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Fractional powers of dissipative operators, II

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The present note is devoted to some supplementary results to be added to the previous paper of the author with the same title (Kato [**1**], which will be quoted as (I) in the following). We follow throughout the notations and terminology of (I).

It was proved in (I), among others, that $\mathfrak{D}(A^{\alpha})=\mathfrak{D}(A^{*\alpha})$ for $0\leqq\alpha<1/2$ whenever A is closed and maximal accretive. But there remained many unsolved questions regarding the case $\alpha=1/2$. The first part of the present note is mainly concerned with this case.

It has been shown by J. L. Lions [**4**] (see the preceding paper by Lions) that $\mathfrak{D}(A^{1/2})=\mathfrak{D}(A^{*1/2})$ is in general not true. The question is still open, however, whether or not this is true when A is *regularly accretive* (see (I)). In particular, it is of considerable interest to decide whether

$$\mathfrak{D}(A^{1/2})=\mathfrak{D}(A^{*1/2})=\mathfrak{D}(\phi) \tag{1}$$

is true (ϕ is the regular sesquilinear form associated with A, see (I)).

It has also been shown by Lions that (1) is true in many important cases in which A is a partial differential operator of elliptic type. For the proof of these results, Lions makes use of the theory of *interpolation spaces*. We shall present here another proof for some of the theorems of Lions (Theorems 1 and 2 below). Also we shall consider the relationship between $\mathfrak{D}(A^{\alpha})$, $\mathfrak{D}(A^{*\alpha})$ and $\mathfrak{D}(\phi)$ for $\alpha\lesseqgtr 1/2$ (Theorem 3); the results will have some applications in the theory of evolution equations. We shall next consider the properties of the powers A^{α} for complex α and, as an application, a new proof of the generalized Heinz inequality will be given.

REMARK. (1) implies that the form ϕ has the representation:

$$\phi[u,v]=(A^{1/2}u, A^{*1/2}v), \qquad u,v\in\mathfrak{D}(\phi). \tag{1a}$$

If $\mathrm{Re}\,\phi$ is strictly positive so that A^{-1} and A^{*-1} are both bounded, (1) implies that $A^{1/2}$ and $A^{*1/2}$ are comparable:

$$m\leqq \|A^{1/2}u\|/\|A^{*1/2}u\|\leqq M, \tag{1b}$$

$M\geqq m>0$ being constants. Furthermore, (1) implies that $A^{1/2}$ and $A^{*1/2}$ have an acute angle:

(1c) $$\mathrm{Re}(A^{1/2}u, A^{*1/2}u) \geqq m_0 \| A^{1/2}u \| \| A^{*1/2}u \|, \qquad m_0 > 0.$$

These results can be proved easily by using the fact that $\mathfrak{D}(\phi) = \mathfrak{D}(H^{1/2})$, where H is the real part of A (see (I)).

§ 1. Some theorems related to $\mathfrak{D}(\phi)$.

LEMMA 1.[1] *Let A be regularly accretive with the real part H. If H is strictly positive, we have*

(2) $$\| H^{1/2}u \| \leqq \| H^{-1/2}Au \| \leqq c \| H^{1/2}u \|, \qquad u \in \mathfrak{D}(A),$$

and similar inequalities with A replaced by A^*; c is a constant depending only on A.

PROOF. H^{-1} is bounded by hypothesis. We have

$$\mathrm{Re}(H^{-1/2}Au, H^{1/2}u) = \mathrm{Re}(Au, u) = \mathrm{Re}\, \phi[u] = \| H^{1/2}u \|^2,$$

whence follows the first inequality of (2). Again,

$$| (H^{-1/2}Au, v) | = | (Au, H^{-1/2}v) | = | \phi[u, H^{-1/2}v] | \leqq (1+\beta) \| H^{1/2}u \| \| v \|$$

for all $v \in H$ (see (2.3) of (I)), whence the second inequality of (2) with $c = 1+\beta$.

THEOREM 1.[2] *Let A be regularly accretive with the associated regular sesquilinear form ϕ. Then the following two conditions are equivalent:*

(3) $$\mathfrak{D}(A^{1/2}) \subset \mathfrak{D}(\phi),$$ (4) $$\mathfrak{D}(A^{*1/2}) \supset \mathfrak{D}(\phi).$$

The same is true when A and A^* are exchanged.

COROLLARY. (1) *is true if both $\mathfrak{D}(A^{1/2})$ and $\mathfrak{D}(A^{*1/2})$ are subsets (or oversets) of $\mathfrak{D}(\phi)$.*

PROOF OF THEOREM 1. Since $A+\varepsilon$ is associated with the form $\phi+\varepsilon$ and since $\mathfrak{D}((A+\varepsilon)^\alpha) = \mathfrak{D}(A^\alpha)$, $0 \leqq \alpha \leqq 1$ (see Lemma A 2 of (I)), we may assume that ϕ, and hence H too, is strictly positive so that A^{-1}, H^{-1} are bounded and Lemma 1 is applicable.

Since $\mathfrak{D}(\phi) = \mathfrak{D}(H^{1/2})$, (3) implies that $H^{1/2}A^{-1/2}$ is bounded. Hence $A^{*-1/2}H^{1/2}$ is bounded, or $\| A^{*-1/2}v \| \leqq \mathrm{const} \| H^{-1/2}v \|$ for $v \in \mathfrak{H}$. On setting $v = A^*w$, $w \in \mathfrak{D}(A^*)$, one obtains by Lemma 1

(5) $$\| A^{*1/2}w \| \leqq \mathrm{const} \| H^{-1/2}A^*w \| \leqq \mathrm{const} \| H^{1/2}w \|.$$

Since $\mathfrak{D}(A^*)$ is a core of $H^{1/2}$ (see (I); this is equivalent to that $\mathfrak{D}(A^*)$ is dense in the Hilbert space $H_\phi = \mathfrak{D}(\phi)$ with the norm $\| H^{1/2}u \| = (\mathrm{Re}\, \phi[u])^{1/2}$), the inequality (5) extends to all $w \in \mathfrak{D}(H^{1/2}) = \mathfrak{D}(\phi)$, the inclusion (4) being thereby

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1) Lemma 1 corresponds to Lions' Proposition (4.4).

2) This corresponds to Lions' Theorem 5.1.

implied.

Conversely, (4) implies that $A^{*1/2}H^{-1/2}$ is bounded. Hence $H^{-1/2}A^{1/2}$ is bounded and $\| H^{-1/2}v \| \leqq \text{const} \| A^{-1/2}v \|$ for $v \in \mathfrak{H}$. On setting $v = Au$, $u \in \mathfrak{D}(A)$ and using Lemma 1, we have

$$\| H^{1/2}u \| \leqq \| H^{-1/2}Au \| \leqq \text{const} \| A^{1/2}u \| . \tag{6}$$

Since $\mathfrak{D}(A)$ is a core of $A^{1/2}$ (see Lemma A 3 of (I)), this again extends to all $u \in \mathfrak{D}(A^{1/2})$, the inclusion (3) being implied.

THEOREM. 2.[3] *Let A, ϕ be as in Theorem 1. The following two conditions are equivalent:*

(7) $\mathfrak{D}(A^{1/2}) \subset \mathfrak{D}(A^{*1/2})$, (8) $\mathfrak{D}(A^{1/2}) \subset \mathfrak{D}(\phi) \subset \mathfrak{D}(A^{*1/2})$.

The same is true when A and A^* are exchanged.

COROLLARY 1. (1) *is true if* $\mathfrak{D}(A^{1/2}) = \mathfrak{D}(A^{*1/2})$;

COROLLARY 2. (1) *is true if* $\mathfrak{D}(A) = \mathfrak{D}(A^*)$.

PROOF OF THEOREM 2. Again we may assume that A^{-1} and A^{*-1} are bounded. Then (7) implies that $A^{*1/2}A^{-1/2}$ and hence $A^{*-1/2}A^{1/2}$ is bounded. Thus $\| A^{*-1/2}v \| \leqq \text{const} \| A^{-1/2}v \|$, $v \in \mathfrak{H}$, and

$$\begin{aligned} \| H^{1/2}u \|^2 &= \operatorname{Re} \phi[u] = \operatorname{Re} (Au, u) = \operatorname{Re} (A^{*-1/2}Au, A^{1/2}u) \\ &\leqq \| A^{*-1/2}Au \| \, \| A^{1/2}u \| \leqq \text{const} \| A^{1/2}u \|^2 , \qquad u \in \mathfrak{D}(A) . \end{aligned}$$

This gives $\mathfrak{D}(A^{1/2}) \subset \mathfrak{D}(H^{1/2}) = \mathfrak{D}(\phi)$ as in the proof of Theorem 1, and $\mathfrak{D}(A^{*1/2}) \supset \mathfrak{D}(\phi)$ follows by Theorem 1.

PROOF OF COROLLARY 2. According to the generalized Heinz inequality (see Kato [**2**]), $\mathfrak{D}(A) = \mathfrak{D}(A^*)$ implies $\mathfrak{D}(A^{1/2}) = \mathfrak{D}(A^{*1/2})$. Thus Corollary 2 follows from Corollary 1.

THEOREM 3. *Let A, ϕ be as in Theorem 1. For $0 \leqq \alpha < 1/2$, we have*

$$\mathfrak{D}(A^{\alpha}) = \mathfrak{D}(A^{*\alpha}) \supset \mathfrak{D}(\phi) , \tag{9}$$

$$\mathfrak{D}(A^{1-\alpha}) \subset \mathfrak{D}(\phi) , \qquad \mathfrak{D}(A^{*1-\alpha}) \subset \mathfrak{D}(\phi) . \tag{10}$$

PROOF. (9) is a direct consequence of Theorem 3.1 of (I), by which $\mathfrak{D}(A^{\alpha}) = \mathfrak{D}(A^{*\alpha}) = \mathfrak{D}(H^{\alpha})$, for $\mathfrak{D}(H^{\alpha}) \supset \mathfrak{D}(H^{1/2}) = \mathfrak{D}(\phi)$. (10) follows from (9) exactly as in the second part of the proof of Theorem 1.

§ 2. Complex powers of accretive operators.

So far we have been mostly concerned with the powers A^{α} of an accretive

3) This corresponds to Lions' Theorem 5.2.

operator A for real α (except when both A and A^{-1} are bounded). Let us now consider A^{α} for complex α in a more general case; in particular we are interested in the case in which α is pure imaginary.

It is easy to give a reasonable definition of A^{α} for complex α even when A is unbounded. For example, the formula (A 7) of (I) can be used to define A^{α} for $0<\operatorname{Re}\alpha<1$. But it appears to be rather difficult to study the properties of A^{α} in this general case. In any case A^{α} is in general a complicated operator, as is seen from the special case in which A is normal in addition to being accretive; then the spectrum of A^{α} consists of a spiral-like band, which, in one direction, coils in to the origin indefinitely and, in the other, coils out to infinity. This band degenerates to a sector in the right semiplane if $0<\alpha<1$, and to a ring domain bounded by two concentric circles if α is pure imaginary; for other α, the spectrum of A is in general not even semibounded. Only in the case in which A is bounded (resp. bounded from below) would this band be bounded in one direction and, accordingly, A^{α} could be bounded or bounded from below.

For this reason, we restrict ourselves to the rather special case in which either the accretive operator A is bounded and $\operatorname{Re}\alpha\geqq 0$ or A^{-1} is bounded and $\operatorname{Re}\alpha\leqq 0$. Since the latter case is reduced to the former by considering A^{-1} instead of A, we shall mainly consider the former case.

THEOREM 4. *Let A be bounded and maximal accretive. Then A^{α} can be extended to complex α in such a way that it is holomorphic for* $\operatorname{Re}\alpha>0$ *and*[4] *($[\xi]$ is the integral part of ξ)*

$$\|A^{\alpha}\|\leqq\frac{\sin\pi\xi'}{\pi\xi'(1-\xi')}\|A\|^{\xi}e^{\frac{\pi|\eta|}{2}}\leqq\frac{4}{\pi}\|A\|^{\xi}e^{\frac{\pi|\eta|}{2}},\qquad \begin{array}{l}\alpha=\xi+i\eta,\\ \xi'=\xi-[\xi].\end{array}\tag{11}$$

If, in particular, A has no eigenvalue zero, A^{α} can be extended to $\operatorname{Re}\alpha\geqq 0$ *in such a way that A^{α} is strongly continuous and* (11) *is true for* $\operatorname{Re}\alpha\geqq 0$. *In particular $A^{i\eta}$ is strongly continuous in real η with $\|A^{i\eta}\|\leqq e^{\frac{\pi|\eta|}{2}}$.*

REMARK. A^{α} can be defined as a holomorphic function for $\operatorname{Re}\alpha\geqq 0$ even when A is a bounded operator in a Banach space and is the infinitesimal generator of a bounded semigroup (this follows from the proof below). The real interest for the case of an accretive operator lies in the estimate (11).

PROOF OF THEOREM 4. A^{α} can be defined for $0<\operatorname{Re}\alpha<1$ by

$$A^{\alpha}=\frac{\sin\pi\alpha}{\pi}\int_{0}^{\infty}\lambda^{\alpha-1}A(\lambda+A)^{-1}\,d\lambda\,.\tag{12}$$

For real α, $0<\alpha<1$, this coincides with (A 7) of (I). Since the A^{α} given by

4) It is not known whether the factor $\sin\pi\xi'/\pi\xi'(1-\xi')$ or $4/\pi$ is the best possible.

(12) is obviously holomorphic in α, this is an analytic extension of the A^α of (I). A^α can then be extended to the semiplane $\operatorname{Re}\alpha>0$ by $A^{n+\alpha'}=A^nA^{\alpha'}$, $n=1,2,\cdots$, $0\leqq\operatorname{Re}\alpha'<1$. There is no difficulty in verifying that A^α is holomorphic for $\operatorname{Re}\alpha>0$ and that $A^{\alpha+\beta}=A^\alpha A^\beta$ ($\alpha\to A^\alpha$ is a holomorphic semigroup).

(12) gives for real ξ, $0<\xi<1$,

$$\|A^\xi\|\leqq\frac{\sin\pi\xi}{\pi}\left\{\int_0^{\|A\|}\lambda^{\xi-1}d\lambda+\|A\|\int_{\|A\|}^{\infty}\lambda^{\xi-2}d\lambda\right\} \tag{13}$$

$$=\frac{\sin\pi\xi}{\pi\xi(1-\xi)}\|A\|^\xi\leqq\frac{4}{\pi}\|A\|^\xi,$$

where we used the inequality $\|A(\lambda+A)^{-1}\|\leqq\min(1,\lambda^{-1}\|A\|)$. We shall now show that, assuming for the moment that $\operatorname{Re}A\geqq\delta>0$ so that A^α is defined for all complex numbers α (see (I)),

$$\|A^{i\eta}\|\leqq e^{\frac{\pi|\eta|}{2}}. \tag{14}$$

Then (11) follows by noting that $A^\alpha=A^{\xi+i\eta}=A^{[\xi]}A^{\xi'}A^{i\eta}$. The general case can then be dealt with by replacing A by $A+\varepsilon$ and letting $\varepsilon\to0$.

To show (14), we note that $A^\alpha=H_\alpha+iK_\alpha$, $A^{*\alpha}=H_\alpha-iK_\alpha$, $\|K_\alpha H_\alpha^{-1}\|\leqq\left|\tan\frac{\pi\alpha}{2}\right|$ (see the proof of Theorem 1.1 of (I)). Hence $\|A^{*\alpha}A^{-\alpha}\|\leqq\left(1+\left|\tan\frac{\pi\alpha}{2}\right|\right)\left(1-\left|\tan\frac{\pi\alpha}{2}\right|\right)^{-1}$. For $\alpha=-i\eta$ this gives $\|A^{i\eta}\|^2=\|A^{*-i\eta}A^{i\eta}\|\leqq e^{\pi|\eta|}$, which proves (14).

To prove the second part of Theorem 4, it suffices to show that, for any $u\in\mathfrak{H}$, $A^\alpha u$ is *uniformly* continuous for $\alpha\in\mathfrak{D}$, where $\mathfrak{D}$ is the semi-open rectangle $0<\xi\leqq1$, $|\eta|\leqq R$, R being any positive number. Since A^α is bounded for $\alpha\in\mathfrak{D}$ by (11), however, it suffices to prove this for u belonging to a dense subset of $\mathfrak{H}$. If A has no eigenvalue zero as assumed, the range of A is dense in $\mathfrak{H}$ (for the proof see Lemma 2 below). Thus it suffices to prove the above proposition for u of the form $u=Av$. But then $A^\alpha u=A^{1+\alpha}v$ and this is obviously uniformly continuous for $\alpha\in\mathfrak{D}$.

LEMMA 2. *Let A be closed and maximal accretive. If A has no eigenvalue zero, then the range of A is dense in $\mathfrak{H}$.*

PROOF. This is an ergodic theorem and is a special case of a general theorem valid in Banach spaces (see Theorem of Kato [**3**]; see also Yosida [**5**]). For an accretive operator A, this follows also from the inequality

$$\|A^*(\lambda+A^*)^{-1}u\|^2\leqq\|A(\lambda+A)^{-1}u\|\,\|u\|,\qquad\lambda>0, \tag{15}$$

which implies that $Au=0$ implies $A^*u=0$. (15) is proved as follows:

$$4\|A^*(\lambda+A^*)^{-1}u\|^2=\|u-(\lambda-A^*)(\lambda+A^*)^{-1}u\|^2$$

$$\leqq2\|u\|^2-2\operatorname{Re}(u,(\lambda-A)(\lambda+A)^{-1}u)=4\operatorname{Re}(u,A(\lambda+A)^{-1}u);$$

note that $\|(\lambda-A^*)(\lambda+A^*)^{-1}\|=\|(\lambda-A)(\lambda+A)^{-1}\|\leqq 1$.

THEOREM 5. *Let A be closed and maximal accretive, with* $\mathrm{Re}(Au, u)\geqq\delta\|u\|^2$, $\delta>0$, *for* $u\in\mathfrak{D}(A)$. *Then $A^{-\alpha}$ can be extended for* $\mathrm{Re}\,\alpha\geqq 0$ *in such a way that it is holomorphic for* $\mathrm{Re}\,\alpha>0$ *and strongly continuous for* $\mathrm{Re}\,\alpha\geqq 0$, *with*

$$\|A^{-\alpha}\|\leqq\delta^{-\xi}e^{\frac{\pi|\eta|}{2}}, \qquad \alpha=\xi+i\eta. \tag{16}$$

PROOF. Only (16) need to be proved, other statements being a direct consequence of Theorem 4 applied to A^{-1}. An inspection of the proof of Theorem 4 shows that it suffices to prove (16) for real ξ, $0<\xi<1$. But this follows immediately from Lemma A 6 of (I).

§ 3. A new proof of the generalized Heinz inequality.

The Heinz inequality for selfadjoint operators was generalized in Kato [2] to the case of accretive operators. In view of its importance in applications, we shall give here another proof of it by using Theorem 4 obtained above. It suffices to prove this inequality in the following weak form (for the unbounded case see Kato [2]).

THEOREM 6. *Let A, B be maximal accretive operators in Hilbert spaces $\mathfrak{H}$, $\mathfrak{H}'$ respectively, all A, B, A^{-1}, B^{-1} being bounded. Let T be a bounded linear operator on H to H' such that* $\|T\|\leqq 1$, $\|BTA\|\leqq 1$. *Then* $\|B^{\xi}TA^{\xi}\|\leqq e^{\pi\sqrt{\xi(1-\xi)}}$ *for* $0\leqq\xi\leqq 1$.

REMARK. In the earlier result (Kato [2]), the exponent $\frac{\pi^2}{2}\xi(1-\xi)$ stands in place of $\pi\sqrt{\xi(1-\xi)}$. Therefore, Theorem 6 is less sharp than the previous one.[5]

PROOF OF THEOREM 6. Consider the operator-valued function

$$F(\alpha)=e^{k\alpha(\alpha-1)}B^{\alpha}TA^{\alpha}, \qquad 0<\mathrm{Re}\,\alpha<1, \tag{17}$$

where k is a positive constant. By Theorem 4, $F(\alpha)$ is holomorphic and bounded in the domain indicated, for ($\alpha=\xi+i\eta$)

$$\begin{aligned}\|F(\alpha)\| &\leqq e^{k\xi(\xi-1)-k\eta^2}\|B^{i\eta}\|\,\|B^{\xi}TA^{\xi}\|\,\|A^{i\eta}\| \\ &\leqq e^{k\xi(\xi-1)-k\eta^2+\pi|\eta|}\|B^{\xi}TA^{\xi}\|\leqq e^{\frac{\pi^2}{4k}}\|B^{\xi}TA^{\xi}\|.\end{aligned} \tag{18}$$

Furthermore, the same inequality shows that both $\|F(i\eta)\|$ and $\|F(1+i\eta)\|$ are

5) The proof of Theorem 6 is similar to the second proof by Heinz of his inequality, while the proof given in Kato [2] follows the method of Cordes. See the bibliography at the end of Kato [2].

bounded by $e^{\frac{\pi^2}{4k}}$ since $\|T\| \leqq 1$, $\|BTA\| \leqq 1$. According to the Phragmén-Lindelöf theorem, it follows that $\|F(\alpha)\| \leqq e^{\frac{\pi^2}{4k}}$ for $0 < \operatorname{Re} \alpha < 1$. For $\eta = 0$, this gives $\|B^\xi T A^\xi\| \leqq e^{\frac{\pi^2}{4k} + k\xi(1-\xi)}$. Since k was arbitrary, the result of the theorem follows by setting $k = \pi/2\sqrt{\xi(1-\xi)}$.

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Bibliography

[1] T. Kato, Fractional powers of dissipative operators, J. Math. Soc. Japan, **13** (1961), 246–274.
[2] T. Kato, A generalization of the Heinz inequality, Proc. Japan Acad., **37** (1961), 305–308.
[3] T. Kato, Remarks on pseudo-resolvents and infinitesimal generators of semi-groups, Proc. Japan Acad., **35** (1959), 467–468.
[4] J. L. Lions, Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs, This Journal,
[5] K. Yosida, Ergodic theorems for pseudo-resolvents, Proc. Japan Acad., **37** (1961), 422–425.