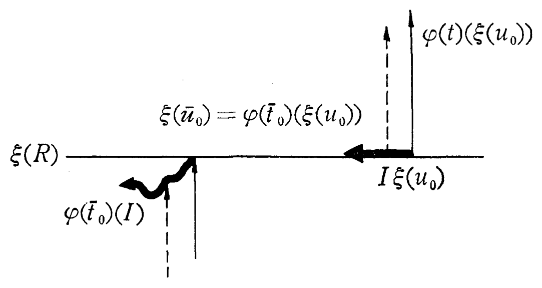

Fig. 1.

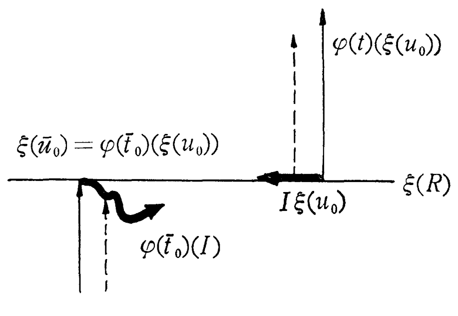

Fig. 2.

Case i. Let $h(u)=\dfrac{\alpha_0(u_0)}{u_0-\bar{u}_0}(u-\bar{u}_0)$ and $\hat{\gamma}(u)=\psi(h(u), u)$ for $u \in [u_0, \bar{u}_0]$. Then, $\hat{\gamma}(u_0)=\xi(\bar{u}_0)=\hat{\gamma}(\bar{u}_0)$. It follows that $\hat{\gamma}([u_0, \bar{u}_0])$ is a closed curve. It will be shown below that $\hat{\gamma}([u_0, \bar{u}_0])$ is a simple closed curve and is a local cross-section.

Let $\hat{\gamma}(u)=\hat{\gamma}(u')$. Then $\psi(|h(u)-h(u')|, [u_0, \bar{u}_0]) \cap \xi([u_0, \bar{u}_0]) \neq \emptyset$. Since $|h(u)-h(u')| \leqq \bar{t}_0$, $|h(u)-h(u')|=0$ or $\bar{t}_0$. If $h(u)=h(u')$, then $u=u'$. If $h(u)-h(u')=\bar{t}_0$, then $h(u)=\bar{t}_0$ and $h(u')=0$, then $\hat{\gamma}(u)=\hat{\gamma}(u')=\xi(\bar{u}_0)$. Thus $\hat{\gamma}([u_0, \bar{u}_0])$ is a simple closed curve. There is no difficulty to verify that $\hat{\gamma}([u_0, \bar{u}_0])$

is a local cross-section, since for any $x \in \hat{\gamma}([u_0, \bar{u}_0])$ there is a neighborhood V of x such that $V \cap \hat{\gamma}([u_0, \bar{u}_0])$ is a local cross-section.

Let $\hat{\alpha}(x) = \min\{t > 0;\ \varphi(t)(x) \in \hat{\gamma}([u_0, \bar{u}_0])\}$, $x \in \hat{\gamma}([u_0, \bar{u}_0])$. From the assumption imposed in Proposition 1, $\hat{\alpha}(x)$ is a well-defined and continuous function. Let $\hat{\beta}(x) = \varphi(\hat{\alpha}(x))(x)$, $x \in \hat{\gamma}([u_0, \bar{u}_0])$. Then $\hat{\beta}$ is continuous injection from $\hat{\gamma}([u_0, \bar{u}_0])$ into itself. Since $\hat{\gamma}([u_0, \bar{u}_0])$ is homeomorphic to a circle, we see that $\hat{\gamma}$ is a homeomorphism. It follows that M is a two dimensional torus.

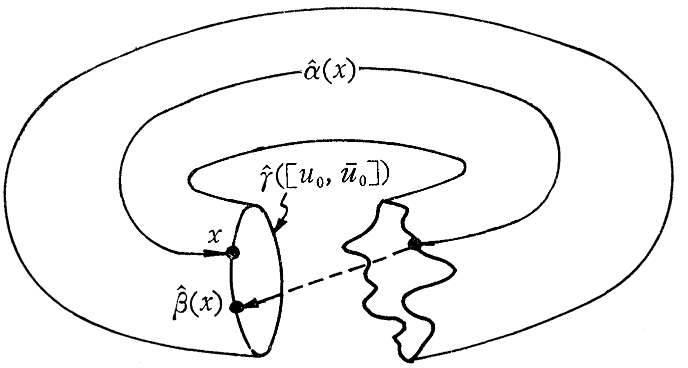

Fig. 3.

Case ii. To simplify the argument below, assume $\bar{u}_0 < u_0$. If $u_0 < \bar{u}_0$, the parallel argument leads to the same conclusion. Under this assumption, $I = [u_0 - \delta_0, u_0]$ and $\beta_0(I) \geqq \bar{u}_0$. Assume, furthermore, that $\beta_0(u_0 - \delta_0) < u_0$, since if not, there is a point $u \in I$ such that $\varphi(R)(\xi(u))$ is homeomorphic to a circle.

Now, it will be defined δ_n, α_n, inductively from δ_0, α_0. There is a positive number δ_n such that $\delta_n > \delta_{n-1}$ and $\psi(\alpha_{n-1}(u_0 - \delta_{n-1}), [u_0 - \delta_n, u_0 - \delta_{n-1}]) \subset \varphi((-K/2, K/2))(\xi(R))$. Then there is a function α_n on $[u_0 - \delta_n, u_0 - \delta_{n-1}]$ defined by

$$\alpha_n(u) = \{t;\ t > 0,\ |t - \alpha_{n-1}(u_0 - \delta_{n-1})| < K/2,\ \psi(t, u) \in \xi(R)\}\,.$$

Since $\alpha_n(u_0 - \delta_{n-1}) = \alpha_{n-1}(u_0 - \delta_{n-1})$, there is a continuous function α_∞ on $(u_0 - \delta_\infty, u_0]$, where $\delta_\infty = \sup \delta_n$.

On putting $\beta_\infty(u) = \xi^{-1}(\psi(\alpha_\infty(u), u))$, if $u_0 - \delta_\infty < \sup \beta_\infty((u_0 - \delta_\infty, u_0])$, then there is $\hat{u} \in (u_0 - \delta_\infty, u_0]$ such that $\beta_\infty(\hat{u}) = \hat{u}$. It follows that $\varphi(R)(\xi(\hat{u}))$ is homeomorphic to a circle. Assume $u_0 - \delta_\infty \geqq \sup \beta_\infty((u_0 - \delta_\infty, u_0])$. If α_∞ is bounded on $(u_0 - \delta_\infty, u_0]$, then there is t_∞ such that $[\bar{u}_0, u_0] \ni \psi(t_\infty, u_0 - \delta_\infty)$ and there is a sequence $\{v_n\} \subset (u_0 - \delta_\infty, u_0]$ converging to $u_0 - \delta_\infty$ and satisfying $\lim \alpha_\infty(v_n) = t_\infty$. Since there is a positive number $\bar{\delta}$ such that $\psi(t, [u_0 - \delta_\infty - \bar{\delta}, u_0 - \delta_\infty + \bar{\delta}]) \subset \psi((-K/2, K/2), R)$, there is a function $\bar{\alpha}$ on $[u_0 - \delta_\infty - \bar{\delta}, u_0 - \delta_\infty + \bar{\delta}]$ defined by $\bar{\alpha}(u) = \{t;\ t > 0,\ |t - t_\infty| < K/2,\ \psi(t, u) \in \xi(R)\}$. It is easy to see that $\bar{\alpha}(v_n) = \alpha_\infty(v_n)$ for sufficiently large n. It follows $\bar{\alpha}(u) = \alpha_\infty(u)$ for $u \in [u_0 - \delta_\infty - \bar{\delta}, u_0 - \delta_\infty + \bar{\delta}] \cap (u_0 - \delta_\infty, u_0]$. Thus, α_∞ can be extended. Thus,

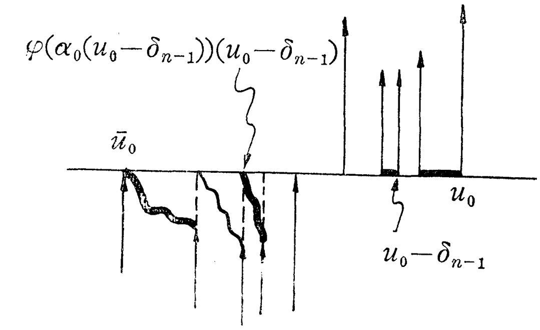

Fig. 4.

assume α_∞ is not bounded. There is a point $\hat{u}\in(u_0-\delta_\infty, u_0]$ such that $\alpha_\infty(\hat{u})=t_0(\hat{u})$, since $t_0(u)$ is bounded and continuous. Since $\tilde{\eta}(u)=\psi(t_0(u), u)\in\psi((-K/2, K/2), R)$, $\alpha_\infty(u)<t_0(u)+K/2$, if $u\leqq\hat{u}$. This is contradiction. It follows that there is $u\in[-L, L]$ such that $\varphi(R)(\xi(u))$ is homeomorphic to a circle.

§2. Proof of Theorem A.

Let M be a two-dimensional manifold with metric ρ. Since M satisfies the second countability axiom, so does $H(M)$. Let φ be a monomorphism from a vector group V into $H(M)$. The relative topology for $\varphi(V)$ induces a topology $\mathfrak{T}$ for V such that (i) $(V, \mathfrak{T})$ is a topological additive group, (ii) $(V, \mathfrak{T})$ satisfies Hausdorff's separation axiom and the first countability axiom, (iii) $\mathfrak{T}$ is weaker than the underlying topology of V.

For a fixed underlying group V, by $T(V, \mathfrak{T}_0)$ is meant the collection of all the pairs of the fixed abstract group V and a topology $\mathfrak{T}$ for V satistying (i)—(iii) above, where $\mathfrak{T}_0$ is the ordinary topology for V. Under these notations, an element $(V, \mathfrak{T})$ of $T(V, \mathfrak{T}_0)$ is called irreducible, if $\mathfrak{T}\neq\mathfrak{T}_0$ and $(V', \mathfrak{T})=(V', \mathfrak{T}_0)$ holds for any proper vector subgroup V' of V, where the topology of $(V', \mathfrak{T})$ is the relative topology in $(V, \mathfrak{T})$. It is easy to see that if $(V, \mathfrak{T})\in T(V, \mathfrak{T}_0)$ and $\mathfrak{T}\neq\mathfrak{T}_0$, then there is a vector subgroup V' of V such that $(V', \mathfrak{T})$ is irreducible.

LEMMA 1. *Let $(V, \mathfrak{T})\in T(V, \mathfrak{T}_0)$ be irreducible and φ be a continuous homomorphism from $(V, \mathfrak{T})$ into $H(M)$. If M is a two-dimensional manifold with metric ρ and if M is not a torus, then every orbit is one-dimensional or a point.*

PROOF. Let $\langle\,,\rangle$ be an ordinary inner product and let $|v|=\sqrt{\langle v, v\rangle}$. If there is a point $x\in M$ such that $\varphi(V)(x)$ is two-dimensional, then $\varphi(V)(x)$ is an open subset of M. Thus, $\varphi(V)(x)$ contains a neighborhood U of x which is homeomorphic to R^2. Since $\mathfrak{T}$ is weaker than $\mathfrak{T}_0$, φ can be considered as a continuous homomorphism from $(V, \mathcal{T}_0)$ into $H(M)$. Since $(V, \mathcal{T}_0)$ and U are sets of second category, there is $\varepsilon/2$-neighborhood W of 0 in $(V, \mathcal{T}_0)$ such that $\varphi(W)(x)$ contains an open subset U' of M. Thus, denoting by W' the ε-neighborhood of 0, $\varphi(W')(x)$ contains an open subset U'' of M which contains x. Since $(V, \mathcal{T})$ is irreducible, there are points $v_1, v_2, \cdots, v_r$, $r=\dim V$ such that $v_1, v_2, \cdots, v_r$ are linearly independent, $|v_i-v_j|>2\varepsilon$ if $i\neq j$, $|v_i|>2\varepsilon$ and $\varphi(v_i)(x)\in U''$ ([**6**], Lemma 3). It follows that there is v_i' for every i such that $v_1'\in v_i+W'$ and $\varphi(v_i')(x)=x$. Thus, $\varphi(V)(x)$ is homeomorphic to a torus. This means that M is a torus.

Now, assumptions being as in Lemma 1 above, assume furthermore that φ is a monomorphism. Let V_x be the isotropy subgroup at x and V_x^0 be its connected component containing 0 under the topology $\mathcal{T}_0$.

Assume there is a point x such that $V_x\neq V_x^0$. Let V_1 be a one-dimensional vector subgroup such that $V_1\not\subset V_x^0$. Since $\varphi(V_1)(x)$ is homeomorphic to S^1, from Lemma 3 in [**6**] and Proposition 1 above, we see that the subset $M(V_1)$ consisting of the points y such that $\varphi(V_1)(y)$ is homeomorphic to a circle is open and not vacuous. Since it is clear that if $\varphi(V_1)(y)\neq\{y\}$, then $\varphi(V)(y)=\varphi(V_1)(y)$, we obtain that the subset M'' consisiting of the points y such that $\varphi(V)(y)$ is homeomorphic to S^1 is an open subset of M. From Theorem B in [**6**], we see that V operates as a circle group on every connected component of M''. That is V_x is constant on every connected component of M''. It follows that for every boundary point x of M'', $\varphi(V)(x)$ is homeomorphic to a circle or a point. Thus, by Theorem B, C in [**6**], $\varphi(V)=S^1$, contradicting the assumption that φ is monomorphic. It follows that $V_x=V_x^0$ for any point.

Let x_0 be a point of M such that $\varphi(V)(x_0)\neq\{x_0\}$. Since $(V_x, \mathcal{T}_0)$ is connected and $(V, \mathcal{T}_0)/(V_x, \mathcal{T}_0)$ is one-dimensional, V_x is continuous on $M'=\{x\in M;\ \varphi(V)(x)\neq\{x\}\}$. That is $\lim_{x_n\to x} V_{x_n}=\{\lim_{n\to\infty} v_n;\ v_n\in V_{x_n}\}=V_x$, where the topology under which lim. is considered is the topology $\mathcal{T}_0$. Thus, there is a $(V, \mathcal{T}_0)$-valued continuous function $n(x)$ on some neighborhood U of x_0 such that $|n(x)|=1$ and $\langle n(x), v\rangle=0$ for any $v\in V_x$. Let $V_1=\{\lambda n(x_0);\ \lambda\in R\}$ and we see that $\varphi(V_1)(x_0)\neq\{x_0\}$. Assume $\varphi(V_1)(x)\neq\{x\}$ for every $x\in U$. Clearly $\varphi(V)(x)=\varphi(V_1)(x)$ for every $x\in U$. There is an injection ξ from R into M' such that $\xi(0)=x_0$ and $\xi(R)$ is a local cross-section of the family of curves $\{\varphi(V_1)(x);\ x\in M'\}$. That is, there is $K>0$ such that $\psi|(-K, K)\times R$ is homeomorphism from $(-K, K)\times R$ into M', where ψ is defined by $\psi(t, u)=\varphi(tn(x_0))(\xi(u))$. Since $\varphi(V_1)(x)\neq\{x\}$ for every $x\in U$, we see $V=V_1+V_x$ (direct sum). Thus, $v=P(v, x)n(x_0)+v'$, $v'\in V_x$. Since $P(v, x)$ is uniquely determined with respect to v, x and V_x is continuous on U, $P(v, x)$ is continuous on $V\times U$.

Let L be a positive number such that $\xi^{-1}(U\cap\xi(R))\supset[-L, L]$. Since

$(V, \mathfrak{T})$ is irreducible, there is $v_0 \in V$ such that $P(v_0, x_0) > K$, $\varphi(v_0)(\xi([-L, L]))$ $\subset \psi((-K/2, K/2), R)$ and $\varphi(v_0)(\xi([-L, L])) \cap \psi((-K/2, K/2), [-L, L]) \neq \emptyset$. Put $\eta(u) = P(v_0, \xi(u))$. Then $\eta(u)$ is continuous on $[-L, L]$. Thus, considering Proposition 2 above, we have only to show that $\eta(u) > K$. Since $\eta(0) > K$, if there is $u \in [-L, L]$ such that $\eta(u) \leqq K$, then there is $\hat{u} \in [-L, L]$ such that $\eta(\hat{u}) = K$. Since $\varphi(v_0)(\xi(\hat{u})) = \varphi(P(v_0, \xi(\hat{u}))n(x_0))(\xi(\hat{u}))$, this is contradiction. It follows $\eta(u) > K$ on $[-L, L]$.

§ 3. Homeomorphisms on a circle.

Let V be a one-dimensional vector group and φ a continuous monomorphism from V into $H(M)$ of a two-dimensional manifold M. From the proof of Proposition 2, we see that if there is a local cross-section C homeomorphic to a circle of the family of the curves $\{\varphi(V)(x);\ x \in M'\}$, and if $\varphi(V)$ is not closed in $H(M)$, then M is a two-dimensional torus. Letting $\hat{\alpha}(x) = \{t > 0, \varphi(t)(x) \cap C \neq \phi;\ x \in C\}$ and $\hat{\beta}(x) = \varphi(\alpha(x))(x)$, we see that $\hat{\beta}$ is a homeomorphism from C onto C. From the assumption that $\varphi(V)$ is not closed, we see easily that the group G generated by β is not closed in $H(C)$.

In this section, it will be proved that $\bar{G}$ is a compact group in $H(C)$.

Let Z be an additive group of the integers and φ a monomorphism from Z into the set of all homeomorphisms $H(S^1)$ on a circle S^1. Assume that $\varphi(Z)$ is not closed in $H(S^1)$. Then there is a sequence z_n, $z_n \in Z$, such that $\varphi(z_n)$ converges to the identity in $H(S_1)$. The integers z_n can be so selected that $z_n > 0$ for all n. This means that for any $\varepsilon > 0$, there is an integer $m > 0$ such that $\rho(\varphi(m)(x), x) < \varepsilon$ for every x in S^1, where ρ is the usual metric on S^1.

Suppose that $T = \varphi(1)$ is orientation preserving and has a fixed point x. $S^1 - \{x\}$ can be identified with the interval $(0, 1)$. Then T is a monotone increasing function such that $T(0) = 0$ and $T(1) = 1$. Let A be a connected component of $\{x;\ T(x) \geqq x\}$ and $a_0 = \sup\{x;\ x \in A\}$. Then a_0 is a fixed point of T and $\lim T^n(x) = a_0$ for every $x \in A$. It follows that $\varphi(Z)$ is closed in $H(S^1)$.

LEMMA 2. *Notations being as above, if $\varphi(Z)$ is not closed in $H(S^1)$, then $\varphi(k)$ has no fixed point for every $k \in Z$.*

PROOF. If $\varphi(k)$ has a fixed point, then $\varphi(2k)$ is orientation preserving and has a fixed point. Thus, $\varphi(2kZ)$ is closed in $H(S^1)$. Since $H(S^1)$ is a set of second category, $\varphi(2kZ)$ is discrete subgroup of $H(S^1)$. Thus, the closure of $\varphi(Z)$ is locally compact and not compact, because $Cl(\varphi(Z))/\varphi(2kZ)$ is compact and $\varphi(2kZ)$ is discrete, where $Cl(A)$ is the closure of A. It follows that $\varphi(Z)$ is discrete under the relative topology in $H(S^1)$ from Lemma 2.3 in [**5**].

LEMMA 3. *Notations being as above, if $\varphi(Z)$ is not closed in $H(S^1)$*

and $\varphi(1)$ *is orientation preserving, then there are a homeomorphism* h *from* S^1 *onto* $\{\exp 2\pi\sqrt{-1}\,\theta,\ \theta\in R\}$ *and an irrational number* α *such that* $h\varphi(1)h^{-1}(\exp 2\pi\sqrt{-1}\,\theta)=\exp 2\pi\sqrt{-1}(\theta+\alpha)$.

PROOF. It suffices to prove that $Cl(\varphi(Z))$ is compact and simply transitive on S^1. For an arbitrarily fixed ε, $\frac{1}{6}>\varepsilon>0$, there is $k\in Z$ such that $\rho(T^k(x), x)<\varepsilon$ for every $x\in S^1$. Put $\delta=\min \rho(T^k(x), x)$. Then, $\delta>0$ from Lemma 1.

For two points x, y with $0<\rho(x, y)<\frac{1}{2}$, (x, y) denotes the connected component of $S^1-\{x, y\}$ whose diameter is smaller than the other. We shall show that $T^j(x, T^k(x))=(T^j(x), T^{j+k}(x))$ for every $j\in Z$. To do this, assume that $T^j(x, T^k(x))=S^1-Cl((T^j(x), T^{j+k}(x))$. Then, $(x, T^k(x))\cap(T^j(x), T^{j+k}(x))\neq\emptyset$. In fact, if $(x, T^k(x))\cap(T^j(x), T^{j+k}(x))=\emptyset$ then $(x, T^k(x))\subset S^1-Cl(T^j(x), T^{j+k}(x))$ $=T^j(x, T^k(x))$. Thus, there is a fixed point of T^j. Since $\rho(z, T^k(z))<\varepsilon$ for every $z\in S^1$ and $3\varepsilon<\frac{1}{2}$, there is $y\in(x, T^k(x))$ such that $\rho(y, T^j(y))=\frac{1}{2}$. Since T^k preserves the orientation of S^1, $T^k(y)\in S^1-(x, T^k(x))$. Thus, $T^{j+k}(y)$ $\in(T^j(x), T^{j+k}(x))$ and then $\rho(y, T^{j+k}(y))\leqq\rho(y, w)+\rho(w, T^{j+k}(y))<2\varepsilon$, where $w\in(x, T^k(x))\cap(T^j(x), T^{j+k}(x))$. It follows

$$\varepsilon>\rho(T^j(y), T^{j+k}(y))\geqq\rho(T^j(y), y)-\rho(y, T^{j+k}(y))\geqq\frac{1}{2}-2\varepsilon>\frac{1}{6}\,.$$

This is a contradiction. Thus, $T^j(x, T^k(x))=(T^j(x), T^{j+k}(x))$ for all $j\in Z$.

Let x, y be points in S^1 such that $\rho(x, y)<\delta$. Then, $(x, y)\subset(x, T^k(x))$ or $(y, T^k(y))$. Assume $(x, y)\subset(x, T^k(x))$ and we have $T^j(x, y)\subset T^j(x, T^k(x))=$ $(T^j(x), T^{j+k}(x))$. It follows that $\rho(T^j(x), T^j(y))<\varepsilon$. Thus, $\{T^j\}$ is equi-continuous. Since S^1 is a compact set, $Cl(\varphi(Z))$ is compact in $H(S^1)$.

If there is $T'\in Cl(\varphi(Z))$ and $x\in S^1$ such that $T'(x)=x$, then from Lemma 2, $\bigcup_{k=-\infty}^{\infty} T'^k$ is discrete in $H(S^1)$, contradicting the fact that $CL(\varphi(Z))$ is compact. Thus, $T'=$ identity. If $Cl(\varphi(Z))(x)\subsetneqq S^1$ for some $x\in S^1$ then the same argument of Lemma 2 leads to the conclusion that $\varphi(Z)$ is discrete in $H(S^1)$. Thus $Cl(\varphi(Z))$ is simply transitive on S^1 and then $Cl(\varphi(Z))\cong S^1$.

§ 4. One-parameter transformation groups on the torus.

In this section, one-parameter groups acting effectively on a two-dimensional torus T^2 are mainly concerned. Considering the above one-dimensional case, it seems to be natural to conjecture that any one-parameter transformation group on T^2 is closed in $H(T^2)$ or on its closure is compact. However, the fact is more complicated. Indeed, there is an example of one-parameter group which acts effectively on T^2 and is not closed in $H(T^2)$ and, futhermore, its closure is not locally compact. This means that the conjecture of Gleason and Palais does not hold in general. In this section, an example of such a one-parameter transformation group on T^2 will be constructed.

Consider the graph Γ of a continuous, positive and periodic function $y=f(x)$ of period 1 on a real line R and take the relatively compact domain D whose boundary is $\Gamma \cup I_0 \cup J_0 \cup J_1$, where $I_0=\{(x, 0);\ 0 \leqq x \leqq 1\}$, $J_0=\{(0, y);\ 0 \leqq y \leqq f(0)\}$ and $J_1=\{(1, y);\ 0 \leqq y \leqq f(1)\}$.

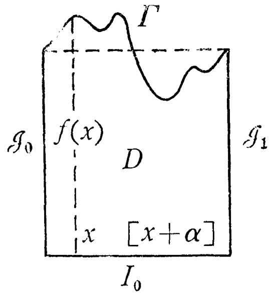

Fig. 5.

Let $[z]$ be the number such that $0 \leqq z < 1$ and $z=[z]$ mod. 1. By identifying $(0, y)$ with $(1, y)$ and $(x, f(x))$ with $([x+\alpha], 0)$ for an arbitrarily fixed irrational number such that $0<\alpha<1$, the domain D turns out into a two-dimensional torus T^2.

A one-parameter transformation group $G=\{g_t\}$ is defined as follows: For every $(x, y) \in D$ and for every $t \in R$, there are an integer m and non-negative number λ such that

$$t=\begin{cases} -y+\sum_{k=0}^{m-1} f(x+k\alpha)+\lambda & \text{if} \quad t \geqq 0 \\ -y-\sum_{k=1}^{m} f(x-k\alpha)+\lambda & \text{if} \quad t<0 \end{cases}$$

and $0 \leqq \lambda < f(x+\operatorname{sgn}(t) m\alpha)$. Such m and λ are determined uniquely with respect to (x, y) and t. Define $g_t(x, y)=([x+\operatorname{sgn}(t) m\alpha], \lambda)$. There is no difficulty to verify the following Lemma.

LEMMA 4. *Notations being as above, G is a transformation group acting effectively and continuously on T^2. If $f(x)$ is differentiable, then g_t is a diffeomorphism of T^2 for every $t \in R$.*

The following properties of G is clear.

a) Any orbit of G is dense in T^2.

b) g_t has no fixed point for every $t \neq 0$.

Now, assume that G is not closed in $H(T^2)$. Then, there is a sequence $\{t_n\}$ such that $t_n \to \infty$ and $g_{t_n} \to id.$ in $H(T^2)$ (Lemma 1.1, [4]). This means that for any $\varepsilon>0$ there is n_0 such that $\rho(g_{t_n}(p), p)<\varepsilon$ for every $p \in T^2$ and $n \geqq n_0$, where ρ is the usual metric on T^2. Assume furthermore that $\varepsilon<\min f(x)$.

For every $x \in I_0$, $m(x)$ is the number of intersection of I_0 with $\{g_t(x);\ 0 \leqq t \leqq t_n+\varepsilon\}$. Since $f(x)$ is continuous and $[0, t_n+\varepsilon]$ is compact, $m(x)$ is constant on a neighborhood of x. It follows that $m(x)$ is constant and equal to m on I_0.

This means that $0<t_n+\varepsilon-\sum_{k=0}^{m-1} f(x+k\alpha)<2\varepsilon$ for every $x \in I_0$. Thus, $|t_n-mE_0| <\varepsilon$, where $E_0=\int_0^1 f(x)dx$.

LEMMA 5. *Notations being as above, G is not closed if and only if there is a sequence of integers $\{m_n\}$, $m_n \to \infty$ such that*

$$\left|\sum_{j=0}^{m_n-1} f(x+j\alpha)-m_n E_0\right| \to 0 \quad \textit{and} \quad \inf\{|m_n\alpha-m|\,;\ m \in Z\} \to 0 .$$

PROOF. Necessity is clear from the above argument. On putting $t_n=m_nE_0$. from the defintion of the operation of $\{g_t\}$, we see that $g_{t_n} \to id.$ in $H(T^2)$.

LEMMA 6. *Let $f_0(x)=f(x)-E_0$. If $Cl(G)$ is compact in $H(T^2)$, then there is a continuous periodic function $g(x)$ of period 1 such that $f_0(x)=-g(x+\alpha)+g(x)$.*

PROOF. Since every orbit of G is dense in T^2, $Cl(G)$ is simply transitive on T^2. Thus, $Cl(G)$ is homeomorphic to T^2, and there is a homeomorphism h from T^2 onto $T=\{(e^{2\pi i\theta}, e^{2\pi i\eta});\ \theta, \eta \in R\} \cong Cl(G)$ such that $hg_th^{-1}(e^{2\pi i\theta}, e^{2\pi i\eta})=(e^{2\pi i(\theta+t\alpha_1)}, e^{2\pi i(\eta+t\alpha_2)})$, where $\alpha_1 \cdot \alpha_2 \neq 0$ and (α_1, α_2) is linearly independent with respect to integral coefficient. Of course the homeomorphism h is defined by $h^{-1}(g)=g(x_0)$ for some point x_0 in T^2, where $g \in T$. Since I_0 is considered as a simple closed curve in T^2, so also is $h(I_0)$ in T. Without loss of generality, assume that $h(I_0)$ contains $\{0\}$ in T. R^2 is a universal covering group of T and the natural projection π is defined by $\pi(\theta, \eta)=(e^{2\pi i\theta}, e^{2\pi i\eta})$. Thus the connected component of $\pi^{-1}(h(I_0))$ containing 0 is a curve C in R^2. Let (m_0, n_0) and (m_1, n_1) be the boundary point of a connected component C' of $C-Z^2$. Without loss of generality we assume that $(m_0, n_0)=(0, 0)$. Let $f'(\theta, \eta)=\min\{t>0;\ (\theta+t\alpha_1, \eta+t\alpha_2) \in \pi^{-1}(h(I_0))\}$. Then $f(x)=f'(\xi(h(x)))$, $x \in I_0$, where $\xi(h(x))=C' \cap \pi^{-1}(h(x))$ if

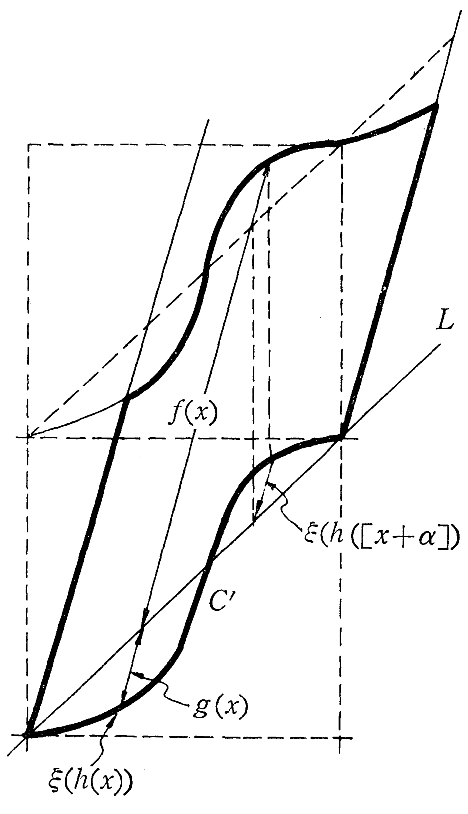

Fig. 6.

$h(x) \neq 0$ and $=0$ if $h(x)=0$. Let $L=\{(\theta, \eta) \in R^2;\ n_1\theta = m_1\eta\}$ and define the function $g'(\theta, \eta)$ so that it may satisfy $(\theta+g'(\theta, \eta)\alpha_1, \eta+g'(\theta, \eta)\alpha_2) \in L$. On putting $g(x)=g'(\xi(h(x)))$, it is easy to see that $f(x)-g(x)+g([x+\alpha])$ is constant (cf. fig. 6). Since

$$\int_0^1 f(x)dx - \int_0^1 g(x)dx + \int_0^1 g([x+\alpha])dx = E_0\,,$$

we see that $f_0(x)-g(x)+g([x+\alpha])=0$. Since $\lim_{x\to 1} g(x)=g(0)$, $g(x)$ can be extended to the whole line R as a periodical function of period 1. This function is denoted by the same notation $g(x)$. We see easily that $g(x)$ is continuous. Thus $g(x)$ is a desired function.

From Lemmas 4-6, we see that if we can find a differentiable and periodic function f_0 of period 1 and an irrational α such that (1) $\int_0^1 f_0(x)dx=0$, (2) for any $\varepsilon>0$, there is an integer m such that $|m\alpha-n|<\varepsilon$ for some integer n, $|\sum_{j=0}^{m-1} f_0(x+j\alpha)|<\varepsilon$, (3) there is no continuous and periodic solution of $f_0(x)=g(x)-g(x+\alpha)$, then there is a one-parameter transformation group with the properties stated in the "Example" in the Introduction.

Now, let $\{a_n\}$ be a series of positive numbers defined by

$$a_n = \sum_{i=0}^{n-1} a_i + (n-1)\,, \qquad a_0 = 1\,.$$

Put $s_n = \sum_{i=0}^{n-1} a_i$. For a positive integer p (for instance $p=10$), take an irrational number $\alpha = \sum_{n=1}^{\infty} p^{-s_n}$ and put $f_0(x) = \sum_{n=1}^{\infty} p^{-a_n} \sin(2\pi p^{s_n} x)$. Then $f_0(x)$ is a periodic function of period 1, continuous and differentiable because $\sum_{n=1} p^{-a_n} p^{s_n} = \sum_{n=1}^{\infty} p^{-(n-1)}$ is absolutely convergent. Moreover, we have $\int_0^1 f_0(x)dx=0$. It will be shown below that f_0 satisfies the conditions (2) and (3) above.

Since $2\pi\theta \geqq |1-e^{2\pi i\theta}| > \pi\theta$ for all $0<\theta<\frac{1}{2}$ and $p^{s_k}\alpha \equiv \sum_{n=k+1}^{\infty} p^{-s_n+s_k} \bmod 1$, we have $|1-e^{2\pi i p^{s_k}\alpha}| > \pi \sum_{n=k+1}^{\infty} p^{-s_n+s_k} > \pi p^{-a_k}$. Thus,

$$\begin{aligned}
|\sum_{j=0}^{m-1} f_0(x+j\alpha)| &= |\sum_{n=1}^{\infty}\sum_{j=0}^{m-1} p^{-a_n} \sin 2\pi p^{s_n}(x+j\alpha)| \\
&\leqq \sum_{n=1}^{\infty} |\sum_{j=0}^{m-1} p^{-a_n} e^{2\pi i p^{s_n}(x+j\alpha)}| \leqq \sum_{n=1}^{\infty} p^{-a_n} \left| \frac{1-e^{2\pi i p^{s_n} m\alpha}}{1-e^{2\pi i p^{s_n}\alpha}} \cdot e^{2\pi i p^{s_n} x} \right| \\
&\leqq \frac{1}{\pi} \sum_{n=1}^{\infty} |1-e^{2\pi i p^{s_n} m\alpha}|\,.
\end{aligned}$$

For the proof of the property (2), we have only to show that for any $\varepsilon>0$, there is M such that $\sum_{n=1}^{\infty}|1-e^{2\pi i p^{s_n}M\alpha}|<\varepsilon$ and there is n such that $|M\alpha-n|<\varepsilon$. Since there is an integer m such that $8\pi\frac{p}{p-1}p^{-(m-1)}<\varepsilon$, on putting $M=p^{s_m}$, we have $|p^{s_m}\alpha-n|<\varepsilon$ for some n and $p^{s_n+s_m}\alpha\equiv p^{s_m}(\sum_{k=n+1}^{\infty}p^{-s_k+s_n})\equiv\sum_{k=n+1}^{\infty}p^{-s_k+s_n+s_m}$ mod 1, if $n\geqq m$, and $p^{s_n+s_m}\alpha\equiv p^{s_n}(\sum_{k=m+1}^{\infty}p^{-s_k+s_m})\equiv\sum_{k=m+1}^{\infty}p^{-s_k+s_n+s_m}$ mod 1, if $m>n$. Since $k>\max\{n, m\}$ in these terms, we have $s_k-s_n-s_m>0$. Thus, $|1-e^{2\pi i p^{s_n+s_m}\alpha}|\leqq 2\pi\sum_{k=n+1}^{\infty}p^{-s_k+s_n+s_m}\leqq 4\pi p^{-a_n+s_m}$ if $n\geqq m$, and $|1-e^{2\pi i p^{s_n+s_m}\alpha}|\leqq 2\pi\sum_{k=m+1}^{\infty}p^{-s_k+s_n+s_m}\leqq 4\pi p^{-a_m+s_n}$ if $m>n$. Since $a_n-s_m\geqq n-1$ for $n\geqq m$ and $a_n-s_m\neq a_{n'}-s_m$ if $n\neq n'$, we see that

$$4\pi\sum_{n=m}^{\infty}p^{-a_n+s_m}\leqq 4\pi p^{-(m-1)}\sum_{k=0}^{\infty}p^{-k}=4\pi p^{-(m-1)}\frac{p}{p-1},$$

and by the same reason we see that

$$4\pi\sum_{n=1}^{m-1}p^{-a_m+s_n}\leqq 4\pi p^{-(m-1)}\sum_{k=0}^{\infty}p^{-k}=4\pi p^{-(m-1)}\frac{p}{p-1}.$$

It follows

$$|1-e^{2\pi i p^{s_n+s_m}\alpha}|\leqq 4\pi\sum_{n=m}^{\infty}p^{-a_n+s_m}+4\pi\sum_{n=1}^{m-1}p^{-a_m+s_n}\leqq 8\pi\frac{p}{p-1}p^{-(m-1)}.$$

Now, assume that there is a solution $g(x)$ of $f_0(x)=g(x)-g(x+\alpha)$ which is continuous and periodic. Without loss of generality, assume $\int_0^1 g(x)dx=0$. Let $\sum_{k=1}^{\infty}(b_k\cos 2\pi kx+c_k\sin 2\pi kx)$ be the Fourier expansion of $g(x)$. Then the Fourier expansion of $g(x)-g(x+\alpha)$ is

$$\sum_{k=1}^{\infty}\{(b_k(1-\cos 2\pi k\alpha)-c_k\sin 2\pi k\alpha)\cos 2\pi kx$$
$$+(b_k\sin 2\pi k\alpha-c_k(1-\cos 2\pi k\alpha))\sin 2\pi kx\}.$$

Thus, the non vanishing terms are given by

$$b_{p^{s_n}}=\frac{1}{2}\frac{\sin 2\pi p^{s_n}\alpha}{1-\cos 2\pi p^{s_n}\alpha}p^{-a_n}=\frac{1}{2}p^{-a_n}\cot\pi p^{s_n}\alpha, \qquad c_{p^{s_n}}=\frac{1}{2}p^{-a_n}.$$

Since $2\theta\geqq\tan\theta$ if $\frac{\pi}{4}\geqq\theta\geqq-\frac{\pi}{4}$, and $p^{s_n}\alpha\equiv\sum_{k=n+1}^{\infty}p^{-s_k+s_n}$ mod 1, we have that if $n\geqq 2$, then

$$p^{-a_n}\cot\pi p^{s_n}\alpha\geqq\frac{p^{-a_n}}{2\sum_{k=n+1}^{\infty}p^{-s_k+s_n}}\geqq\frac{p^{-a_n}}{4p^{-a_n}}=\frac{1}{4}.$$

Thus,

$$\int_0^1 g(x)^2dx=2\sum_{n=1}^{\infty}\left(\frac{p^{-a_n}}{2}\right)^2+\sum_{n=1}^{\infty}(p^{-a_n}\cot\pi p^{s_n}\alpha)^2\geqq\sum_{n=1}^{\infty}\frac{1}{16}=\infty.$$

This contradicts the assumption that $g(x)$ is continuous on $[0, 1]$. It follows that there is no continuous and periodic solution for $f_0(x)=g(x)-g(x+\alpha)$.

Thus, putting $f(x)=f_0(x)+E$, $E>\max f_0(x)$, we have a one-parameter transformation group $G=\{g_t\}$ whose closure in $H(T^2)$ is not locally compact.

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