ON THE TORSE-FORMING DIRECTIONS IN FINSLER SPACES

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0. The torse-forming directions in Riemann spaces have the property that, when we develop an arbitrary continuous curve $x^i = x^i(t)$ on the tangential space at a point of the curve, a vector p^i defined on each point of the curve form a torse (or a developable surface). This concept was treated by K. Yano at first. But the torse-forming directions ought to be generally defined in affinely connected spaces or projectively connected spaces. This paper deals with the torse-forming derections in Finsler spaces. In this paper the fundamental metric function $L(x, x')$ and the coefficients of the connection Γ^{*j}_{ik}, $\Gamma_i{}^j{}_k$, $A_i{}^j{}_k$, $C_i{}^j{}_k$ have the same meaning as those of E. Cartan.

1. We assume that a vector field $p^i(x^1, \ldots, x^n)$ is independent upon the linear elements. Then a point on the vector p^i defined on each point of the curve is represented by $x^i + \alpha p^i$, where α is a scalar function. If the vector p^i is a torse-forming direction then the vector p^i should envelop a curve $x^i(t) + \alpha(t)p^i$, the edge of regression of the torse. Then, we get the following relations for the enveloping curve

$$D(x^i + \alpha p^i) = \beta p^i, \tag{1.1}$$

where $\beta = \beta_k dx^k + \beta'_k de^k$ and D means the covariant differention. Therefore

$$dx^i + \alpha_k dx^k + \alpha Dp^i = (\beta_k dx^k + \beta'_k De^k)p^i.$$

If we put $Dp^i = p^i_{|k}\, dx^k + p^i_{||k} De^k$, then we get:

$$\begin{cases} \delta^i_k + \alpha_k p^i + \alpha p^i_{|k} = \beta_k p^i, \\ \alpha p^i_{||k} = \beta'_k p^i \end{cases} \tag{1.2}$$

At this time we assume $p^i_{||k} = 0$. For if $p^i_{||k} \neq 0$ then the torse depends of the linear elements and it is difficult to deal with such a problem in Finsler spaces. Accordingly $\alpha \neq 0$, $\beta'_k = 0$. Generally we can assume that p^i is a unit vector without loss of generality. Hence $p^i_{|k}p_i = 0$ and we get $\alpha_k - \beta_k = -p_k$ from (1. 2). $p^i_{||k} = 0$ implies that p_i are the function of x^1, $\ldots, x^n$. Now we obtain as necessary and sufficient conditions in order that the vector p^i being a torse forming direction the following relations:

$$\begin{cases} p^i_{|k} = H(\delta^i_k - p^i p_k), \left(H = -\dfrac{1}{\alpha}\right) \\ p^i_{||k} = LeC^i_{jk}p^j = 0. \end{cases} \tag{1.3}$$

2. (1.3) are written by using the covariant components p_i as follows:

$$(2.1)\qquad \begin{cases} p_{i|k} = H(g_{ik} - p_i p_k), \\ p_{i||k} = Le\, C^j_{ik} p_j = 0. \end{cases}$$

Then we can easily find that $p_{i|k}$ are symmetric with respect to i, j. Hence we get:

$$\frac{\partial p_k}{\partial x^i} = \frac{\partial p_i}{\partial x^k}$$

The above relations mean that p_i is a gradient vector. Hence if we choose a suitable scalar function $F(x^1, \ldots, x^n)$ then p_i are represented by $p_i = \frac{\partial F}{\partial x^i}$. Now we put

$$(2.2)\qquad F(x^1, \ldots, x^n) = C \quad (C = \text{const}).$$

(2.2) is a system of ∞ hypersurfaces, which are transversal to p^i. We per adopt a regular representation of a hypersurface of (2.2) i.e. :

$$(2.3)\qquad x^i = x^i(u^1, \ldots, u^{n-1}).$$

If we put $\partial x^i/\partial u^a = Q^i_a$ $(a, b, c, \ldots = 1, 2, \ldots, n-1)$ then we get:

$$p_i Q^i_a = 0.$$

In the above relations we can see that the vector Q^i_a are transversal to p^i. Following to the general theory of Finsler spaces, we introduce the fundamental metric tensor $\bar{g}_{ab}$ on the hypersurface (2.3) as follows:

$$\bar{g}_{ab} = Q_a{}^i Q_b{}^j g_{ij}.$$

If we adopt p^i as a linear element then we have $-Dp_i Q^i_a = p_i Q^i_a = H_{ab} du^b$ or $p_{ik} Q^i_a Q^k_b = H_{ab}$ from (2.1). Hence we have:

$$(2.4)\qquad H_{ab} = H\bar{g}_{ab}.$$

The tensor H_{ab} is the fundamental tensor of the second kind and the hypersurfaces (2.2) are totally umbilical. Also we obtain $H = \frac{1}{n-1} H^a_a$ from (2.5) and H is the mean curvature of the hypersurface (2.3). It is well known that the curves which traverse the hypersurfaces (2.2) in the direction p^i are all geodesics on account of $p^i_{|k} p_i = 0$. Therefore we get the following results.

THEOREM. *In a Finsler space which admits a torse-forming vector field, there exists a system of ∞^1 totally umbilical hypersurfaces whose trajectories are all geodesics.*

3. We choose a suitable coordinate system so that x^n=const. represent the hypersurfaces $F = C$ and x^a = const. represent the trajectories. Then we know :

$$(3.1)\qquad g^{an} = g_{an} = 0.$$

In this coordinate system, the vector p^i is represented by δ^i_n $(0, \ldots, 1)$.

Accordingly, putting $k = a, i = a$, in $(1.3)_1$ we get the following relations

$$\Gamma^{*a}_{nn} = 0. \tag{3.2}$$

(3.2) implies $\partial g_{nn}/\partial x^a = 0$. Hence we obtain

(i) *g_{nn} does not contain $x^1, \ldots, x^{n-1}$.*

Also we have $C^i_{nk} = 0$ from $(1.3)_2$. Hence we have the following results (ii), (iii) by making use of $C_{nik} = g_{ij}C^j_{nk} = \partial^3\mathfrak{F}/\partial x^{n'}\partial x^{i'}\partial x^{k'}$, $(\mathfrak{F} = \frac{1}{2}\mathfrak{L}^2)$.

(ii) *g_{nn} does not contain $x^{1'}, \ldots, x^{n'}$.*

(iii) *g_{ab} do not contain $x^{n'}$.*

Also we have the following relations from $(1.3)_2$:

$$\Gamma^{*n}_{ab} = H'g_{ab}, \quad (H' = H/g_{nn}). \tag{3.4}$$

The above relations are written in full detail as follows :

$$\frac{1}{2} g^{nn} \frac{\partial g_{ab}}{\partial x^n} + g^{nn}C_{abr}\frac{\partial G^r}{\partial x^n} = H'g_{ab}.$$

But these relations are also true for $n = a$. Hence if we mutiply the above equations by $x^{a'}$ and sum about a then we get

$$\frac{1}{2} g^{nn} \frac{\partial^2\mathfrak{F}}{\partial x^n \partial x^{b'}} = \frac{\partial\mathfrak{F}}{\partial x^{b'}} H'$$

where H' does not contain $x^{1'}, \ldots, x^{n'}$. Hence we get $\partial g_{ab}/\partial x^n = H''g_{ab}$ $(H'' = H/(g_{nn})^2)$ and H'' does not contain $x^{1'}, \ldots . x^{n'}$, Thus we obtain the following results (iv).

(iv) *g_{ab} are written as follows :*

$$g_{ab} = f(x^1, \ldots, x^n)\, g^*_{ab}(x^1, \ldots, x^{(n-1)}; x^{1'}, \ldots, x^{(n-1)'}).$$

Consequently we obtain the following theorem from (i) (ii) (iii) (iv).

THEOREM. *In a Finslar space which admits a torse-forming vector field, we can choose a suitable coordinate system such as:*

$$ds^2 = f(x^1, \ldots, x^n)\, g^*_{ab}(x^1, \ldots, x^{n-1}; x^{1'}, \ldots, x^{(n-1)'})dx^a dx^b + (dx^n)^2.$$

where the last term is obtained by putting $\bar{x}^n = \int\sqrt{g_{nn}(x^n)}\,dx^n$. The converse is also true

4. As a particular case of the torse-forming vector field. We can consider a concurrent vector field. Then we get $D(x^i + \alpha p^i) = 0$ from (2.1). Hence we know $\beta_k = 0, \alpha_k = -p_k$. Accordingly we get (1.3). If we adopt the coordinate system as before then we know that $\alpha^k(= -p_k)$ is a gradient vector and $-\alpha = F(x^1, \ldots, x^n)$. Therefore $H(= -1/\alpha)$ is constant along a hypersurface of (2.2). S. Tachibana has shown that in a Finsler space which admits a concurrent vector field, we can choose a coordinate system such that

$$ds^2 = (x^n)^2\, g^{ab}(x^1, \ldots, x^{n-1}; x^{1'}, \ldots, x^{(n-1)'})dx^a dx^b + (dx^n)^2,$$

and the converse is also true.

Also as a particular case of the concurrent vector field, we can consider a parallel vector field. Then we get $p^i_{|k} = 0$ (or $Dp^i = 0$) by putting $\alpha \rightarrow \infty$ in (1.3). It is shown that we can choose a coordinate system such that

$$ds^2 = g_{ab}(x^1, \ldots\ldots x^{n-1}; x^{1'}, \ldots, x^{(n-1)'})dx^a dx^b + (dx^n)^2.$$

At this time $H = 0$. Therefore the hypersurfaces (2.2) are all totally geodesic hypersurfaces. The existence of such a coordinate system in a Finsler space is the necessary and sufficient condition in order to admit a parallel vector field.

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