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A REMARK ON ALMOST SURE GLOBAL WELL-POSEDNESS OF THE ENERGY-CRITICAL DEFOCUSING NONLINEAR WAVE EQUATIONS IN THE PERIODIC SETTING

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Abstract. In this note, we prove almost sure global well-posedness of the energy-critical defocusing nonlinear wave equation on $\mathbb{T}^d$, $d = 3, 4$, and 5, with random initial data below the energy space.

1. Introduction.

1.1. Energy-critical nonlinear wave equations. We consider the Cauchy problem for the energy-critical defocusing nonlinear wave equation (NLW) on the d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$, $d = 3, 4$ or 5:

$$\begin{cases} \partial_t^2 u - \Delta u + |u|^{\frac{4}{d-2}} u = 0 \\ (u, \partial_t u)\big|_{t=0} = (u_0, u_1)\,, \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{T}^d\,, \tag{1.1}$$

where u is a real-valued function on $\mathbb{R} \times \mathbb{T}^d$. In particular, we prove almost sure global well-posedness of (1.1) with randomized initial data below the energy space.

NLW on the Euclidean space $\mathbb{R}^d$ has been studied extensively from both applied and theoretical points of view. Due to its analytical difficulty, the energy-critical defocusing NLW (1.1) on $\mathbb{R}^d$ has attracted a tremendous amount of attention over the last few decades. After substantial efforts by many mathematicians, it is known that (1.1) on $\mathbb{R}^d$ is globally well-posed in the energy space and all finite energy solutions scatter [29, 13, 14, 27, 28, 16, 12, 2, 1, 21, 22, 31]. Thanks to the finite speed of propagation, these global well-posedness results of (1.1) on $\mathbb{R}^d$ in the energy space immediately yield the corresponding global well-posedness of (1.1) on $\mathbb{T}^d$ in the energy space. We point out that these well-posedness results in the energy space are sharp in the sense that the energy-critical NLW (1.1) on $\mathbb{R}^d$ is known to be ill-posed below the energy space [10].

In recent years, there has been a significant development in incorporating non-deterministic points of view in the study of the Cauchy problems for hyperbolic and dispersive PDEs below certain regularity thresholds, in particular a scaling critical regularity. For example, the methodology developed in [6, 8, 4, 25] readily yields almost sure local well-posedness of (1.1) with respect to randomized initial data below the energy space. There

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are also results on almost sure global well-posedness that go beyond the deterministic thresholds. Burq-Tzvetkov [9] considered the energy-subcritical defocusing cubic NLW on $\mathbb{T}^3$ and established almost sure global well-posedness below the scaling critical regularity. Subsequently, Lührmann-Mendelson [19] applied the probabilistic high-low method developed in [11] and proved almost sure global well-posedness for some energy-subcritical NLW on $\mathbb{R}^3$ below the scaling critical regularity. See [20] for a recent improvement on this work.[1] More recently, the authors [25, 23] incorporated the deterministic energy-critical theory and proved almost sure global well-posedness below the energy space of the energy-critical defocusing NLW (1.1) on $\mathbb{R}^d$, $d = 3, 4$, and 5. Our main goal in this paper is to consider the energy-critical defocusing NLW (1.1) on $\mathbb{T}^d$ in the probabilistic setting and prove almost sure global well-posedness below the energy space. In the classical deterministic setting, the finite speed of propagation immediately allows us to transfer a deterministic global well-posedness result of NLW on $\mathbb{R}^d$ to the corresponding deterministic global well-posedness result on $\mathbb{T}^d$. This finite speed of propagation also plays an important role in our probabilistic setting. As we see below, however, the probabilistic results on $\mathbb{R}^d$ in [25, 23] are not directly transferrable to the periodic setting and some care must be taken.

1.2. Main result. The energy-critical NLW (1.1) on $\mathbb{R}^d$ is known to enjoy the following dilation symmetry: $u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x)$. Namely, if u is a solution to (1.1) on $\mathbb{R}^d$, then u_λ is also a solution to (1.1) on $\mathbb{R}^d$ with rescaled initial data. It is easy to check that the $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$-norm and the conserved energy $E(u)$ defined by

$$E(u) = E(u, \partial_t u) := \int \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 + \frac{d-2}{2d}|u|^{\frac{2d}{d-2}} dx$$

are invariant under this dilation symmetry. Note that by Sobolev's inequality, $E(u, \partial_t u) < \infty$ if and only if $(u, \partial_t u) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. For this reason, the space $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ is called the energy space. While there is no dilation symmetry on $\mathbb{T}^d$, we still refer to $H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ as the energy space for (1.1) posed on $\mathbb{T}^d$.

Our main goal is to prove almost sure global well-posedness of (1.1) on $\mathbb{T}^d$ below the energy space. We use the following shorthand notation for products of Sobolev spaces:

$$\mathcal{H}^s(M) := H^s(M) \times H^{s-1}(M),$$

where $M = \mathbb{T}^d$ or $\mathbb{R}^d$.

Given $s < 1$, fix a pair $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^d)$ of real valued functions. In terms of the Fourier series, we have

$$u_j(x) = \sum_{n \in \mathbb{Z}^d} \widehat{u}_j(n) e^{in\cdot x}, \quad j = 0, 1,$$

such that $\widehat{u}_j(-n) = \overline{\widehat{u}_j(n)}$. We introduce a randomization (u_0^ω, u_1^ω) of (u_0, u_1) as follows. For $j = 0, 1$, let $\{g_{n,j}\}_{n\in\mathbb{Z}^d}$ be a sequence of mean zero complex-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that $g_{-n,j} = \overline{g_{n,j}}$ for all $n \in \mathbb{Z}^d$, $j = 0, 1$. In

[1] There is also a recent work by Sun-Xia [30] on almost sure global well-posedness for some energy-subcritical NLW on $\mathbb{T}^3$.

particular, $g_{0,j}$ is real-valued. Moreover, we assume that $\{g_{0,j}, \operatorname{Re} g_{n,j}, \operatorname{Im} g_{n,j}\}_{n\in\mathcal{I}, j=0,1}$ are independent, where the index set $\mathcal{I}$ is defined by

$$\mathcal{I} := \bigcup_{k=0}^{d-1} \mathbb{Z}^k \times \mathbb{Z}_+ \times \{0\}^{d-k-1} . \tag{1.2}$$

Note that $\mathbb{Z}^d = \mathcal{I} \cup (-\mathcal{I}) \cup \{0\}$. Then, we define the randomization (u_0^ω, u_1^ω) of (u_0, u_1) by

$$(u_0^\omega, u_1^\omega) := \bigg(\sum_{n\in\mathbb{Z}^d} g_{n,0}\widehat{u}_0(n)e^{in\cdot x}, \sum_{n\in\mathbb{Z}^d} g_{n,1}\widehat{u}_1(n)e^{in\cdot x} \bigg). \tag{1.3}$$

In particular, if $\{g_{0,j}, \operatorname{Re} g_{n,j}, \operatorname{Im} g_{n,j}\}_{n\in\mathcal{I}, j=0,1}$ are independent standard complex-valued Gaussian random variables, then the randomization (1.3) corresponds the white noise randomization: $(u_0^\omega, u_1^\omega) = (\Xi_0 * u_0, \Xi_1 * u_1)$, where Ξ_0 and Ξ_1 are independent Gaussian white noise on $\mathbb{T}^d$. See [23] for more on this.

In the following, we also make the following assumption on the probability distributions $\mu_{n,j}$ of $g_{n,j}$; there exists $c > 0$ such that

$$\int e^{\gamma\cdot x} d\mu_{n,j}(x) \le e^{c|\gamma|^2} , \quad j = 0, 1 , \tag{1.4}$$

for all $n \in \mathbb{Z}^d$, (i) all $\gamma \in \mathbb{R}$ when $n = 0$, and (ii) all $\gamma \in \mathbb{R}^2$ when $n \in \mathbb{Z}^d \setminus \{0\}$. Note that (1.4) is satisfied by standard complex-valued Gaussian random variables, standard Bernoulli random variables, and any random variables with compactly supported distributions.

Our main result reads as follows.

THEOREM 1.1. *For $d = 3, 4$, or 5, let $s \in \mathbb{R}$ satisfy*

(i) $\frac{1}{2} < s < 1$ *when* $d = 3$, (ii) $0 < s < 1$ *when* $d = 4$, (iii) $0 \le s < 1$ *when* $d = 5$.

Given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^d)$, let (u_0^ω, u_1^ω) be the randomization defined in (1.3)*, satisfying* (1.4)*. Then, the energy-critical defocusing NLW* (1.1) *on $\mathbb{T}^d$ is almost surely globally well-posed. More precisely, there exists a set $\Omega_{(u_0,u_1)} \subset \Omega$ of probability 1 such that, for every $\omega \in \Omega_{(u_0,u_1)}$, there exists a unique solution u^ω to* (1.1) *with $(u^\omega, \partial_t u^\omega)|_{t=0} = (u_0^\omega, u_1^\omega)$ in the class:*

$$\big(S_{\mathrm{per}}(t)(u_0^\omega, u_1^\omega), \partial_t S_{\mathrm{per}}(t)(u_0^\omega, u_1^\omega)\big) + C(\mathbb{R}; \mathcal{H}^1(\mathbb{T}^d)) \subset C(\mathbb{R}; \mathcal{H}^s(\mathbb{T}^d)) .$$

Here, $S_{\mathrm{per}}(t)$ denotes the propagator for the linear wave equation on $\mathbb{T}^d$ given by

$$S_{\mathrm{per}}(t)\,(f_0, f_1) := \cos(t|\nabla|)f_0 + \frac{\sin(t|\nabla|)}{|\nabla|} f_1 .$$

This is the first result on almost sure global existence of unique solutions to energy-critical hyperbolic/dispersive PDEs in the periodic setting. In particular, when $d = 4$, Theorem 1.1 provides an affirmative answer to a question posed in [7]. When $d = 4$, Burq-Thomann-Tzvetkov [7] previously proved almost sure global existence (without uniqueness) of weak solutions to (1.1) on $\mathbb{T}^4$ for $0 < s < 1$. Moreover, the continuity (of the nonlinear part) of the solution constructed in [7] was obtained only in a weaker topology. Their main approach was to establish a probabilistic energy estimate and apply a compactness argument. The lack of uniqueness in [7] comes from the use of the compactness argument. Theorem 1.1 allows us to upgrade the weak solutions in [7] to strong solutions.[2]

In the Euclidean setting, we introduced in [25, 23] the probabilistic perturbation theory and proved almost sure global existence of unique solutions to (1.1) on $\mathbb{R}^d$, $d = 3, 4$, and 5. Let us briefly discuss the randomization of real-valued functions on $\mathbb{R}^d$ employed in [25, 23]. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\operatorname{supp} \psi \subset [-1, 1]^d$, $\psi(-\xi) = \overline{\psi(\xi)}$, and

$$\sum_{n\in\mathbb{Z}^d} \psi(\xi - n) \equiv 1 \quad \text{for all } \xi \in \mathbb{R}^d. \tag{1.5}$$

Then, any function u on $\mathbb{R}^d$ can be written as

$$u = \sum_{n\in\mathbb{Z}^d} \psi(D - n)u, \tag{1.6}$$

where $\psi(D - n)$ denotes the Fourier multiplier operator with symbol $\psi(\cdot - n)$. We then consider a randomization adapted to the decomposition (1.6). More precisely, given a pair (u_0, u_1) of functions on $\mathbb{R}^d$, we define the Wiener randomization (u_0^ω, u_1^ω) of (u_0, u_1) by

$$(u_0^\omega, u_1^\omega) := \bigg(\sum_{n\in\mathbb{Z}^d} g_{n,0}(\omega)\psi(D - n)u_0, \sum_{n\in\mathbb{Z}^d} g_{n,1}(\omega)\psi(D - n)u_1 \bigg). \tag{1.7}$$

This randomization is based on the uniform decomposition of the frequency space $\mathbb{R}^d_\xi$ into the unit cubes, called the Wiener decomposition [32]. In [25, 23], we proved that, given $s < 1$ satisfying the condition in Theorem 1.1 and any $(u_0, u_1) \in \mathcal{H}^s(\mathbb{R}^d)$, the energy-critical defocusing NLW on $\mathbb{R}^d$ is almost surely globally well-posed with respect to the Wiener randomization (u_0^ω, u_1^ω) defined in (1.7). See [19, 4, 5, 20] for other results utilizing the Wiener randomization (1.7).

Our basic strategy for the proof of Theorem 1.1 is to make use of the finite speed of propagation of solutions and reduce the problem on $\mathbb{T}^d \cong \big[-\frac{1}{2}, \frac{1}{2}\big)^d$ to a problem in the Euclidean setting. Fix $\eta \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ such that $\eta \equiv 1$ on $[-1, 1]^d$. Given $T > 0$, let

$$\eta_T(x) = \eta\big(\langle T\rangle^{-1}x\big), \tag{1.8}$$

where $\langle\cdot\rangle = 1 + |\cdot|$. Let $\mathbf{u}$ be a solution to the following energy-critical defocusing NLW on $\mathbb{R}^d$:

$$\begin{cases} \partial_t^2\mathbf{u} - \Delta\mathbf{u} + |\mathbf{u}|^{\frac{4}{d-2}}\mathbf{u} = 0 \\ (\mathbf{u}, \partial_t\mathbf{u})\big|_{t=0} = (\mathbf{u}_{0,T}, \mathbf{u}_{1,T}) := (\eta_T u_0, \eta_T u_1), \end{cases} \qquad (t, x) \in [0, T] \times \mathbb{R}^d, \tag{1.9}$$

where we view (u_0, u_1) as periodic functions on $\mathbb{R}^d$ with period 1. Then, by the finite speed of propagation, we see that $u := \mathbf{u}|_{[0,T]\times\mathbb{T}^d}$ is a solution to the periodic NLW (1.1) on the time interval $[0, T]$ with initial data (u_0, u_1). In the classical deterministic setting, this allows us to transfer global well-posedness on NLW on $\mathbb{R}^d$ to the corresponding global well-

[2]Here, we are indeed referring to the nonlinear part of a solution u.

posedness of the periodic NLW on $\mathbb{T}^d$. In our current probabilistic setting, however, this is not so straightforward. In particular, under such a reduction from the periodic setting to the Euclidean setting, our random initial data (u_0^ω, u_1^ω) on $\mathbb{T}^d$ of the form (1.3) does not give rise to an appropriate random initial data on $\mathbb{R}^d$ of the form (1.7) such that the results in [25, 23] are directly applicable.

Fix a pair (u_0, u_1) of real-valued functions defined on $\mathcal{H}^s(\mathbb{T}^d)$. Given $T > 0$, define a pair $(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)$ of random functions on $\mathbb{R}^d$ by setting

$$
\begin{aligned}
(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega) &:= (\eta_T u_0^\omega, \eta_T u_1^\omega) \\
&= \bigg(\sum_{n\in\mathbb{Z}^d} \eta_T(x) g_{n,0}(\omega)\widehat{u}_0(n)e^{in\cdot x}, \sum_{n\in\mathbb{Z}^d} \eta_T(x) g_{n,1}(\omega)\widehat{u}_1(n)e^{in\cdot x}\bigg),
\end{aligned}
\tag{1.10}
$$

where η_T is as in (1.8) and (u_0^ω, u_1^ω) is the randomization of (u_0, u_1) defined in (1.3), satisfying (1.4). Then, in order to prove Theorem 1.1, we need to prove almost sure well-posedness of (1.9) on $[0, T] \times \mathbb{R}^d$ with $(\mathbf{u}, \partial_t \mathbf{u})\big|_{t=0} = (\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)$ for some sequence of $T \to \infty$. First, note that the randomized initial data $(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)$ in (1.10) depends on T. Moreover, it is not of the form (1.7). Indeed, we have

$$
\widetilde{\mathbf{u}_{j,T}^\omega}(\xi) = \widetilde{\eta_T u_j^\omega}(\xi) = \sum_{n\in\mathbb{Z}^d} \widetilde{\eta}_T(\xi - n) g_{n,j}(\omega)\widehat{u}_j(n), \quad j = 0, 1.
\tag{1.11}
$$

In particular, the Fourier transform $\widetilde{\mathbf{u}_{j,T}^\omega}(\xi)$ depends on infinitely many $g_{n,j}$'s for each $\xi \in \mathbb{R}^d$. See Remark 1.2 below.

The proof of almost sure global well-posedness of (1.1) on $\mathbb{R}^d$ in [25, 23] consists of two disjoint parts: (i) a probabilistic part and (ii) a deterministic part. We can apply the deterministic part of the argument without any change. Therefore, our main task is to adapt the probabilistic part to our current problem. In particular, we will establish probabilistic Strichartz estimates (Propositions 4.1 and 4.4 below) that allow us to control random linear profiles on $\mathbb{R}^d$ in terms of functions on $\mathbb{T}^d$. See Section 4. We then need to adjust the argument in [25, 23] suitably to our setting.

We conclude this introduction by stating several remarks.

REMARK 1.2. If there were a function $\eta \in L^2(\mathbb{R}^d)$ with the properties (i) $\eta(x) \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2})^d$ and (ii) its Fourier transform $\widetilde{\eta}$ has a compact support, then we could basically apply the arguments in [25, 23] to study (1.9) with random initial data $(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)$ defined in (1.10). However, Paley-Wiener Theorem (Theorems IX.11 and IX.12 in [26]) states that there is no such function $\eta \in L^2(\mathbb{R}^d)$ satisfying both (i) and (ii).

REMARK 1.3. The uniqueness statement in Theorem 1.1 holds in the following sense. The existence part of Theorem 1.1 states that given any $\omega \in \Omega_{(u_0,u_1)}$, there exists a global solution u^ω to (1.1). Now, we fix one such $\omega \in \Omega_{(u_0,u_1)}$ and let $f^\omega := S_{\mathrm{per}}(\cdot)(u_0^\omega, u_1^\omega)$.

Setting $v^\omega := u^\omega - f^\omega$, we see that v^ω is a global solution to the perturbed NLW on $\mathbb{T}^d$:

$$\begin{cases} \partial_t^2 v^\omega - \Delta v^\omega + |v^\omega + f^\omega|^{\frac{4}{d-2}}(v^\omega + f^\omega) = 0 \\ (v^\omega, \partial_t v^\omega)|_{t=0} = (0,0)\,. \end{cases} \tag{1.12}$$

Then, the uniqueness in Theorem 1.1 holds for v^ω in

$$X(\mathbb{R}) := \big\{(v, \partial_t v) : (v, \partial_t v) \in C(\mathbb{R}, \dot{\mathcal{H}}^1(\mathbb{T}^d)),\ v \in L^{\frac{d+2}{d-2}}_{\text{loc}}(\mathbb{R}, L^{\frac{2(d+2)}{d-2}}(\mathbb{T}^d))\big\}. \tag{1.13}$$

This follows from a standard deterministic analysis of the perturbed NLW (1.12) on $\mathbb{T}^d$. See Appendix B. In terms of u^ω, the uniqueness holds in

$$\big(S_{\text{per}}(t)(u_0^\omega, u_1^\omega), \partial_t S_{\text{per}}(t)(u_0^\omega, u_1^\omega)\big) + X(\mathbb{R})\,.$$

Lastly, note that the almost sure global solutions constructed in [25, 23] also satisfy the same kind of uniqueness.

REMARK 1.4. Let $\mathbf{u_0} : \Omega \to \mathcal{H}^s(\mathbb{T}^d)$ be the map given by $\mathbf{u_0}(\omega) := (u_0^\omega, u_1^\omega)$, where (u_0^ω, u_1^ω) is as in (1.3). Then, the map $\mathbf{u_0}$ induces a probability measure $\mu = \mu_{(u_0,u_1)} = P \circ \mathbf{u_0}^{-1}$ on $\mathcal{H}^s(\mathbb{T}^d)$. Now, let $\Sigma_{(u_0,u_1)} = \mathbf{u_0}(\Omega_{(u_0,u_1)})$, where $\Omega_{(u_0,u_1)}$ is as in Theorem 1.1. Then, while $\mu(\Sigma_{(u_0,u_1)}) = 1$, it is possible that $\mu\big(\Phi(t)(\Sigma_{(u_0,u_1)})\big)$ becomes smaller for some $t \neq 0$ and even tends to 0, where $\Phi(t)$ denotes the solution map of (1.1). Arguing as in [25], we can strengthen the statement in Theorem 1.1 and show that there exists another set of μ-full measure $\Sigma \subset \mathcal{H}^s(\mathbb{T}^d)$ such that (a) for any $(\phi_0, \phi_1) \in \Sigma$, there exists a unique global solution u to (1.1) with initial data $(u, \partial_t u)\big|_{t=0} = (\phi_0, \phi_1)$ and (b) $\mu\big(\Phi(t)(\Sigma)\big) = 1$ for any $t \in \mathbb{R}$. Namely, the measure of our new initial data set Σ does not become smaller under the dynamics of (1.1). See [9, 24, 25] for related discussions in this direction.

2. Notations. Given a periodic function f on $\mathbb{T}^d$, we use $\widehat{f}(n) = \mathcal{F}_{\mathbb{T}^d}(f)(n)$ to denote the Fourier coefficient of f on $\mathbb{T}^d$. Given a function f on $\mathbb{R}^d$, we use $\widetilde{f}(\xi) = \mathcal{F}_{\mathbb{R}^d}(f)(\xi)$ to denote the Fourier transform of f on $\mathbb{R}^d$. Let f be a periodic function on $\mathbb{T}^d$. By viewing f as a tempered distribution on $\mathbb{R}^d$ we have

$$\widetilde{f}(\xi) = \sum_{n \in \mathbb{Z}^d} \delta(\xi - n)\widehat{f}(n)\,.$$

Moreover, given $\eta \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\widetilde{\eta f}(\xi) = \sum_{n \in \mathbb{Z}^d} \widetilde{\eta}(\xi - n)\widehat{f}(n)\,. \tag{2.1}$$

Given $n \in \mathbb{Z}^d$, let Q_n be the unit cube $Q_n := n + \big[-\frac{1}{2}, \frac{1}{2}\big)^d$ centered at n.

Next, we briefly go over the Littlewood-Paley theory on $\mathbb{R}^d$. Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth bump function supported on $[-\frac{8}{5}, \frac{8}{5}]$ and $\varphi \equiv 1$ on $\big[-\frac{5}{4}, \frac{5}{4}\big]$. Given dyadic $N \geq 1$, we set $\varphi_1(\xi) = \varphi(|\xi|)$ and

$$\varphi_N(\xi) = \varphi\big(\tfrac{|\xi|}{N}\big) - \varphi\big(\tfrac{2|\xi|}{N}\big)$$

for $N \geq 2$. Then, we define the Littlewood-Paley projection $\mathbf{P}_N$ as the Fourier multiplier operator with symbol φ_N. Moreover, we define $\mathbf{P}_{\leq N}$ and $\mathbf{P}_{\geq N}$ by $\mathbf{P}_{\leq N} = \sum_{1 \leq M \leq N} \mathbf{P}_M$ and $\mathbf{P}_{>N} = \sum_{M>N} \mathbf{P}_M$. For a periodic function f on $\mathbb{T}^d$, we define $\mathbf{P}_N$ to be the projection onto the frequencies $\{\frac{1}{2}N < |n| \leq N\}$ if $N \geq 2$ and $\{|n| \leq 1\}$ if $N = 1$. In the following, we use $\mathbf{P}_N$ to denote the Littlewood-Paley projection for both functions on $\mathbb{R}^d$ and $\mathbb{T}^d$, depending on the context.

We use $S(t)$ to denote the propagator for the linear wave equation on $\mathbb{R}^d$ given by

$$S(t)\,(f_0, f_1) := \cos(t|\nabla|) f_0 + \frac{\sin(t|\nabla|)}{|\nabla|} f_1\,. \tag{2.2}$$

We say that u is a solution to the following nonhomogeneous wave equation on $\mathbb{R}^d$:

$$\begin{cases} \partial_t^2 u - \Delta u + F = 0 \\ (u, \partial_t u)|_{t=t_0} = (\phi_0, \phi_1) \end{cases} \tag{2.3}$$

on a time interval I containing t_0, if u satisfies the following Duhamel formulation:

$$u(t) = S(t - t_0)(\phi_0, \phi_1) - \int_{t_0}^{t} \frac{\sin((t - t')|\nabla|)}{|\nabla|} F(t') dt' \tag{2.4}$$

for $t \in I$. We now recall the Strichartz estimates for wave equations on $\mathbb{R}^3$. We say that (q, r) is an s-wave admissible pair if $q \geq 2$, $2 \leq r < \infty$,

$$\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4}, \quad \text{and} \quad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s\,.$$

Then, we have the following Strichartz estimates. See [12, 18, 17] for more discussions on the Strichartz estimates.

LEMMA 2.1. *Let $s > 0$. Let (q, r) and $(\tilde{q}, \tilde{r})$ be s- and $(1-s)$-wave admissible pairs, respectively. Then, we have*

$$\|(u, \partial_t u)\|_{L_t^\infty(I; \dot{\mathcal{H}}_x^s(\mathbb{R}^d))} + \|u\|_{L_t^q(I; L_x^r)} \lesssim \|(\phi_0, \phi_1)\|_{\dot{\mathcal{H}}^s(\mathbb{R}^d)} + \|F\|_{L_t^{\tilde{q}'}(I; L_x^{\tilde{r}'}(\mathbb{R}^d))} \tag{2.5}$$

for all solutions u to (2.3) *on a time interval $I \ni t_0$.*

In our argument, we will only use the following wave admissible pairs: $\left(\frac{d+2}{d-2}, \frac{2(d+2)}{d-2}\right)$ with $s = 1$ and $(\infty, 2)$ with $s = 0$. For simplicity, we denote the space $L_t^q(I; L_x^r)$ by $L_I^q L_x^r$ or $L_T^q L_x^r$ if $I = [0, T]$.

In the following, constants in various estimates depend on the smooth cutoff function η, appearing in (1.8). Since we fix such η once and for all, we suppress the dependence on η. Lastly, in view of the time reversibility of the equation, we only consider positive times in the following.

3. Reduction to the Euclidean setting. We first reduce Theorem 1.1 to the following proposition on "almost" almost sure global well-posedness of (1.1).

PROPOSITION 3.1. *Let* (s,d) *be as in Theorem* 1.1. *Given* $(u_0,u_1) \in \mathcal{H}^s(\mathbb{T}^d)$, *let* (u_0^ω, u_1^ω) *be the randomization defined in* (1.3), *satisfying* (1.4). *Then, for any given* $T \geq 1$ *and* $\varepsilon > 0$, *there exists a set* $\Omega_{T,\varepsilon} \subset \Omega$ *with* $P(\Omega_{T,\varepsilon}^c) < \varepsilon$ *such that, for every* $\omega \in \Omega_{T,\varepsilon}$, *there exists a unique solution* u^ω *to* (1.1) *with* $(u^\omega, \partial_t u^\omega)|_{t=0} = (u_0^\omega, u_1^\omega)$ *in the class:*

$$(3.1)\quad \big(S_{\mathrm{per}}(t)(u_0^\omega,u_1^\omega), \partial_t S_{\mathrm{per}}(t)(u_0^\omega,u_1^\omega)\big) + C([0,T];\mathcal{H}^1(\mathbb{T}^d)) \subset C([0,T];\mathcal{H}^s(\mathbb{T}^d)).$$

It is easy to see that Proposition 3.1 implies Theorem 1.1. See, for example, [11, 25]. Therefore, in the remaining part of the paper, we focus on the proof of Proposition 3.1 for each fixed $T \geq 1$ and $\varepsilon > 0$.

Given $(u_0,u_1) \in \mathcal{H}^s(\mathbb{T}^d)$ and $T \geq 1$, let $(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)$ be the random functions on $\mathbb{R}^d$ defined in (1.10). Consider the following Cauchy problem:

$$(3.2)\qquad \begin{cases} \partial_t^2 \mathbf{u}^\omega - \Delta \mathbf{u}^\omega + |\mathbf{u}^\omega|^{\frac{4}{d-2}}\mathbf{u}^\omega = 0 \\ (\mathbf{u}^\omega, \partial_t \mathbf{u}^\omega)\big|_{t=0} = (\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)\,, \end{cases} \qquad (t,x) \in [0,T]\times\mathbb{R}^d\,.$$

In view of the finite speed of propagation, Proposition 3.1 follows once we prove the following proposition. See Appendix A for this part of the reduction.

PROPOSITION 3.2. *Let* (s,d) *be as in Theorem* 1.1. *Given* $(u_0,u_1) \in \mathcal{H}^s(\mathbb{T}^d)$ *and* $T \geq 1$, *let* $(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)$ *be the random functions on* $\mathbb{R}^d$ *defined in* (1.10), *satisfying* (1.4). *Then, for any* $\varepsilon > 0$, *there exists a set* $\widetilde{\Omega}_{T,\varepsilon} \subset \Omega$ *with* $P(\widetilde{\Omega}_{T,\varepsilon}^c) < \varepsilon$ *such that, for every* $\omega \in \widetilde{\Omega}_{T,\varepsilon}$, *there exists a unique solution* $\mathbf{u}^\omega$ *to* (3.2) *with* $(\mathbf{u}^\omega, \partial_t \mathbf{u}^\omega)|_{t=0} = (\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)$ *in the class:*

$$\big(S(t)(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega), \partial_t S(t)(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)\big) + C([0,T];\mathcal{H}^1(\mathbb{R}^d)) \subset C([0,T];\mathcal{H}^s(\mathbb{R}^d))\,.$$

Moreover, the nonlinear part $\mathbf{v}^\omega := \mathbf{u}^\omega - S(\cdot)(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega)$ *of the solution satisfies the bounds*

$$(3.3)\qquad \|\mathbf{v}^\omega\|_{L_t^q([0,T],L_x^r(\mathbb{R}^d))} \leq C(T,\varepsilon,\|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{T}^d)})\,,$$

for all 1*-wave admissible pairs* (q,r).

The main idea is to adapt the argument in [25, 23] on almost sure global well-posedness of (1.1) on $\mathbb{R}^d$ with random initial data of the form (1.7). Denoting the linear and nonlinear parts of the solution $\mathbf{u}^\omega$ to (3.2) by

$$(3.4)\qquad \mathbf{z}^\omega(t) = \mathbf{z}_T^\omega(t) := S(t)(\mathbf{u}_{0,T}^\omega, \mathbf{u}_{1,T}^\omega) \quad \text{and} \quad \mathbf{v}^\omega := \mathbf{u}^\omega - \mathbf{z}^\omega\,,$$

we can reformulate (3.2) as the following perturbed NLW:

$$(3.5)\qquad \begin{cases} \partial_t^2 \mathbf{v}^\omega - \Delta \mathbf{v}^\omega + F(\mathbf{v}^\omega + \mathbf{z}^\omega) = 0 \\ (\mathbf{v}^\omega, \partial_t \mathbf{v}^\omega)|_{t=0} = (0,0)\,, \end{cases}$$

where $F(u) = |u|^{\frac{4}{d-2}}u$. As mentioned above, the argument in [25, 23] can be divided into two parts: (i) the probabilistic part and (ii) the deterministic study of the perturbed NLW:

$$(3.6)\qquad \begin{cases} \partial_t^2 \mathbf{v} - \Delta \mathbf{v} + F(\mathbf{v} + f) = 0 \\ (\mathbf{v}, \partial_t \mathbf{v})\big|_{t=0} = (\mathbf{v}_0, \mathbf{v}_1)\,, \end{cases}$$

where f is a deterministic function, satisfying some a priori space-time bounds. This deterministic part (Proposition 4.3 in [25] and Proposition 5.2 in [23]) can be applied to our problem without any change, and hence we take it as a black box in this paper. Therefore, our main task is to appropriately modify the probabilistic part of the argument.

In the next section, we prove new probabilistic Strichartz estimates (Propositions 4.1 and 4.4), controlling the size of the random linear solution $S(t)(\mathbf{u}_{0,T}^{\omega}, \mathbf{u}_{1,T}^{\omega})$ on $\mathbb{R}^d$ in terms of the deterministic initial data (u_0, u_1) on $\mathbb{T}^d$. Then, in Section 5, we briefly discuss how to modify the argument in [25, 23] to prove Proposition 3.2. In Appendix A, we consider the issue on the finite speed of propagation on random solutions at a low regularity and show how to deduce Proposition 3.1 from Proposition 3.2. Finally, we sketch the uniqueness part of Theorem 1.1 in Appendix B.

Let us conclude this section by stating a lemma, which allows us to compare the H^s-norms of a periodic function on $\mathbb{R}^d$ and $\mathbb{T}^d$ through the multiplication by η_T.

LEMMA 3.3. *Let* $0 \le s < 1$. *Then, there exists* $C > 0$ *such that*

$$\frac{1}{C}\langle T\rangle^{\frac{d}{2}}\|f\|_{H^s(\mathbb{T}^d)}\| \le \|\eta_T f\|_{H^s(\mathbb{R}^d)} \le C\langle T\rangle^{\frac{d}{2}}\|f\|_{H^s(\mathbb{T}^d)}, \tag{3.7}$$

for any $T > 0$ *and any periodic function* $f \in H^s(\mathbb{T}^d)$.

PROOF. Given $m \in \mathbb{Z}^d$ and $T > 0$, set $\langle T\rangle Q_m := \{\xi \in \mathbb{R}^d : \langle T\rangle^{-1}\xi \in Q_m\}$. Then, for $s \ge 0$, it follows from (1.8) that

$$\|\langle \cdot \rangle^s \widetilde{\eta}_T\|_{L^2(Q_m)} \le \langle T\rangle^{\frac{d}{2}}\|\langle \cdot \rangle^s \widetilde{\eta}\|_{L^2(\langle T\rangle Q_m)} \lesssim \langle T\rangle^{\frac{d}{2}} \sum_{k\in\mathbb{Z}^d\cap\langle T\rangle Q_m} \langle k\rangle^s \|\psi(D-k)\eta\|_{L^2(\mathbb{R}^d)}, \tag{3.8}$$

where ψ is as in (1.5). Then, by (2.1), the triangle inequality (with $s \ge 0$), Minkowski's integral inequality, Young's inequality, and (3.8), we have

$$\begin{aligned}
\|\eta_T f\|_{H^s(\mathbb{R}^d)} &= \bigg(\int \langle\xi\rangle^{2s}\bigg|\sum_{n\in\mathbb{Z}^d}\widetilde{\eta}_T(\xi-n)\widehat{f}(n)\bigg|^2 d\xi\bigg)^{\frac{1}{2}}\\
&\lesssim \bigg(\int\bigg(\sum_{n\in\mathbb{Z}^d}\langle\xi-n\rangle^s|\widetilde{\eta}_T(\xi-n)|\cdot\langle n\rangle^s|\widehat{f}(n)|\bigg)^2 d\xi\bigg)^{\frac{1}{2}}\\
&\lesssim \bigg(\sum_{m\in\mathbb{Z}^d}\int_{Q_m}\bigg(\sum_{n\in\mathbb{Z}^d}\langle\xi-n\rangle^s|\widetilde{\eta}_T(\xi-n)|\cdot\langle n\rangle^s|\widehat{f}(n)|\bigg)^2 d\xi\bigg)^{\frac{1}{2}}\\
&\lesssim \bigg(\sum_{m\in\mathbb{Z}^d}\bigg(\sum_{n\in\mathbb{Z}^d}\|\langle\cdot\rangle^s\widetilde{\eta}_T\|_{L^2(Q_{m-n})}\langle n\rangle^s|\widehat{f}(n)|\bigg)^2\bigg)^{\frac{1}{2}}\\
&\le \|\langle\cdot\rangle^s\widetilde{\eta}_T\|_{\ell^1_m L^2(Q_m)}\|f\|_{H^s(\mathbb{T}^d)} \lesssim \langle T\rangle^{\frac{d}{2}}\|\eta\|_{M^s_{2,1}}\|f\|_{H^s(\mathbb{T}^d)}.
\end{aligned} \tag{3.9}$$

Here, $M^s_{2,1}$ denotes the (weighted) modulation space defined by the norm

$$\|\eta\|_{M^s_{2,1}} = \big\|\langle n\rangle^s\|\psi(D-n)\eta\|_{L^2(\mathbb{R}^d)}\big\|_{\ell^1(\mathbb{Z}^d)}.$$

Let $\mathbb{T}^d_T := \big[-T-\frac{1}{2}, T+\frac{1}{2}\big)^d$. Then, by the definition of η_T, we have

$$\langle T\rangle^{\frac{d}{2}} \|f\|_{L^2(\mathbb{T}^d)} \sim \|f\|_{L^2(\mathbb{T}^d_T)} \le \|\eta_T f\|_{L^2(\mathbb{R}^d)}. \tag{3.10}$$

By the characterization of the $\dot{H}^s$-norms on the physical side (see, for example, [15] and [3] on $\mathbb{R}^d$ and $\mathbb{T}^d$, respectively), the periodicity of f, and the definition of η_T, we have

$$\begin{aligned}
\langle T\rangle^{\frac{d}{2}} \|f\|_{\dot{H}^s(\mathbb{T}^d)} &\sim \langle T\rangle^{\frac{d}{2}} \bigg(\int_{\mathbb{T}^d}\int_{Q_0} \frac{|f(x+y)-f(x)|^2}{|y|^{d+2s}} dydx\bigg)^{\frac{1}{2}}\\
&\sim \bigg(\int_{\mathbb{T}^d_T}\int_{Q_0} \frac{|f(x+y)-f(x)|^2}{|y|^{d+2s}} dydx\bigg)^{\frac{1}{2}}\\
&\le \bigg(\int_{\mathbb{R}^d}\int_{Q_0} \frac{|\eta_T(x+y)f(x+y)-\eta_T(x)f(x)|^2}{|y|^{d+2s}} dydx\bigg)^{\frac{1}{2}}\\
&\le \bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\eta_T(x+y)f(x+y)-\eta_T(x)f(x)|^2}{|y|^{d+2s}} dydx\bigg)^{\frac{1}{2}}\\
&\sim \|\eta_T f\|_{\dot{H}^s(\mathbb{R}^d)}
\end{aligned} \tag{3.11}$$

for $0 < s < 1$. Hence, (3.7) follows from (3.9), (3.10), and (3.11). □

4. Probabilistic Strichartz estimates. In this section, we state and prove the crucial probabilistic Strichartz estimates that build a bridge between the random linear solution $S(t)(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})$ on $\mathbb{R}^d$ and the deterministic initial data (u_0, u_1) on $\mathbb{T}^d$. In [25, 23], we studied the probabilistic Strichartz estimates on $\mathbb{R}^d$ with random initial data of the form (1.7) (Proposition 2.3 in [25] and Proposition 3.3 in [23]). The following propositions (Propositions 4.1 and 4.4) are suitable replacements for our problem at hand. In particular, we have the $\mathcal{H}^s(\mathbb{T}^d)$-norm of (u_0, u_1) on $\mathbb{T}^d$ on the right-hand side of (4.1) and (4.12).

PROPOSITION 4.1. *Let $T > 0$. Given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^d)$, let $(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})$ be the randomization on $\mathbb{R}^d$ defined in* (1.10)*, satisfying* (1.4)*. Then, given $1 \le q < \infty$, $2 \le r \le \infty$, there exist $C, c > 0$ such that*

$$\begin{aligned}
&P\Big(\|S(t)(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})\|_{L^q_t(I;L^r_x(\mathbb{R}^d))} > \lambda\Big)\\
&\qquad\le C\exp\bigg(-c\frac{\lambda^2}{\max(1,b^2)\langle T\rangle^d |I|^{\frac{2}{q}} \|(u_0,u_1)\|^2_{\mathcal{H}^s(\mathbb{T}^d)}}\bigg)
\end{aligned} \tag{4.1}$$

for any compact time interval $I = [a,b] \subset [0,T]$, provided (i) $s = 0$ *if* $r < \infty$ *and* (ii) $s > 0$ *if* $r = \infty$.

REMARK 4.2. Let $(\mathbf{u}_{0,T}, \mathbf{u}_{1,T}) := (\eta_T u_0, \eta_T u_1)$ as in (1.9). Then, in view of Lemma 3.3, we can rewrite (4.1) as

$$P\Big(\|S(t)(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})\|_{L^q_t(I;L^r_x(\mathbb{R}^d))} > \lambda\Big)$$

$$\leq C \exp\bigg(- c \frac{\lambda^2}{\max(1, b^2)|I|^{\frac{2}{q}} \|(\mathbf{u}_{0,T}, \mathbf{u}_{1,T})\|^2_{\mathcal{H}^s(\mathbb{R}^d)}}\bigg). \tag{4.2}$$

We point out that (4.2) is more in the spirit of the statement of Proposition 2.3 (ii) and (iii) in [25].

Before presenting the proof of Proposition 4.1, we first recall the following probabilistic estimate. See [8] for the proof.

LEMMA 4.3. *Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be a sequence of mean zero complex-valued random variables such that $g_{-n} = \overline{g_n}$ for all $n \in \mathbb{Z}^d$. With $\mathcal{I}$ as in* (1.2)*, assume that g_0,* Re g_n*, and* Im g_n*, $n \in \mathcal{I}$, are independent. Moreover, assume that* (1.4) *is satisfied. Then, there exists $C > 0$ such that the following holds:*

$$\Big\| \sum_{n\in\mathbb{Z}^d} g_n(\omega) c_n \Big\|_{L^p(\Omega)} \leq C\sqrt{p} \|c_n\|_{\ell^2_n(\mathbb{Z}^d)}$$

for any $p \geq 2$ and any sequence $\{c_n\} \in \ell^2(\mathbb{Z}^d)$ satisfying $c_{-n} = \overline{c_n}$ for all $n \in \mathbb{Z}^d$.

PROOF OF PROPOSITION 4.1. The proof is analogous to that of Proposition 2.3 in [25]. There are, however, important differences due to the fact that $\widetilde{\eta}$ does not have a compact support and that we use the $\mathcal{H}^s(\mathbb{T}^d)$-norm of (u_0, u_1) on the right-hand side of (4.1).

• **Case 1:** We first consider the case $r < \infty$. Given $1 \leq q < \infty$ and $2 \leq r < \infty$, let $p \geq \max(q, r)$.

Let T_m be the Fourier multiplier operator with a bounded multiplier m. Let $\beta = \frac{(r-2)}{2r} d + \varepsilon \leq \frac{d}{2}$ for some small $\varepsilon > 0$. Then, by Hausdorff-Young's inequality and Hölder's inequality with $\frac{1}{r'} = \frac{1}{2} + \frac{r-2}{2r}$, we have

$$\begin{aligned} \|T_m(\eta_T e^{inx})\|_{L^r_x(\mathbb{R}^d)} &\leq \|m(\xi)\widetilde{\eta}_T(\xi - n)\|_{L^{r'}_\xi(\mathbb{R}^d)} \lesssim \|\langle \xi - n\rangle^\beta m(\xi)\widetilde{\eta}_T(\xi - n)\|_{L^2_\xi(\mathbb{R}^d)} \\ &\lesssim \|m\|_{L^\infty(\mathbb{R}^d)} \|\eta_T\|_{H^{\frac{d}{2}}(\mathbb{R}^d)} \lesssim \langle T\rangle^{\frac{d}{2}} \|m\|_{L^\infty(\mathbb{R}^d)} \|\eta\|_{H^{\frac{d}{2}}(\mathbb{R}^d)} \end{aligned} \tag{4.3}$$

for each $n \in \mathbb{Z}^d$. Note that we have

$$|\cos(t|\xi|)| \leq 1 \quad \text{and} \quad \left| \frac{\sin(t|\xi|)}{|\xi|} \right| \leq t \tag{4.4}$$

for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and all $t \in \mathbb{R}$. Then, by Minkowski's integral inequality, Lemma 4.3 with (1.10), and (4.3) with (4.4), we have

$$\begin{aligned} \Big(\mathbb{E}\big\| \cos(t|\nabla|)\mathbf{u}^\omega_{0,T}\big\|^p_{L^q_t(I; L^r_x(\mathbb{R}^d))}\Big)^{\frac{1}{p}} &\leq \Big\| \|\cos(t|\nabla|)\mathbf{u}^\omega_{0,T}\|_{L^p(\Omega)}\Big\|_{L^q_I L^r_x} \\ &\lesssim \sqrt{p} \Big\| \big\|\widehat{u}_0(n)\cos(t|\nabla|)(\eta_T e^{inx})\big\|_{\ell^2_n}\Big\|_{L^q_I L^r_x} \leq \sqrt{p} \Big\| \big\|\cos(t|\nabla|)(\eta_T e^{inx})\big\|_{L^r_x} \cdot \widehat{u}_0(n)\Big\|_{L^q_I \ell^2_n} \\ &\lesssim \sqrt{p} \langle T\rangle^{\frac{d}{2}} |I|^{\frac{1}{q}} \|\eta\|_{H^{\frac{d}{2}}(\mathbb{R}^d)} \|u_0\|_{L^2(\mathbb{T}^d)}. \end{aligned} \tag{4.5}$$

When $|\xi| \gtrsim 1$, it follows from the triangle inequality: $\langle n\rangle \leq \langle \xi\rangle\langle \xi - n\rangle$ that

$$\left|\frac{\sin(t|\xi|)}{|\xi|}\right| \lesssim \frac{1}{\langle \xi\rangle} \leq \frac{\langle \xi - n\rangle}{\langle n\rangle} \tag{4.6}$$

for all $n \in \mathbb{Z}^d$. On the other hand, when $|\xi| \ll 1$, it follows from (4.4) that

$$\left|\frac{\sin(t|\xi|)}{|\xi|}\right| \lesssim \frac{t}{\langle \xi\rangle} \leq t\frac{\langle \xi - n\rangle}{\langle n\rangle} \tag{4.7}$$

for all $n \in \mathbb{Z}^d$. Hence, proceeding as before with (4.6) and (4.7), we have

$$\begin{aligned}\left(\mathbb{E}\left\|\frac{\sin(t|\nabla|)}{|\nabla|}\mathbf{u}^\omega_{1,T}\right\|^p_{L^q_t(I;L^r_x(\mathbb{R}^d))}\right)^{\frac{1}{p}} &\lesssim \sqrt{p}\left\|\left\|\frac{\sin(t|\nabla|)}{|\nabla|}\big(\eta_T e^{inx}\big)\right\|_{L^r_x}\cdot \widehat{u}_1(n)\right\|_{L^q_I\ell^2_n}\\ &\lesssim \sqrt{p}\max(1,b)\left\|\|\langle \xi - n\rangle\widetilde{\eta}_T(\xi - n)\|_{L^{r'}_x}\cdot\frac{\widehat{u}_1(n)}{\langle n\rangle}\right\|_{L^q_I\ell^2_n}\\ &\lesssim \sqrt{p}\max(1,b)\langle T\rangle^{\frac{d}{2}}|I|^{\frac{1}{q}}\|\eta\|_{H^{\frac{d}{2}+1}(\mathbb{R}^d)}\|u_1\|_{H^{-1}(\mathbb{T}^d)}. \end{aligned}\tag{4.8}$$

Then, (4.1) follows from (4.5), (4.8), and a standard argument using Chebyshev's inequality. See [4, 25] for details.

• **Case 2:** Next, we consider the case $r = \infty$. In this case, given small $s > 0$, choose $\widetilde{r} \gg 1$ such that $s\widetilde{r} > d$. Then, by Sobolev embedding theorem, we have

$$\big\|S(t)(\mathbf{u}^\omega_{0,T},\mathbf{u}^\omega_{1,T})\big\|_{L^q_t(I;L^\infty_x)} \lesssim \big\|\langle\nabla\rangle^s S(t)(\mathbf{u}^\omega_{0,T},\mathbf{u}^\omega_{1,T})\big\|_{L^q_t(I;L^{\widetilde{r}}_x)}.$$

We now proceed as in Case 1 with $\widetilde{r}$ instead of r. With the triangle inequality $\langle\xi\rangle^s \lesssim \langle \xi - n\rangle^s\langle n\rangle^s$, we have

$$\left(\mathbb{E}\big\|\langle\nabla\rangle^s\cos(t|\nabla|)\mathbf{u}^\omega_{0,T}\big\|^p_{L^q_t(I;L^{\widetilde{r}}_x)}\right)^{\frac{1}{p}} \lesssim \sqrt{p}\langle T\rangle^{\frac{d}{2}}|I|^{\frac{1}{q}}\|\eta\|_{H^{\frac{d}{2}+s}(\mathbb{R}^d)}\|u_0\|_{H^s(\mathbb{T}^d)} \tag{4.9}$$

and

$$\begin{aligned}&\left(\mathbb{E}\left\|\langle\nabla\rangle^s\frac{\sin(t|\nabla|)}{|\nabla|}\ \mathbf{u}^\omega_{1,T}\right\|^p_{L^q_t(I;L^{\widetilde{r}}_x(\mathbb{T}^d))}\right)^{\frac{1}{p}}\\ &\qquad\lesssim \sqrt{p}\max(1,b^2)\langle T\rangle^{\frac{d}{2}}|I|^{\frac{1}{q}}\|\eta\|_{H^{\frac{d}{2}+1+s}(\mathbb{R}^d)}\|u_0\|_{H^{s-1}(\mathbb{T}^d)}.\end{aligned}\tag{4.10}$$

Once again, (4.1) follows from (4.9), (4.10), and a standard argument using Chebyshev's inequality. □

Next, we prove a probabilistic estimate involving the L^∞_t-norm. This proposition replaces Proposition 3.3 in [23] and plays an important role in treating the three-dimensional case. Define an operator $\widetilde{S}(t)$ on a pair (f_0, f_1) of functions on $\mathbb{R}^d$ by

$$\widetilde{S}(t)(f_0,f_1) := -\frac{|\nabla|}{\langle\nabla\rangle}\sin(t|\nabla|)f_0 + \frac{\cos(t|\nabla|)}{\langle\nabla\rangle}f_1. \tag{4.11}$$

Namely, we have $\partial_t S(t)(f_0,f_1) = \langle\nabla\rangle\widetilde{S}(t)(f_0,f_1)$.

PROPOSITION 4.4. *Let $T \geq 1$. Given a pair (u_0, u_1) of real-valued functions defined on $\mathbb{T}^d$, let $(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})$ be the randomization on $\mathbb{R}^d$ defined in* (1.10)*, satisfying* (1.4)*. Let $S^*(t) = S(t)$ or $\widetilde{S}(t)$ defined in* (2.2) *and* (4.11)*, respectively. Then, for $2 \leq r \leq \infty$, we have*

$$P\Big(\|S^*(t)(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})\|_{L^\infty_t([0,T];L^r_x(\mathbb{R}^d))} > \lambda\Big) \leq C\langle T\rangle \exp\bigg(-c\frac{\lambda^2}{\langle T\rangle^{d+2}\|(u_0,u_1)\|^2_{\mathcal{H}^\varepsilon(\mathbb{T}^d)}}\bigg) \tag{4.12}$$

for any $\varepsilon > 0$, where the constants C and c depend only on r and ε.

Proposition 4.4 follows as a corollary to the following lemma. Let $S_+(t)$ and $S_-(t)$ be the linear propagators for the half wave equations on $\mathbb{R}^d$ defined by

$$S_\pm(t)f := \mathcal{F}^{-1}_{\mathbb{R}^d}\big(e^{\pm i|\xi|t}\widetilde{f}(\xi)\big).$$

Given $\phi \in H^s(\mathbb{T}^d)$ and $T \geq 1$, we define its randomization $\boldsymbol{\phi}^\omega_T$ on $\mathbb{R}^d$ by

$$\boldsymbol{\phi}^\omega_T := \sum_{n\in\mathbb{Z}^d} \eta_T(x) g_{n,0}(\omega)\widehat{\phi}(n)e^{inx},$$

as in the first component of (1.10). Then, we have the following tail estimate on the size of $S_\pm(t)\boldsymbol{\phi}^\omega_T$ over a time interval of length 1.

LEMMA 4.5. *Let $T \geq 1$ and $2 \leq r \leq \infty$. Given any $\varepsilon > 0$, there exist constants $C, c > 0$, depending only on r and ε, such that*

$$P\Big(\|S_\pm(t)\boldsymbol{\phi}^\omega_T\|_{L^\infty_t([j,j+1];L^r_x(\mathbb{R}^d))} > \lambda\Big) \leq C\exp\bigg(-c\frac{\lambda^2}{\langle T\rangle^d\|\phi\|^2_{H^\varepsilon(\mathbb{T}^d)}}\bigg), \tag{4.13}$$

$$P\Big(\|\langle\nabla\rangle^{-1}S_\pm(t)\boldsymbol{\phi}^\omega_T\|_{L^\infty_t([j,j+1];L^r_x(\mathbb{R}^d))} > \lambda\Big) \leq C\exp\bigg(-c\frac{\lambda^2}{\langle T\rangle^d\|\phi\|^2_{H^{\varepsilon-1}(\mathbb{T}^d)}}\bigg), \tag{4.14}$$

$$P\bigg(\bigg\|\frac{\sin(t|\nabla|)}{|\nabla|}\boldsymbol{\phi}^\omega_T\bigg\|_{L^\infty_t([j,j+1];L^r_x(\mathbb{R}^d))} > \lambda\bigg) \leq C\exp\bigg(-c\frac{\lambda^2}{\max(1,j^2)\langle T\rangle^d\|\phi\|^2_{H^{\varepsilon-1}(\mathbb{T}^d)}}\bigg) \tag{4.15}$$

for any $[j, j+1] \subset [0, T]$.

Assuming Lemma 4.5, we first present the proof of Proposition 4.4.

PROOF OF PROPOSITION 4.4. We first consider the case $S^*(t) = S(t)$ and $T \geq 1$. By subadditivity, (4.13), and (4.15), we have

$$\begin{aligned} &P\Big(\|S(t)(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})\|_{L^\infty_t([0,T];L^r_x(\mathbb{R}^d))} > \lambda\Big) \\ &\quad \leq P\Big(\max_{j=0,\dots,[T]}\|S(t)(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})\|_{L^\infty_t([j,j+1];L^r_x(\mathbb{R}^d))} > \lambda\Big)\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=0}^{[T]} P\Big(\|S(t)(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})\|_{L^\infty_t([j,j+1];L^r_x(\mathbb{R}^d))} > \lambda\Big)\\
&\le \sum_{j=0}^{[T]} P\bigg(\|\cos(t|\nabla|)\mathbf{u}^\omega_{0,T}\|_{L^\infty_t([j,j+1];L^r_x(\mathbb{R}^d))} > \frac{\lambda}{2}\bigg)\\
&\quad + \sum_{j=0}^{[T]} P\bigg(\bigg\|\frac{\sin(t|\nabla|)}{|\nabla|}\mathbf{u}^\omega_{1,T}\bigg\|_{L^\infty([j,j+1];L^r_x(\mathbb{R}^d))} > \frac{\lambda}{2}\bigg)\\
&\le C\langle T\rangle \exp\bigg(-c\frac{\lambda^2}{\langle T\rangle^{d+2}\|(u_0,u_1)\|^2_{\mathcal{H}^\varepsilon(\mathbb{T}^d)}}\bigg).
\end{aligned}
$$

When $S^*(t) = \widetilde{S}(t)$, (4.12) follows from (4.13) and (4.14). In this case, we obtain $\langle T\rangle^d$ instead of $\langle T\rangle^{d+2}$ on the right-hand side of (4.12). □

Finally, we present the proof of Lemma 4.5.

PROOF OF LEMMA 4.5. We first prove (4.13). In the following, we only consider the case of $S_+(t)$. Set $\mathbf{z}^\omega(t) = \mathbf{z}^\omega_T(t) := S_+(t)\boldsymbol{\phi}^\omega_T$.

Part 1 (a): We first consider the case $r < \infty$. The first half of the reduction (up to (4.18)) is exactly the same as that in the proof of Lemma 3.4 in [23]. We decided to include it for reader's convenience. Without loss of generality, assume $j = 0$. For $k \in \mathbb{N}\cup\{0\}$, let $\{t_{\ell,k} : \ell = 0, 1, \dots, 2^k\}$ be 2^k+1 equally spaced points on $[0, 1]$, i.e. $t_{0,k} = 0$ and $t_{\ell,k} - t_{\ell-1,k} = 2^{-k}$ for $\ell = 1, \dots, 2^k$. Then, given $t \in [0, 1]$, we have

$$
\mathbf{z}^\omega(t) = \sum_{k=1}^\infty \big(\mathbf{z}^\omega(t_{\ell_k,k}) - \mathbf{z}^\omega(t_{\ell_{k-1},k-1})\big) + \mathbf{z}^\omega(0) \tag{4.16}
$$

for some $\ell_k = \ell_k(t) \in \{0, \dots, 2^k\}$.

By the square function estimate and Minkowski's integral inequality with (4.16), we have

$$
\begin{aligned}
&\|\mathbf{z}^\omega\|_{L^\infty_t([0,1];L^r_x(\mathbb{R}^d))}\\
&\quad\lesssim \bigg(\sum_{\substack{N\ge 1\\ \text{dyadic}}} \bigg(\sum_{k=1}^\infty \max_{0\le \ell_k\le 2^k} \big\|\mathbf{P}_N\big(\mathbf{z}^\omega(t_{\ell_k,k}) - \mathbf{z}^\omega(t_{\ell'_{k-1},k-1})\big)\big\|_{L^r_x(\mathbb{R}^d)}\bigg)^2\bigg)^{\frac12} + \|\mathbf{z}^\omega(0)\|_{L^r_x(\mathbb{R}^d)},
\end{aligned}
$$

where $t_{\ell'_{k-1},k-1}$ is one of the $2^{(k-1)}$+1 equally spaced points such that

$$
|t_{\ell_k,k} - t_{\ell'_{k-1},k-1}| \le 2^{-k}. \tag{4.17}
$$

Hence, for $p \ge 2$, we have

$$
\Big(\mathbb{E}\big[\|\mathbf{z}^\omega\|_{L^\infty_t([0,1];L^r_x(\mathbb{R}^d))}\big]^p\Big)^{\frac1p}
$$

$$\lesssim \Bigg(\sum_{\substack{N\geq 1\\ \text{dyadic}}} \bigg(\sum_{k=1}^{\infty} \Big(\mathbb{E}\Big[\max_{0\leq \ell_k\leq 2^k} \big\| \mathbf{P}_N\big(\mathbf{z}^\omega(t_{\ell_k,k}) - \mathbf{z}^\omega_\pm(t_{\ell'_{k-1},k-1})\big)\big\|_{L^r_x(\mathbb{R}^d)}\Big]^p\Big)^{\frac{1}{p}}\bigg)^2\Bigg)^{\frac12}$$

$$+ \Big(\mathbb{E}\big[\|\mathbf{z}^\omega(0)\|_{L^r_x(\mathbb{R}^d)}\big]^p\Big)^{\frac1p}. \tag{4.18}$$

Proceeding as in (4.5) with Lemma 4.3 and (4.3), the second term on the right-hand side of (4.18) can be bounded by

$$\begin{aligned}\Big(\mathbb{E}\big[\|\mathbf{z}^\omega(0)\|_{L^r_x(\mathbb{R}^d)}\big]^p\Big)^{\frac1p} &\leq \Big\| \|\boldsymbol{\phi}^\omega_T\|_{L^p(\Omega)}\Big\|_{L^r_x} \lesssim \sqrt{p}\Big\| \|\eta_T(x)\widehat{\phi}(n)e^{inx}\|_{\ell^2_n}\Big\|_{L^r_x}\\ &\leq \sqrt{p}\Big\| \|\eta_T(x)e^{inx}\|_{L^r_x}\cdot\widehat{\phi}(n)\Big\|_{\ell^2_n}\\ &\lesssim \sqrt{p}\langle T\rangle^{\frac d2}\|\eta\|_{H^{\frac d2}(\mathbb{R}^d)}\|\phi\|_{L^2_x(\mathbb{T}^d)}\end{aligned} \tag{4.19}$$

for $p\geq r\geq 2$.

In the following, we first estimate

$$I_N := \sum_{k=1}^{\infty}\Big(\mathbb{E}\Big[\max_{0\leq\ell_k\leq 2^k}\big\|\mathbf{P}_N\big(\mathbf{z}^\omega(t_{\ell_k,k}) - \mathbf{z}^\omega(t_{\ell'_{k-1},k-1})\big)\big\|_{L^r_x(\mathbb{R}^d)}\Big]^p\Big)^{\frac1p}$$

for each dyadic $N\geq 1$. Let

$$q_k := \max(\log 2^k, p, r) \sim \log 2^k + p + r\,. \tag{4.20}$$

Then, we have

$$I_N \leq \sum_{k=1}^{\infty}\bigg(\sum_{\ell_k=0}^{2^k}\mathbb{E}\big\|\mathbf{P}_N\big(\mathbf{z}^\omega(t_{\ell_k,k}) - \mathbf{z}^\omega(t_{\ell'_{k-1},k-1})\big)\big\|^{q_k}_{L^r_x}\bigg)^{\frac{1}{q_k}}.$$

Noting that $(2^k+1)^{\frac{1}{q_k}}\lesssim 1$ and applying Lemma 4.3,

$$\begin{aligned}&\lesssim \sum_{k=1}^{\infty}\max_{0\leq\ell_k\leq 2^k}\Big(\mathbb{E}\big\|\mathbf{P}_N\big(\mathbf{z}^\omega(t_{\ell_k,k}) - \mathbf{z}^\omega(t_{\ell'_{k-1},k-1})\big)\big\|^{q_k}_{L^r_x}\Big)^{\frac{1}{q_k}}\\ &\lesssim \sum_{k=1}^{\infty}\sqrt{q_k}\max_{0\leq\ell_k\leq 2^k}\Big\|\big\|\mathbf{P}_N\big(S_+(t_{\ell_k,k}) - S_+(t_{\ell'_{k-1},k-1})\big)(\eta_T e^{inx})\big\|_{L^r_x}\cdot\widehat{\phi}(n)\Big\|_{\ell^2_n}.\end{aligned} \tag{4.21}$$

For $|\xi|\sim N$, it follows from (4.17) that

$$\big|e^{i|\xi|t_{\ell_k,k}} - e^{i|\xi|t_{\ell'_{k-1},k-1}}\big| \lesssim \min(1, 2^{-k}N)\,. \tag{4.22}$$

We now proceed as in (4.3). With (4.22) and the triangle inequality, we have

$$\begin{aligned}\big\|\mathbf{P}_N\big(S_+(t_{\ell_k,k}) - S_+(t_{\ell'_{k-1},k-1})\big)(\eta_T e^{inx})\big\|_{L^r_x(\mathbb{R}^d)} &\lesssim \min(1,2^{-k}N)\|\widetilde{\eta}_T(\xi-n)\|_{L^{r'}_{|\xi|\sim N}(\mathbb{R}^d)}\\ &\lesssim \langle n\rangle^{\varepsilon}N^{-\varepsilon}\min(1,2^{-k}N)\|\langle\xi-n\rangle^{\beta+\varepsilon}\widetilde{\eta}_T(\xi-n)\|_{L^2_\xi(\mathbb{R}^d)}\end{aligned}$$

$$\lesssim \langle n\rangle^{\varepsilon} N^{-\varepsilon} \min(1, 2^{-k}N)\|\eta_T\|_{H^{\frac{d}{2}}(\mathbb{R}^d)}$$

$$\lesssim \langle T\rangle^{\frac{d}{2}} \langle n\rangle^{\varepsilon} N^{-\varepsilon} \min(1, 2^{-k}N)\|\eta\|_{H^{\frac{d}{2}}(\mathbb{R}^d)} \tag{4.23}$$

as long as $\varepsilon > 0$ is sufficiently small such that $\beta + \varepsilon = \frac{(r-2)}{2r}d + 2\varepsilon \leq \frac{d}{2}$. Hence, from (4.21) and (4.23), we obtain

$$I_N \lesssim \langle T\rangle^{\frac{d}{2}} \|\eta\|_{H^{\frac{d}{2}}(\mathbb{R}^d)} \sum_{k=1}^{\infty} \sqrt{q_k} N^{-\varepsilon} \min(1, 2^{-k}N)\|\phi\|_{H^{\varepsilon}(\mathbb{T}^d)}. \tag{4.24}$$

Separating the summation (in k) into $2^{-k}N \geq 1$ and $2^{-k}N < 1$ and applying (4.20), we have

$$\sum_{k=1}^{\infty} \sqrt{q_k} N^{-\varepsilon} \min(1, 2^{-k}N) \leq C_{r,\varepsilon}\sqrt{p} N^{-\frac{\varepsilon}{2}}. \tag{4.25}$$

See [23] for details. Finally, putting (4.18), (4.19), (4.24), and (4.25), together, we obtain

$$\Big(\mathbb{E}\big[\|\mathbf{z}^{\omega}\|_{L^{\infty}_t([0,1];L^r_x(\mathbb{R}^d))}\big]^p\Big)^{\frac{1}{p}} \leq C_{r,\varepsilon}\sqrt{p}\langle T\rangle^{\frac{d}{2}}\|\eta\|_{H^{\frac{d}{2}}(\mathbb{R}^d)}\|\phi\|_{H^{\varepsilon}(\mathbb{T}^d)}$$

for all $p \geq r$ and sufficiently small $\varepsilon > 0$. The rest follows from a standard argument using Chebyshev's inequality.

Part 1 (b): Next, we consider the case $r = \infty$. It follows from Sobolev embedding that, given any small $\varepsilon > 0$, there exists large $\tilde{r} \gg 1$ with $\varepsilon\tilde{r} > d$ such that

$$P\Big(\|S_{\pm}(t)\boldsymbol{\phi}^{\omega}_T\|_{L^{\infty}_t([j,j+1];L^{\infty}_x(\mathbb{R}^d))} > \lambda\Big) \leq P\Big(\|\langle\nabla\rangle^{\varepsilon} S_{\pm}(t)\boldsymbol{\phi}^{\omega}_T\|_{L^{\infty}_t([j,j+1];L^{\tilde{r}}_x(\mathbb{R}^d))} > C\lambda\Big).$$

Then, the rest follows from the triangle inequality $\langle\xi\rangle^{\varepsilon} \lesssim \langle n\rangle^{\varepsilon}\langle\xi - n\rangle^{\varepsilon}$ and the argument in Part 1 (a).

Part 2: Next, we consider (4.14). By proceeding as in Part 1 (a), the only essential modifications appear only in (4.19) and (4.23). With (4.22) and the triangle inequality: $\langle\xi\rangle^{-1} \leq \langle n\rangle^{-1}\langle\xi - n\rangle$, we have

$$\begin{aligned}
&\big\|\mathbf{P}_N\langle\nabla\rangle^{-1}\big(S_+(t_{\ell_k,k}) - S_+(t_{\ell'_{k-1},k-1})\big)(\eta_T e^{inx})\big\|_{L^r_x(\mathbb{R}^d)}\\
&\quad\lesssim \langle n\rangle^{-1}\min(1, 2^{-k}N)\|\langle\xi - n\rangle\widetilde{\eta}_T(\xi - n)\|_{L^{r'}_{|\xi|\sim N}(\mathbb{R}^d)}\\
&\quad\lesssim \langle n\rangle^{\varepsilon-1}N^{-\varepsilon}\min(1, 2^{-k}N)\|\langle\xi - n\rangle^{\beta+\varepsilon+1}\widetilde{\eta}_T(\xi - n)\|_{L^2_{\xi}(\mathbb{R}^d)}\\
&\quad\lesssim \langle T\rangle^{\frac{d}{2}}\langle n\rangle^{\varepsilon-1}N^{-\varepsilon}\min(1, 2^{-k}N)\|\eta\|_{H^{\frac{d}{2}+1}(\mathbb{R}^d)}.
\end{aligned}$$

This shows how one modifies (4.23), while (4.19) can be modified similarly. Then, the rest follows as in Part 1 (a), yielding (4.14).

Part 3: Finally, we prove (4.15) when $r < \infty$. The modification needed for the case $r = \infty$ is straightforward as in Part 1 (b). Define $\mathbf{Z}^{\omega}(t)$ by

$$\mathbf{Z}^{\omega}(t) := \frac{\sin(t|\nabla|)}{|\nabla|}\boldsymbol{\phi}^{\omega}_T.$$

Repeating the argument in Part 1 (a) (but on $[j, j+1]$ instead of $[0, 1]$), we have

$$\begin{aligned}
&\Big(\mathbb{E}\big[\|\mathbf{Z}^\omega\|_{L_t^\infty([j,j+1];L_x^r(\mathbb{R}^d))}\big]^p\Big)^{\frac{1}{p}} \\
&\lesssim \bigg(\sum_{\substack{N\geq 1\\ \text{dyadic}}}\bigg(\sum_{k=1}^\infty\Big(\mathbb{E}\Big[\max_{0\leq \ell_k\leq 2^k}\big\|\mathbf{P}_N\big(\mathbf{Z}^\omega(t_{\ell_k,k})-\mathbf{Z}^\omega(t_{\ell'_{k-1},k-1})\big)\big\|_{L_x^r(\mathbb{R}^d)}\Big]^p\Big)^{\frac{1}{p}}\bigg)^2\bigg)^{\frac{1}{2}} \\
&\quad + \Big(\mathbb{E}\big[\|\mathbf{Z}^\omega(j)\|_{L_x^r(\mathbb{R}^d)}\big]^p\Big)^{\frac{1}{p}} =: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

When $j = 0$, then we have $\mathrm{II} = 0$. When $j \geq 1$, proceeding as in (4.8), we have

$$\mathrm{II} \lesssim \sqrt{p}\cdot j\langle T\rangle^{\frac{d}{2}}\|\eta\|_{H^{\frac{d}{2}+1}(\mathbb{R}^d)}\|\phi\|_{H^{-1}(\mathbb{T}^d)} \tag{4.26}$$

for $p \geq r$. As for I, we simply repeat the computations in Part 1 (a) with a modification in (4.23). For non-zero $|\xi| \sim N$, it follows from Mean Value Theorem with (4.17) and the triangle inequality that

$$\bigg|\frac{\sin(t_{\ell_k,k}|\xi|)-\sin(t_{\ell'_{k-1},k-1}|\xi|)}{|\xi|}\bigg| \lesssim \min(1, 2^{-k}N)\frac{\langle \xi - n\rangle}{\langle n\rangle}. \tag{4.27}$$

Proceeding as in Part 1 with (4.27), we obtain

$$\mathrm{I} \leq C_{r,\varepsilon}\sqrt{p}\langle T\rangle^{\frac{d}{2}}\|\eta\|_{H^{\frac{d}{2}+1}(\mathbb{R}^d)}\|\phi\|_{H^{\varepsilon-1}(\mathbb{T}^d)}. \tag{4.28}$$

Then, the desired estimate (4.15) follows from (4.26) and (4.28). □

5. Proof of Proposition 3.2.

In this section, we present the proof of Proposition 3.2. In Subsection 5.1, we treat the higher dimensional case $d = 4, 5$. Then, we briefly discuss some components of the proof for the $d = 3$ case in Subsection 5.2.

5.1. Higher dimensional case.

In this subsection, we consider the case $d = 4, 5$. In this case, the following probabilistic a priori energy bound plays an essential role, replacing Proposition 5.2 in [25].

LEMMA 5.1 (Probabilistic energy bound). *Let $d = 4$ or 5 and $s < 1$ satisfy the condition in Theorem 1.1. Given $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^d)$ and $T \geq 1$, let $(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})$ be the randomization on $\mathbb{R}^d$ defined in* (1.10)*, satisfying* (1.4)*. Suppose that $\mathbf{v}^\omega$ is a solution to the Cauchy problem* (3.5) *on $[0, T]$. Then, given small $\varepsilon > 0$, there exists a set $\widetilde{\Omega}_{T,\varepsilon} \subset \Omega$ with $P(\widetilde{\Omega}^c_{T,\varepsilon}) < \frac{\varepsilon}{2}$, such that for all $\omega \in \widetilde{\Omega}_{T,\varepsilon}$, we have*

$$\sup_{t\in[0,T]} E(\mathbf{v}^\omega(t)) \leq C\Big(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{T}^d)}\Big), \tag{5.1}$$

and thus also

$$\big\|(\mathbf{v}^\omega, \partial_t\mathbf{v}^\omega)\big\|_{L_t^\infty([0,T];\mathcal{H}^1(\mathbb{R}^d))} \leq C_0\Big(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{T}^d)}\Big).$$

PROOF. The proof of this lemma follows closely the proof of Proposition 5.2 in [25] and thus we only sketch the proof of (5.1) when $d = 4$.

Taking the time derivative of the energy $E(v^\omega(t))$ with (3.5) and integrating by parts, we have

$$\frac{d}{dt}E(\mathbf{v}^\omega(t)) = \int_{\mathbb{R}^4} \partial_t \mathbf{v}^\omega \Big(\partial_t^2 \mathbf{v}^\omega - \Delta \mathbf{v}^\omega + (\mathbf{v}^\omega)^3\Big) dx = \int_{\mathbb{R}^4} \partial_t \mathbf{v}^\omega \Big((\mathbf{v}^\omega)^3 - (\mathbf{v}^\omega + \mathbf{z}^\omega)^3\Big) dx\,.$$

By Hölder's inequality, we have

$$\left|\frac{d}{dt}E(\mathbf{v}^\omega(t))\right| \le C\Big(E(\mathbf{v}^\omega(t))\Big)^{\frac{1}{2}}\Big(\|\mathbf{z}^\omega\|^3_{L^6_x(\mathbb{R}^4)} + \|\mathbf{z}^\omega\|_{L^\infty_x(\mathbb{R}^4)}\|\mathbf{v}^\omega\|^2_{L^4_x(\mathbb{R}^4)}\Big).$$

Noting that $E(v^\omega(0)) = 0$, integration in time then yields

$$\Big(E(\mathbf{v}^\omega(t))\Big)^{\frac{1}{2}} \le C\|\mathbf{z}^\omega\|^3_{L^3_T L^6_x} + C\int_0^t \|\mathbf{z}^\omega(t')\|_{L^\infty_x}\Big(E(\mathbf{v}^\omega(t'))\Big)^{\frac{1}{2}} dt'.$$

By Gronwall's inequality, we obtain

$$\sup_{t\in[0,T]}\Big(E(\mathbf{v}^\omega(t))\Big)^{\frac{1}{2}} \le C\|\mathbf{z}^\omega\|^3_{L^3_T L^6_x} e^{C\|\mathbf{z}^\omega\|_{L^1_T L^\infty_x}}. \tag{5.2}$$

Then, by choosing $\lambda = K\langle T\rangle^4\|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{T}^4)}$ and $K = K(\varepsilon) \gg 1$, it follows from Proposition 4.1 that there exists $\widetilde{\Omega}_{T,\varepsilon} \subset \Omega$ with $P(\widetilde{\Omega}^c_{T,\varepsilon}) < \frac{\varepsilon}{2}$ such that for all $\omega \in \widetilde{\Omega}_{T,\varepsilon}$, we have

$$\|\mathbf{z}^\omega\|_{L^3_T L^6_x} + \|\mathbf{z}^\omega\|_{L^1_T L^\infty_x} \le K\langle T\rangle^4\|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{T}^d)}\,. \tag{5.3}$$

Combining this with (5.2) yields (5.1). □

The key deterministic ingredient in the proof of Proposition 3.2 is the following "good" local well-posedness result of the perturbed NLW (3.6). In particular, the time of local existence is characterized only in terms of the $\dot{\mathcal{H}}^1$-norm of the initial data (v_0, v_1) and the size of the perturbation f.

LEMMA 5.2 (Proposition 4.3 in [25]). *Let $d = 4$ or 5 and $(\mathbf{v}_0,\mathbf{v}_1) \in \dot{\mathcal{H}}^1(\mathbb{R}^d)$. Then, there exists a function $\tau : [0,\infty)\times\mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}_+$, non-increasing in the first two arguments, such that if f satisfies the condition*

$$\|f\|_{L_t^{\frac{d+2}{d-2}} L_x^{\frac{2(d+2)}{d-2}}([t_0,t_0+\tau_*]\times\mathbb{R}^d)} \le K\tau_*^\theta \tag{5.4}$$

for some $K,\theta > 0$ and $\tau_* \le \tau = \tau\big(\|(\mathbf{v}_0,\mathbf{v}_1)\|_{\dot{\mathcal{H}}^1(\mathbb{R}^d)}, K, \theta\big) \ll 1$, then there exists a unique solution $(\mathbf{v},\partial_t\mathbf{v}) \in C([t_0,t_0+\tau_*];\dot{\mathcal{H}}^1(\mathbb{R}^d))$ to (3.6). Moreover,

$$\|\mathbf{v}\|_{L^q_t([t_0,t_0+t_*];L^r_x(\mathbb{R}^d))} \le C(\|(\mathbf{v}_0,\mathbf{v}_1)\|_{\dot{\mathcal{H}}^1(\mathbb{R}^d)})\,, \tag{5.5}$$

for all 1-admissible pairs (q,r).

Now, we are ready to present the proof of Proposition 3.2 for $d = 4, 5$.

PROOF OF PROPOSITION 3.2. Let $T \geq 1$ and $\varepsilon > 0$. Given $(\mathbf{u}^\omega_{0,T}, \mathbf{u}^\omega_{1,T})$, let $\mathbf{z}^\omega$ and $\mathbf{v}^\omega$ be as in (3.4). By Lemma 5.1, there exists a set Ω_1 with

$$P(\Omega_1^c) < \frac{\varepsilon}{2} \tag{5.6}$$

such that

$$\sup_{t\in[0,T]} \|(\mathbf{v}^\omega(t), \partial_t \mathbf{v}^\omega(t))\|_{\mathcal{H}^1(\mathbb{R}^d)} \leq C_0 := C_0(T, \varepsilon, \|(u_0, u_1)\|_{\mathcal{H}^s(\mathbb{T}^d)}) < \infty\,, \tag{5.7}$$

for each $\omega \in \Omega_1$.

Let $\tau = \tau\big(C_0, K, \theta\big)$ be as in Lemma 5.2, where $K = \|(u_0, u_1)\|_{\mathcal{H}^0(\mathbb{T}^d)}$ and $\theta = \frac{d-2}{2(d+2)}$. Fix $\tau_* \leq \tau$ to be chosen later. By writing $[0, T] = \bigcup_{j=0}^{[T/\tau_*]} I_j$ with $I_j = [j\tau_*, (j+1)\tau_*] \cap [0, T]$, define Ω_2 by

$$\Omega_2 := \bigg\{\omega \in \Omega : \|\mathbf{z}^\omega\|_{L^{\frac{d+2}{d-2}}_{I_j} L_x^{\frac{2(d+2)}{d-2}}} \leq K|I_j|^\theta,\ j = 0, \dots, \big[\tfrac{T}{\tau_*}\big]\bigg\}. \tag{5.8}$$

Then, by Proposition 4.1 with $|I_j| \leq \tau_*$, we have

$$P(\Omega_2^c) \leq \sum_{j=0}^{[\frac{T}{\tau_*}]} P\Big(\|\mathbf{z}^\omega\|_{L^{\frac{d+2}{d-2}}_{I_j} L_x^{\frac{2(d+2)}{d-2}}} > K|I_j|^\theta\Big) \lesssim \frac{T}{\tau_*} \exp\bigg(-\frac{c}{\langle T\rangle^{d+2}\tau_*^{2\theta}}\bigg).$$

By making τ_* smaller if necessary,

$$\lesssim \frac{T}{\tau_*}\tau_* \exp\bigg(-\frac{c}{2\langle T\rangle^{d+2}\tau_*^{2\theta}}\bigg) = T \exp\bigg(-\frac{c}{2\langle T\rangle^{d+2}\tau_*^{2\theta}}\bigg).$$

Hence, by choosing $\tau_* = \tau_*(T, \varepsilon)$ sufficiently small, we conclude that

$$P(\Omega_2^c) < \frac{\varepsilon}{2}. \tag{5.9}$$

Let $\widetilde{\Omega}_{T,\varepsilon} := \Omega_1 \cap \Omega_2$. Then, from (5.6) and (5.9), we have $P(\widetilde{\Omega}^c_{T,\varepsilon}) < \varepsilon$. Moreover, it follows from Lemma 5.2 applied iteratively with (5.7) and (5.8) on the intervals I_j, $j = 0, \dots, [\frac{T}{\tau_*}]$, that for each $\omega \in \widetilde{\Omega}_{T,\varepsilon}$, there exists a unique solution $\mathbf{v}^\omega$ to (3.5) on $[0, T]$. Hence, for $\omega \in \widetilde{\Omega}_{T,\varepsilon}$, there exists a unique solution $\mathbf{u}^\omega = \mathbf{z}^\omega + \mathbf{v}^\omega$ to (3.2) on $[0, T]$. Moreover, (3.3) follows from (5.5). □

5.2. Three-dimensional case. In the following, we briefly sketch the idea of the proof of Proposition 3.2 when $d = 3$. In this case, the additional difficulty comes from the lack of a probabilistic a priori energy bound (Lemma 5.1). Therefore, as in [23], we need to establish a uniform probabilistic energy bound for approximating random solutions.

Let $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$ with $\frac{1}{2} < s < 1$ and $T \geq 1$. Given $N \geq 1$ dyadic, define $\mathbf{u}^\omega_{j,T,N}$, $j = 0, 1$, by

$$\mathbf{u}^\omega_{j,T,N} := \mathbf{P}_{\leq N}\mathbf{u}^\omega_{j,T}\,. \tag{5.10}$$

Let $\mathbf{u}_N$ be the smooth global solution to (3.2) on $\mathbb{R}^3$ with initial data $(\mathbf{u}_N, \partial_t\mathbf{u}_N)|_{t=0} = (\mathbf{u}^\omega_{0,T,N}, \mathbf{u}^\omega_{1,T,N}) \in \mathcal{H}^\infty(\mathbb{R}^3)$. Denote the linear and nonlinear parts of $\mathbf{u}_N$ by $\mathbf{z}_N = \mathbf{z}^\omega_N$ and

$\mathbf{v}_N = \mathbf{v}_N^\omega$, respectively. In particular, $\mathbf{v}_N$ is the smooth global solution to the following perturbed NLW on $\mathbb{R}^3$:

$$\begin{cases} \partial_t^2 \mathbf{v}_N - \Delta \mathbf{v}_N + (\mathbf{v}_N + \mathbf{z}_N)^5 = 0\,, \\ (\mathbf{v}_N, \partial_t \mathbf{v}_N)|_{t=0} = (0,0)\,. \end{cases} \tag{5.11}$$

While we have $\|(\mathbf{v}_N^\omega, \partial_t \mathbf{v}_N^\omega)\|_{L_t^\infty(\mathbb{R};\dot{\mathcal{H}}^1(\mathbb{R}^3))} \le C(N,\omega) < \infty$ for each $N \in \mathbb{N}$, there is no uniform (in N) control on the H^1-norm of $\mathbf{v}_N$. The following lemma establishes a uniform (in N) bound on the H^1-norm of $\mathbf{v}_N$ in a probabilistic manner.

LEMMA 5.3. *Let $s \in (\frac{1}{2}, 1)$ and $N \ge 1$ dyadic. Given $T, \varepsilon > 0$, there exists $\widetilde{\Omega}_{N,T,\varepsilon} \subset \Omega$ such that*

(i) *$P(\widetilde{\Omega}^c_{N,T,\varepsilon}) < \varepsilon$.*
(ii) *There exists a finite constant $C(T, \varepsilon, \|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{T}^3)}) > 0$ such that the following energy bound holds:*

$$\sup_{t\in[0,T]} \|(\mathbf{v}_N^\omega(t), \partial_t \mathbf{v}_N^\omega(t))\|_{\mathcal{H}^1(\mathbb{R}^3)} \le C(T, \varepsilon, \|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{T}^3)})\,, \tag{5.12}$$

for all solutions $\mathbf{v}_N^\omega$ to (5.11) *on* $[0,T]$ *with $\omega \in \widetilde{\Omega}_{N,T,\varepsilon}$.*

Note that the constant $C(T, \varepsilon, \|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{R}^3)})$ is independent of dyadic $N \ge 1$.

Lemma 5.3 plays the role of Proposition 4.1 in [23] and is a suitable substitute of the probabilistic a priori energy estimate (Lemma 5.1) when $d = 3$. One can prove Lemma 5.3 exactly in the same manner as Proposition 4.1 in [23], by simply replacing the probabilistic Strichartz estimates on $\mathbb{R}^d$ (Lemma 3.2 and Proposition 3.3 in [23]) with the appropriate probabilistic Strichartz estimates for our problem (Propositions 4.1 and 4.4 above). Therefore, we omit details.

The following lemma is the key deterministic ingredient in this case. Given $f \in L^5_{t,\mathrm{loc}} L^{10}_x$, let $f_N = \mathbf{P}_{\le N} f$ for dyadic $N \ge 1$. Consider the following perturbed NLW:

$$\begin{cases} \partial_t^2 \mathbf{v}_N - \Delta \mathbf{v}_N + (\mathbf{v}_N + f_N)^5 = 0 \\ (\mathbf{v}_N, \partial_t \mathbf{v}_N)|_{t=0} = (0,0)\,. \end{cases} \tag{5.13}$$

LEMMA 5.4 (Proposition 5.2 in [23]). *Let f, f_N, and $\mathbf{v}_N$ be as above. Given finite $T > 0$, assume that the following conditions hold:*

(i) *There exist $K, \theta > 0$ such that*

$$\|f\|_{L_t^5 L_x^{10}(I\times\mathbb{R}^3)} \le K|I|^\theta$$

for any compact interval $I \subset [0,T]$.

(ii) *For each dyadic $N \ge 1$, a solution $\mathbf{v}_N$ to* (5.13) *exists on* $[0,T]$ *and satisfies the following uniform a priori energy bound:*

$$\sup_N \sup_{t\in[0,T]} \|(\mathbf{v}_N(t), \partial_t \mathbf{v}_N(t))\|_{\mathcal{H}^1(\mathbb{R}^3)} < C_0(T) < \infty\,.$$

(iii) *There exists* $\alpha > 0$ *such that*

$$\|f - f_N\|_{L^5_T L^{10}_x} < C_1(T) N^{-\alpha}$$

for all dyadic $N \geq 1$.

Then, there exists a unique solution $(\mathbf{v}, \partial_t \mathbf{v}) \in C([0, T]; \mathcal{H}^1(\mathbb{R}^3))$ *to* (3.6) *with* $(\mathbf{v}, \partial_t \mathbf{v})|_{t=0} = (0, 0)$*, satisfying*

$$\sup_{t \in [0,T]} \|(\mathbf{v}(t), \partial_t \mathbf{v}(t))\|_{\mathcal{H}^1(\mathbb{R}^3)} < 2C_0(T) < \infty .$$

Finally, with Proposition 4.1, Lemmas 5.3 and 5.4, one can prove Proposition 3.2, following the proof of Proposition 6.1 in [23]. Since the argument is identical, we omit details.

Appendix A. On the finite speed of propagation. In this appendix, we discuss the issues related to the finite speed of propagation. In particular, we provide details of the reduction from Proposition 3.2 on $\mathbb{R}^d$ to Proposition 3.1 on $\mathbb{T}^d$. For simplicity of the presentation, we only consider the case $d = 4$.

In the following, fix $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^4)$ with $0 < s < 1$, $T \geq 1$, and $\varepsilon > 0$. By Proposition 3.2, there exists $\widetilde{\Omega}_{T, \frac{1}{2}\varepsilon}$ with $P(\widetilde{\Omega}^c_{T, \frac{1}{2}\varepsilon}) < \frac{1}{2}\varepsilon$ and, for each $\omega \in \widetilde{\Omega}_{T, \frac{1}{2}\varepsilon}$, there exists a unique solution $\mathbf{v}^\omega$ to (3.5) on $[0, T]$, satisfying the energy bound (5.7).

Given $N \in \mathbb{N}$, define periodic functions $u^\omega_{j,N}$ on $\mathbb{T}^d$, $j = 0, 1$, by

$$u^\omega_{j,N} := \mathbf{P}_{\leq N} u^\omega_j = \sum_{|n| \leq N} g_{n,j}(\omega) \widehat{u}_j(n) e^{inx}$$

and set $(\mathbf{u}^\omega_{0,N,T}, \mathbf{u}^\omega_{1,N,T}) = (\eta_T u^\omega_{0,N}, \eta_T u^\omega_{1,N})$. Note that $\mathbf{u}^\omega_{j,N,T}$ is different from $\mathbf{u}^\omega_{j,T,N}$ defined in (5.10). It follows from an analogue of Lemma 3.3 that $(\mathbf{u}^\omega_{0,N,T}, \mathbf{u}^\omega_{1,N,T}) \in \mathcal{H}^\infty(\mathbb{R}^4)$ almost surely. Therefore, there exists a unique (smooth) global solution $\mathbf{u}^\omega_N$ to the following Cauchy problem on $\mathbb{R}^4$:

$$\begin{cases} \partial_t^2 \mathbf{u}^\omega_N - \Delta \mathbf{u}^\omega_N + (\mathbf{u}^\omega_N)^3 = 0 \\ (\mathbf{u}^\omega_N, \partial_t \mathbf{u}^\omega_N)\big|_{t=0} = (\mathbf{u}^\omega_{0,N,T}, \mathbf{u}^\omega_{1,N,T}) . \end{cases}$$

By the finite speed of propagation (for smooth solutions), $u^\omega_N := \mathbf{u}^\omega_N|_{[0,T]\times\mathbb{T}^4}$ is a solution to the periodic NLW (1.1) on the time interval $[0, T]$ with initial data $(u^\omega_{0,N}, u^\omega_{1,N})$.

Denote the linear and nonlinear parts of $\mathbf{u}_N$ by

$$\mathbf{z}_N = \mathbf{z}^\omega_N := S(t)(\mathbf{u}^\omega_{0,N,T}, \mathbf{u}^\omega_{1,N,T}) \quad \text{and} \quad \mathbf{v}_N := \mathbf{u}^\omega_N - \mathbf{z}^\omega_N .$$

Then, $\mathbf{v}_N$ is the smooth global solution to the following perturbed NLW on $\mathbb{R}^4$:

$$\begin{cases} \partial_t^2 \mathbf{v}_N - \Delta \mathbf{v}_N + (\mathbf{v}_N + \mathbf{z}_N)^3 = 0 \\ (\mathbf{v}_N, \partial_t \mathbf{v}_N)|_{t=0} = (0, 0) . \end{cases}$$

Also, define $z^\omega_{\mathrm{per},N}$ and z^ω_{per} by

$$z^\omega_{\mathrm{per},N} := S_{\mathrm{per}}(t)(u^\omega_{0,N}, u^\omega_{1,N}) \quad \text{and} \quad z^\omega_{\mathrm{per}} := S_{\mathrm{per}}(t)(u^\omega_0, u^\omega_1) .$$

Note that, by the finite speed of propagation for the linear solutions, we have

$$\mathbf{z}_N^\omega|_{[0,T]\times\mathbb{T}^4} = z_{\mathrm{per},N}^\omega \quad \text{and} \quad \mathbf{z}^\omega|_{[0,T]\times\mathbb{T}^4} = z_{\mathrm{per}}^\omega\,, \tag{A.1}$$

where $\mathbf{z}^\omega$ is as in (3.4). In particular, $v_N := \mathbf{v}_N|_{[0,T]\times\mathbb{T}^4}$ is the smooth global solution to the following perturbed NLW on $\mathbb{T}^4$:

$$\begin{cases} \partial_t^2 v_N - \Delta v_N + (v_N + z_{\mathrm{per},N})^3 = 0\,, \\ (v_N, \partial_t v_N)|_{t=0} = (0,0)\,. \end{cases} \tag{A.2}$$

By Proposition 4.1, we have the following probabilistic estimate on $\mathbf{z}^\omega - \mathbf{z}_N^\omega$.

LEMMA A.1. *Let $T > 0$ and $N \in \mathbb{N}$. Given $1 \le q < \infty$, $2 \le r \le \infty$, there exist $C, c > 0$ such that*

$$P\Big(\|\mathbf{z}^\omega - \mathbf{z}_N^\omega\|_{L_t^q([0,T];L_x^r(\mathbb{R}^d))} > \lambda \Big) \le C \exp\bigg(- c \frac{\lambda^2}{\langle T\rangle^{d+2+\frac{2}{q}} \|\mathbf{P}_{>N}(u_0,u_1)\|^2_{\mathcal{H}^s(\mathbb{T}^d)}} \bigg),$$

provided (i) $s = 0$ *if* $r < \infty$ *and* (i) $s > 0$ *if* $r = \infty$.

Noting that $\|\mathbf{P}_{>N}(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{T}^4)} \to 0$ as $N \to \infty$, given $k \in \mathbb{N}$, there exists $N_k \in \mathbb{N}$ and $\widetilde{\Omega}_k \subset \Omega$ with $P(\widetilde{\Omega}_k^c) < \frac{\varepsilon}{2^{k+1}}$ such that for all $\omega \in \widetilde{\Omega}_k$, we have

$$\sup_{(q,r)\in\mathcal{A}} \|\mathbf{z}^\omega - \mathbf{z}_{N_k}^\omega\|_{L_T^q L_x^r} \le \frac{1}{k}\,, \tag{A.3}$$

where $\mathcal{A} = \{(3,6),(1,\infty)\}$. Now, define $\Omega_{T,\varepsilon}$ by

$$\Omega_{T,\varepsilon} = \widetilde{\Omega}_{T,\frac{1}{2}\varepsilon} \cap \bigg(\bigcap_{k=1}^\infty \widetilde{\Omega}_k \bigg).$$

Then, we have $P(\Omega_{T,\varepsilon}^c) < \varepsilon$. Recall that $\Omega_{T,\varepsilon} \subset \widetilde{\Omega}_{T,\frac{\varepsilon}{2}} \subset \Omega_1$, where Ω_1 was defined in the proof of Proposition 3.2 in Subsection 5.1 such that (5.3) and (5.7) hold for all $\omega \in \Omega_1$. Then, by repeating the proof of Lemma 5.1 with (A.3), there exists $k_0 \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \|(\mathbf{v}_{N_k}^\omega(t), \partial_t \mathbf{v}_{N_k}^\omega(t))\|_{\mathcal{H}^1(\mathbb{R}^4)} \le 2C_0(T,\varepsilon,\|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{T}^4)}) < \infty\,,$$

for all $\omega \in \Omega_{T,\varepsilon}$ and all $k \ge k_0$. Moreover, it follows from (3.3), (A.1), and the fact that $\Omega_{T,\varepsilon} \subset \widetilde{\Omega}_{T,\frac{\varepsilon}{2}} \subset \Omega_2$ with Ω_2 defined in (5.8) that there exists $k_1 \in \mathbb{N}$ such that

$$\|\mathbf{v}^\omega\|_{L_T^3 L_x^6(\mathbb{R}^4)}, \|\mathbf{v}_{N_k}^\omega\|_{L_T^3 L_x^6(\mathbb{R}^4)} \le C_1(T,\varepsilon,\|(u_0,u_1)\|_{\mathcal{H}^s(\mathbb{T}^4)}) < \infty\,, \tag{A.4}$$

for all $\omega \in \Omega_{T,\varepsilon}$ and all $k \ge k_1$.

In the following, we fix $\omega \in \Omega_{T,\varepsilon}$. Given an interval I, let $X(I) = \{(\mathbf{w}, \partial_t \mathbf{w}) : (\mathbf{w}, \partial_t \mathbf{w}) \in C_I \dot{\mathcal{H}}_x^1(\mathbb{R}^4),\ \mathbf{w} \in L_I^3 L_x^6(\mathbb{R}^4)\}$. By Monotone Convergence Theorem with (A.4), we can further subdivide the intervals I_j in (5.8) and relabel them such that

$$\|\mathbf{v}^\omega\|_{L_{I_j}^3 L_x^6} \le \gamma \tag{A.5}$$

for some sufficiently small $\gamma > 0$, where $[0, T] = \bigcup_{j=0}^{J} I_j$ with $I_j = [t_j, t_{j+1}]$, $t_0 = 0 < t_1 < \cdots < t_J = T$, and $J < \infty$. Moreover, it follows from (5.8) and (A.3) that there exists $k_2 \in \mathbb{N}$ such that

$$\|\mathbf{z}^\omega\|_{L^3_{I_j} L^6_x}, \|\mathbf{z}^\omega_{N_k}\|_{L^3_{I_j} L^6_x} \le \gamma \ll 1\,, \tag{A.6}$$

for all $k \ge k_2$.

Let $k \ge \max(k_0, k_1, k_2)$. By Monotone Convergence Theorem with (A.4), we have

$$\|\mathbf{v}^\omega_{N_k}\|_{L^3_t([0,\delta]; L^6_x)} \le 4\gamma \ll 1 \tag{A.7}$$

for some small $\delta = \delta(k, \omega) > 0$ with $[0, \delta] \subset I_0$. Then, by Lemma 2.1 with (A.5), (A.6), and (A.7), we have

$$\|\mathbf{v}^\omega - \mathbf{v}^\omega_{N_k}\|_{X([0,\delta])} \le \frac{1}{2}\|\mathbf{v}^\omega - \mathbf{v}^\omega_{N_k}\|_{L^3_t([0,\delta]; L^6_x)} + \frac{1}{2}\|\mathbf{z}^\omega - \mathbf{z}^\omega_{N_k}\|_{L^3_t([0,\delta]; L^6_x)}\,. \tag{A.8}$$

It follows from (A.3), (A.5), and (A.8) that there exists $K_0 \ge \max(k_0, k_1, k_2)$ such that

$$\|\mathbf{v}^\omega_{N_k}\|_{L^3_t([0,\delta]; L^6_x)} \le 2\gamma$$

for all $k \ge K_0$. Then, a continuity argument with (A.3), (A.5), (A.6), and (A.8) yields

$$\|\mathbf{v}^\omega_{N_k}\|_{L^3_{I_0} L^6_x} \le 2\gamma \quad \text{and} \quad \|\mathbf{v}^\omega - \mathbf{v}^\omega_{N_k}\|_{X(I_0)} \le \|\mathbf{z}^\omega - \mathbf{z}^\omega_{N_k}\|_{L^3_{I_0} L^6_x} \tag{A.9}$$

for all $k \ge K_0$.

Once again, by Monotone Convergence Theorem with (A.4), we have

$$\|\mathbf{v}^\omega_{N_k}\|_{L^3_t([t_1, t_1+\delta]; L^6_x)} \le 4\gamma \ll 1 \tag{A.10}$$

for some small $\delta = \delta(k, \omega) > 0$ with $[t_1, t_1 + \delta] \subset I_1$. By Lemma 2.1 with (A.5), (A.6), and (A.10), we have

$$\begin{aligned} \|\mathbf{v}^\omega - \mathbf{v}^\omega_{N_k}\|_{X([t_1, t_1+\delta])} &\le C\|\mathbf{v}^\omega(t_1) - \mathbf{v}^\omega_{N_k}(t_1)\|_{\dot{\mathcal{H}}^1} + \frac{1}{2}\|\mathbf{v}^\omega - \mathbf{v}^\omega_{N_k}\|_{L^3_t([t_1, t_1+\delta]; L^6_x)} \\ &\quad + \frac{1}{2}\|\mathbf{z}^\omega - \mathbf{z}^\omega_{N_k}\|_{L^3_t([t_1, t_1+\delta]; L^6_x)}\,. \end{aligned} \tag{A.11}$$

Hence, by (A.9) and (A.11), we have

$$\|\mathbf{v}^\omega - \mathbf{v}^\omega_{N_k}\|_{X([t_1, t_1+\delta])} \le 2C\|\mathbf{z}^\omega - \mathbf{z}^\omega_{N_k}\|_{L^3_{I_0} L^6_x} + \|\mathbf{z}^\omega - \mathbf{z}^\omega_{N_k}\|_{L^3_t([t_1, t_1+\delta]; L^6_x)}\,. \tag{A.12}$$

Applying the continuity argument again with (A.3), (A.5), (A.6), and (A.12), it follows that there exists $K_1 \ge K_0$ such that

$$\|\mathbf{v}^\omega_{N_k}\|_{L^3_{I_1} L^6_x} \le 2\gamma \quad \text{and} \quad \|\mathbf{v}^\omega - \mathbf{v}^\omega_{N_k}\|_{X(I_1)} \le (2C+1)\|\mathbf{z}^\omega - \mathbf{z}^\omega_{N_k}\|_{L^3_T L^6_x}$$

for all $k \ge K_1$.

By arguing inductively, we conclude that there exists $K_J \in \mathbb{N}$ such that

$$\|\mathbf{v}^\omega - \mathbf{v}^\omega_{N_k}\|_{X([0,T])} \le C_T\|\mathbf{z}^\omega - \mathbf{z}^\omega_{N_k}\|_{L^3_T L^6_x} < \frac{C_T}{k}$$

for all $k \geq K_J$. In particular, $v^\omega_{N_k} = \mathbf{v}^\omega_{N_k}|_{[0,T]\times\mathbb{T}^4}$ converges to $v^\omega := \mathbf{v}^\omega|_{[0,T]\times\mathbb{T}^4}$ in

$$L^3_t([0,T];\, L^6_x(\mathbb{T}^4)) \cap C([0,T];\, \dot{H}^1_x(\mathbb{T}^4))\,. \tag{A.13}$$

Moreover, $\partial_t v^\omega_{N_k}$ converges to $\partial_t v^\omega$ in $C([0,T];\, L^2_x(\mathbb{T}^4))$. It follows from (A.1), (A.13), and the fact that $\mathbf{v}^\omega$ satisfies (3.5) on $[0,T]\times\mathbb{R}^4$ that v^ω is a distributional solution to the following perturbed NLW on $\mathbb{T}^4$:

$$\begin{cases} \partial_t^2 v^\omega - \Delta v^\omega + (v^\omega + z^\omega_{\text{per}})^3 = 0 \\ (v^\omega, \partial_t v^\omega)|_{t=0} = (0,0)\,, \end{cases} \qquad (t,x) \in [0,T]\times\mathbb{T}^4\,.$$

Moreover, v^ω satisfies the following Duhamel formulation:

$$v^\omega(t) = -\int_0^t S_{\text{per}}(t-t')(v^\omega(t') + z^\omega_{\text{per}}(t'))^3 dt'$$

for $t \in [0,T]$. This can be seen from the fact that $v^\omega_{N_k}$ satisfies the corresponding Duhamel formulation for (A.2), the convergence of $v^\omega_{N_k}$ to v^ω in (A.13), and the convergence of z_{per,N_k} to z_{per} given by (A.1) and (A.3). Therefore, $u^\omega := z^\omega_{\text{per}} + v^\omega$ is a solution to (1.1) on $[0,T]\times\mathbb{T}^4$ in the class (3.1). This shows how Proposition 3.1 follows from Proposition 3.2.

REMARK A.2. In the above argument, we only controlled the homogeneous $\dot{H}^1$-norm of v^ω for simplicity. One can easily control the nonhomogeneous H^1-norm of v^ω by estimating the L^2-norm of v^ω from the control on the L^2-norm of $\partial_t v^\omega$ and Cauchy-Schwarz inequality (in time). Since this is standard, we omit details.

Appendix B. On uniqueness. We briefly discuss the issue on uniqueness mentioned in Remark 1.3. It follows from the proof of Theorem 1.1 that the set $\Omega_{(u_0,u_1)}$ can be written as $\Omega_{(u_0,u_1)} = \bigcup_{\varepsilon>0} \Omega_\varepsilon$ with $P(\Omega_\varepsilon^c) < \varepsilon$ such that (i) there exists a global solution u^ω to (1.1) and (ii) given any $T > 0$, we have

$$\|f^\omega\|_{L^{\frac{d+2}{d-2}}([0,T];\, L^{\frac{2(d+2)}{d-2}}(\mathbb{T}^d))} \leq C_1(T) < \infty\,, \tag{B.1}$$

for all $\omega \in \Omega_\varepsilon$, where $f^\omega := S_{\text{per}}(\cdot)(u^\omega_0, u^\omega_1)$. Now, we fix such $\omega \in \Omega_\varepsilon$ and suppress the dependence on ω in the following. Letting $v = u - f$, we see that v is a global solution to the perturbed NLW on $\mathbb{T}^d$:

$$\begin{cases} \partial_t^2 v - \Delta v + F(v+f) = 0 \\ (v, \partial_t v)|_{t=0} = (0,0)\,. \end{cases} \tag{B.2}$$

Suppose that $v_1, v_2 \in X(\mathbb{R})$ are two global solutions to (B.2), where $X(\mathbb{R})$ is as in (1.13). Then, for each $T > 0$, we have

$$\|v_j\|_{L^{\frac{d+2}{d-2}}([0,T];\, L^{\frac{2(d+2)}{d-2}}(\mathbb{T}^d))} \leq C_2(T) < \infty\,, \quad j = 1,2\,. \tag{B.3}$$

In view of (B.1) and (B.3), we can write $[0, T] = \bigcup_{j=0}^{J} I_j$ with $I_j = [t_j, t_{j+1}]$, $t_0 = 0 < t_1 < \cdots < t_J = T$, and $J < \infty$ such that

$$\|f^\omega\|_{L_{I_j}^{\frac{d+2}{d-2}} L_x^{\frac{2(d+2)}{d-2}}} + \sum_{j=1}^{2} \|v_j\|_{L_{I_j}^{\frac{d+2}{d-2}} L_x^{\frac{2(d+2)}{d-2}}} \leq \gamma \ll 1. \tag{B.4}$$

Given a finite interval I, let $X(I) = \{(w, \partial_t w) : (w, \partial_t w) \in C_I \dot{\mathcal{H}}_x^1(\mathbb{T}^d),\ w \in L_I^{\frac{d+2}{d-2}} L_x^{\frac{2(d+2)}{d-2}}(\mathbb{T}^d)\}$. Then, by a standard deterministic local-in-time analysis with Lemma 2.1 and (B.4), we obtain

$$\|v_1 - v_2\|_{X(I_0)} \leq C(\gamma)\|v_1 - v_2\|_{L_{I_0}^{\frac{d+2}{d-2}} L_x^{\frac{2(d+2)}{d-2}}} \leq \frac{1}{2}\|v_1 - v_2\|_{L_{I_0}^{\frac{d+2}{d-2}} L_x^{\frac{2(d+2)}{d-2}}}.$$

Therefore, we conclude that $v_1 = v_2$ on I_0. In particular, we have $v_1(t_1) = v_2(t_1)$. Thus, we can iterate the above argument and conclude that $v_1 = v_2$ on I_j, $j = 1, 2, \dots, J$. Namely, $v_1 = v_2$ on $[0, T]$. Since the choice of T was arbitrary, we conclude that $v_1 = v_2$ on $[0, \infty)$. Clearly, the same argument works for negative times.

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REFERENCES

[1] H. BAHOURI AND P. GÉRARD, High frequency approximation of solutions to critical nonlinear wave equations, Amer. J. Math. 121 (1999), no. 1, 131–175.

[2] H. BAHOURI AND J. SHATAH, Decay estimates for the critical semilinear wave equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998), no. 6, 783–789.

[3] Á. BÉNYI AND T. OH, The Sobolev inequality on the torus revisited, Publ. Math. Debrecen, 83 (2013), no. 3, 359–374.

[4] Á. BÉNYI, T. OH AND O. POCOVNICU, Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS, Excursions in Harmonic Analysis, Vol. 4, 3–25, Appl. Numer. Harmon. Anal., Birkhüser/Springer, Cham, 2015.

[5] Á. BÉNYI, T. OH AND O. POCOVNICU, On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$, $d \geq 3$, Trans. Amer. Math. Soc. Ser. B 2 (2015), 1–50.

[6] J. BOURGAIN, Invariant measures for the 2D-defocusing nonlinear Schrödinger equation, Comm. Math. Phys. 176 (1996), no. 2, 421–445.

[7] N. BURQ, L. THOMANN AND N. TZVETKOV, Global infinite energy solutions for the cubic wave equation, Bull. Soc. Math. France. 143 (2015), no. 2, 301–313.

[8] N. BURQ AND N. TZVETKOV, Random data Cauchy theory for supercritical wave equations. I. Local theory, Invent. Math. 173 (2008), no. 3, 449–475.

[9] N. BURQ AND N. TZVETKOV, Probabilistic well-posedness for the cubic wave equation, J. Eur. Math. Soc. 16 (2014), no. 1, 1–30.

[10] M. CHRIST, J. COLLIANDER AND T. TAO, Ill-posedness for nonlinear Schrödinger and wave equations, arXiv:math/0311048 [math.AP].

[11] J. COLLIANDER AND T. OH, Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$, Duke Math. J. 161 (2012), no. 3, 367–414.

[12] J. GINIBRE, A. SOFFER AND G. VELO, The global Cauchy problem for the critical nonlinear wave equation, Jour. Func. Anal. 110 (1992), 96–130.

[13] M. GRILLAKIS, Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity, Ann. of Math. 132 (1990), 485–509.

[14] M. GRILLAKIS, Regularity for the wave equation with a critical nonlinearity, Commun. Pure Appl. Math. 45 (1992), 749–774.

[15] L. HÖRMANDER, The analysis of linear partial differential operators. I. Distribution theory and Fourier analysis, Reprint of the second (1990) edition. Classics in Mathematics. Springer-Verlag, Berlin, 2003. x+440 pp.

[16] L. KAPITANSKI, Global and unique weak solutions of nonlinear wave equations, Math. Res. Letters 1 (1994), 211–223.

[17] M. KEEL AND T. TAO, Endpoint Strichartz estimates, Amer. J. Math. 120 (1998), no. 5, 955–980.

[18] H. LINDBLAD AND C. SOGGE, On existence and scattering with minimal regularity for semilinear wave equations, J. Funct. Anal. 130 (1995), no. 2, 357–426.

[19] J. LÜHRMANN AND D. MENDELSON, Random data Cauchy theory for nonlinear wave equations of power-type on $\mathbb{R}^3$, Comm. Partial Differential Equations 39 (2014), no. 12, 2262–2283.

[20] J. LÜHRMANN AND D. MENDELSON, On the almost sure global well-posedness of energy sub-critical nonlinear wave equations on $\mathbb{R}^3$, New York J. Math. 22 (2016), 209–227.

[21] K. NAKANISHI, Unique global existence and asymptotic behaviour of solutions for wave equations with noncoercive critical nonlinearity, Comm. Partial Differential Equations 24 (1999), no. 1–2, 185–221.

[22] K. NAKANISHI, Scattering Theory for Nonlinear Klein-Gordon Equation with Sobolev Critical Power, Internat. Math. Res. Not. 1 (1999), 31–60.

[23] T. OH AND O. POCOVNICU, Probabilistic well-posedness of the energy-critical defocusing quintic nonlinear wave equation on $\mathbb{R}^3$, J. Math. Pures Appl. (9) 105 (2016), no. 3, 342–366.

[24] T. OH AND J. QUASTEL, On Cameron-Martin theorem and almost sure global existence, Proc. Edinb. Math. Soc. 59 (2016), 483–501.

[25] O. POCOVNICU, Almost sure global well-posedness for the energy-critical defocusing nonlinear wave equation on $\mathbb{R}^d$, $d = 4$ and 5, J. Eur. Math. Soc. (JEMS) 19 (2017), no. 8, 2521–2575.

[26] M. REED AND B. SIMON, Methods of modern mathematical physics. II. Fourier analysis, self-adjointness, Academic Press, New York-London, 1975. xv+361 pp.

[27] J. SHATAH AND M. STRUWE, Regularity results for nonlinear wave equations, Ann. of Math. (2) 138 (1993), no. 3, 503–518.

[28] J. SHATAH AND M. STRUWE, Well-posedness in the energy space for semilinear wave equations with critical growth, Inter. Math. Research Not. 7 (1994), 303–309.

[29] M. STRUWE, Globally regular solutions to the u^5 Klein-Gordon equation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 15 (1988), 495–513.

[30] C. SUN AND B. XIA, Probabilistic well-posedness for supercritical wave equation on $\mathbb{T}^3$, arXiv:1508.00228 [math.AP].

[31] T. TAO, Spacetime bounds for the energy-critical nonlinear wave equation in three spatial dimensions, Dyn. Partial Differ. Equ. 3 (2006), no. 2, 93–110.

[32] N. WIENER, Tauberian theorems, Ann. of Math. (2) 33 (1932), no. 1, 1–100.

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