ON THE GENERATION OF STRONGLY ERGODIC SEMI-GROUPS OF OPERATORS II*)

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(Received September 20, 1954)

1. Introduction. This paper is concerned with the problem of determining necessary and sufficient conditions that a linear operator is the infinitesimal generator of a semi-group of bounded linear operators. The first results in this direction were published independently by E. Hille [2][1] and K. Yosida [10] for semi-groups of operators satisfying the following conditions :

(c_1) *$T(\xi)$ is strongly continuous at $\xi = 0$,*

(c_2) *$\|T(\xi)\| \leqq 1 + \beta\xi$ for sufficiently small ξ, where β is a constant.*

Their results were later generalized to semi-groups of operators satisfying only the condition (c_1) by R. S. Phillips [8] and the present author [4], independently. Further this result has been generalized to strongly measurable semi-groups of operators by W. Feller [1]. R. S. Phillips [9] and the present author [5] have recently given necessary and sufficient conditionsthat a given operator generates a semi-group of class $(1, A)$ or of class $(1, C_1)$. In the present paper we give a necessary and sufficient condition that a given operator generates a semi-group of class $(0, A)$ or of class $(0, C_\alpha)$.

2. Definitions and preliminary theorems. Let $\{T(\xi);\ 0 \leqq \xi < \infty\}$ be a semi-group of operators satisfying the following conditions:

(a) For each ξ, $0 \leqq \xi < \infty$, $T(\xi)$ is a bounded linear operator from a complex Banach space X into itself and

$$T(\xi + \eta) = T(\xi)T(\eta) \qquad \text{for } \xi, \eta \geqq 0, \tag{2.1}$$
$$T(0) = I\ (= \text{the identity}).$$

(b) $T(\xi)$ is strongly measurable on $(0, \infty)$ (see [2, Definition 3. 3. 2]).

$$(c) \qquad \int_0^1 \|T(\xi)x\|\, d\xi < \infty \qquad \text{for each } x \in X.$$

A consequence of (a) and (b) is that $T(\xi)$ is strongly continuous for $\xi > 0$ (see [3] and [7]). If $T(\xi)$ satisfies the condition

$$(d) \qquad \lim_{\lambda\to\infty} \lambda \int_0^\infty e^{-\lambda\xi}\, T(\xi)x d(\xi) = x \qquad \text{for each } x \in X,$$

then $T(\xi)$ is said to be of *class* $(0, A)$. If, instead of (d), $T(\xi)$ satisfies the

*) This paper is the continuation of the paper with the same title in this Journal, vol. 6 (1954).

1) Numbers in brackets refer to the references at the end of this paper.

stronger condition

$$\text{(e)} \qquad \lim_{\tau \to 0} \alpha \tau^{-\alpha} \int_0^{\tau} (\tau - \xi)^{\alpha - 1} T(\xi) x d\xi = x \qquad \text{for each } x \in X,$$

then $T(\xi)$ is said to be of *class* $(0, C_\alpha)$. If (c) is replaced by the stronger condition

$$(c') \qquad \int_0^1 \| T(\xi) \| d\xi < \infty,$$

then these classes become $(1, A)$ and $(1, C_\alpha)$, respectively.

DEFINITION. *The operator A_0 which is defined by*

$$(2.2) \qquad A_0 x = \lim_{h \to 0} \frac{1}{h} [T(h) - I] x$$

whenever the limit on the right hand side exists, is said to be the infinitesimal generator of $\{T(\xi); 0 \leqq \xi < \infty\}$.

It follows from (d) that if $T(\xi)$ is a semi-group of class $(0, A)$, the domain of A_0 is dense in X (see [5] or [9]). We denote by A the smallest closed linear extension of the infinitesimal generator A_0 which is called *the complete infinitesimal generator* (c. i. g.).

Since $\| T(\xi) \|$ is lower semi-continuous, $\log \| T(\xi) \|$ is a measurable subadditive function. Then it follows that

$$\omega_0 = \inf_{\xi > 0} \log \| T(\xi) \| / \xi = \lim_{\xi \to \infty} \log \| T(\xi) \| / \xi,$$

where $-\infty \leqq \omega_0 < \infty$ [2, Theorem 6. 6. 1].

We shall now define $R(\lambda; A)$, for each λ with $\Re(\lambda)$[2] $> \omega_0$, by

$$(2.3) \qquad R(\lambda; A) x = \int_0^{\infty} e^{-\lambda \xi} T(\xi) x \, d\xi \qquad \text{for each } x \in X.$$

It is clear that this integral converges absolutely for λ with $\Re(\lambda) > \omega_0$.

THEOREM 2.1 *For each λ with $\Re(\lambda) > \omega_0$, $R(\lambda; A)$ is a bounded linear operator on X into itself with the following properties*:

$$R(\lambda; A)(\lambda - A_0) x = x \qquad \textit{for each } x \in D(A_0)^{3)},$$

$$(\lambda - A_0) R(\lambda; A) x = x \qquad \textit{for each } x \textit{ such that}$$

$$\lim_{\tau \to 0} \tau^{-1} \int_0^{\tau} T(\xi) x \, d\xi = x.$$

For the proof of this theorem, see [5] or [9].

The following theorems are due to R. S. Phillips [9].

THEOREM 2.2 *If $\{T(\xi); 0 \leqq \xi < \infty\}$ is of class* (0, A), *then there exists the complete infinitesimal generator A whose resolvent is $R(\lambda; A)$ for λ with* $\Re(\lambda)$

2) $\Re(\lambda)$ denotes the real part of λ.

3) The notation $D(B)$ denotes the domain of the operator B.

$> \omega_0$.

THEOREM 2.3 *If $f(\xi) \in \mathfrak{B}([0, d])$[4] for every finite d and if*

$$\int_0^\infty e^{-\omega\sigma} \| f(\sigma) \| d\sigma < \infty$$

for some real ω, then

$$\lim_{\lambda\to\infty} e^{-\lambda\xi} \sum_{n=0}^\infty \frac{(-1)^n (\lambda^2\xi)^{n+1}}{n!(n+1)!} F^{(n)}(\lambda) = f(\xi)$$

at all points ξ such that

$$\int_\xi^\sigma \| f(\xi) - f(\eta) \| d\eta = o(|\xi - \sigma|),$$

and hence for almost all ξ, where

$$F(\lambda) = \int_0^\infty e^{-\lambda\xi} f(\xi) d\xi$$

and $F^{(n)}(\lambda)$ denotes the n-th derivative of $F(\lambda)$. If $f(\xi)$ is continuous in some open interval, then this limit exists uniformly in every compact subinterval.

3. Generation of semi-groups of operators. We shall consider the problem for the generation of semi-groups, namely, what properties should an operator A possess in order that it is the c. i. g. of a semi-group of a given class?

Since $\xi^{-1}\log \|T(\xi)\|$ tends to a finite limit or to $-\infty$ as $\xi \to \infty$ (see section 2) and we can always replace $\{T(\xi); 0 \leqq \xi < \infty\}$ by the equivalent semi-group of operators $\{e^{-\omega\xi} T(\xi); 0 \leqq \xi < \infty\}$, we may assume without loss of generality that

$$\int_0^\infty \| T(\xi)x \| d\xi < \infty \qquad \text{for each } x \in X.$$

THEOREM 3.1 *A necessary and sufficient condition that a closed linear operator A is the* c. i. g. *of a semi-group $\{T(\xi); 0 \leqq \xi < \infty\}$ of class* (0, A) *with $\int_0^\infty \| T(\xi)x \| d\xi < \infty$ for each $x \in X$, is that*

(i) *the spectrum of A is located in $\Re(\lambda) \leqq 0$,*

(ii) *$D(A)$ is a dense linear subset in X,*

(iii) *there exists a finite positive constant M such that*

$$\| \lambda R(\lambda; A) \| \leqq M$$

for all real $\lambda \geqq 1$, where $R(\lambda; A)$ is the resolvent of A,

4) A function $f(\xi)$ on $[0,d]$ into X is said to belong to $\mathfrak{B}([0,d])$ if $f(\xi)$ is strongly measurable in $[0, d]$ and $\int_0^d \| f(\xi) \| d\xi < \infty$.

(iv) *there exists a non-negative function* $f(\xi, x)$ *defined on the product space* $(0, \infty) \times X$ *having the following properties*:

(a′) *for each* $x \in X, f(\xi, x)$ *is continuous for* $\xi > 0$ *and is integrable on* $(0, \infty)$,

(b′) $\|R^{(k)}(\lambda; A)x\| \leqq (-1)^k F^{(k)}(\lambda, x)$

for each $x \in X$, *all real* $\lambda > 0$ *and all integers* $k \geqq 0$, *where* $F(\lambda, x)$ *is defined by*

$$F(\lambda, x) = \int_0^\infty e^{-\lambda\xi} f(\xi, x) d\xi$$

for each $x \in X$ *and for all* $\lambda > 0$, *and* $R^{(k)}(\lambda; A)$, $F^{(k)}(\lambda, x)$ *denote the k-th derivative of* $R(\lambda; A), F(\lambda, x)$ *with respect to* λ, *respectively.*

Then $\|T(\xi)x\| \leqq f(\xi, x)$ *for each* $x \in X$ *and for all* $\xi > 0$, *and*

$$T(\xi)x = \lim_{\lambda \to \infty} \exp \xi(-\lambda + \lambda^2 R(\lambda; A))x$$

for each $x \in X$ *and for all* $\xi > 0$.

PROOF. Suppose that $\{T(\xi); 0 \leqq \xi < \infty\}$ is of class $(0, A)$ with $\int_0^\infty \|T(\xi)x\| d\xi < \infty$ for each $x \in X$. We shall define $R(\lambda; A)$ for each $\lambda > 0$ by

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda\xi} T(\xi)x \, d\xi \qquad \text{for each } x \in X. \tag{3.1}$$

Then it follows from Theorem 2.2 that $R(\lambda; A)$ is the resolvent of the c. i. g. A for all λ with $\Re(\lambda) > 0$, from which we get the property (i). As we have already remarked, the domain of the infinitesimal generator is dense in X, so that $D(A)$ is dense in X. The property (iii) is immediately obtained from the strong Abel-ergodicity of $T(\xi)$ (the condition (*d*)) and the uniform boundedness theorem [2, Theorem 2.12.2]. Finally we have, for each $x \in X$,

$$R^{(k)}(\lambda; A)x = (-1)^k \int_0^\infty e^{-\lambda\xi} \xi^k T(\xi)x \, d\xi$$

for all real $\lambda > 0$ and for all integers $k \geqq 0$. Setting $f(\xi, x) = \|T(\xi)x\|$, we obtain the property (iv).

Conversely, suppose that the conditions (i)-(iv) are satisfied. Since A is a closed linear operator, we get by (i)

$$\begin{cases} (\lambda - A) R(\lambda; A)x = x & \text{for } x \in X, \\ R(\lambda; A)(\lambda - A)x = x & \text{for } x \in D(A). \end{cases} \tag{3.2}$$

Hence we obtain the functional equation

$$R(\lambda; A) - R(\mu; A) = -(\lambda - \mu) R(\lambda; A) R(\mu; A), \tag{3.3}$$

so that

$$R^{(k-1)}(\lambda; A) = (-1)^{k-1}(k-1)! [R(\lambda; A)]^k.$$

Since by the definition of $F(\lambda, x)$

$$(-1)^{k-1}F^{(k-1)}(\lambda, x) = \int_0^\infty e^{-\lambda\xi}\xi^{k-1}f(\xi, x)\,d\xi,$$

we have

$$\|[\lambda R(\lambda\,;A)]^k x\| \leqq \frac{\lambda^k}{(k-1)!}\int_0^\infty e^{-\lambda\xi}\xi^{k-1}f(\xi, x)d\xi \tag{3.4}$$

for $k=1, 2, \ldots$ and for each $x \in X$. Let us put, for any real $\lambda > 0$,

$$T_\lambda(\xi) = \exp \xi(-\lambda + \lambda^2 R(\lambda\,; A)) \equiv \exp(-\lambda\xi)\sum_{k=0}^\infty \frac{(\lambda^2\xi)^k}{k\,!}[R(\lambda;A)]^k. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{cases} \|T_\lambda(\xi)x\| \leqq e^{-\lambda\xi}\|x\| + e^{-\lambda\xi}\displaystyle\sum_{k=0}^\infty \frac{(\lambda^2\xi)^{k+1}}{(k+1)!}\|[R(\lambda\,;A)]^{k+1}x\| \\ \qquad \leqq e^{-\lambda\xi}\|x\| + e^{-\lambda\xi}\displaystyle\sum_{k=0}^\infty \frac{(\lambda^2\xi)^{k+1}}{k\,!\,(k+1)\,!}\int_0^\infty e^{-\lambda\eta}\eta^k f(\eta, x)d\eta \\ \qquad = e^{-\lambda\xi}\|x\| + e^{-\lambda\xi}\displaystyle\sum_{k=0}^\infty \frac{(-1)^k(\lambda^2\xi)^{k+1}}{k!\,(k+1)\,!}F^{(k)}(\lambda, x). \end{cases} \tag{3.6}$$

Since, for each $x \in X$, $f(\xi, x)$ is continuous for $\xi > 0$ and is integrable on $(0, \infty)$, it follows from Theorem 2.3 that $f_\lambda(\xi, x)$ converges uniformly to $f(\xi, x)$ in every closed interval $[\varepsilon, 1/\varepsilon]$, $\varepsilon > 0$, where

$$f_\lambda(\xi, x) = e^{-\lambda\xi}\sum_{k=0}^\infty \frac{(-1)^k(\lambda^2\xi)^{k+1}}{k\,!\,(k+1)\,!}F^{(k)}(\lambda, x).$$

Thus, for each $x \in X$ and $\varepsilon > 0$, there exists a positive constant $M_{\epsilon,x}$ such that

$$\sup_{\lambda \geqq 1, 1/\epsilon \geqq \xi \geqq \epsilon} f_\lambda(\xi, x) \leqq M_{\epsilon,x},$$

so that by (3.6)

$$\sup_{\lambda \geqq 1, 1/\epsilon \geqq \xi \geqq \epsilon} \|T_\lambda(\xi)x\| \leqq \|x\| + M_{\epsilon,x}.$$

We obtain by the uniform boundedness theorem [2, Theorem 2.12.2] that

$$\sup_{\lambda \geqq 1, 1/\epsilon \geqq \xi \geqq \epsilon} \|T_\lambda(\xi)\| = M_\epsilon < \infty \tag{3.7}$$

for each $\varepsilon > 0$.

By the condition (iii) and (3.2)

$$\|\lambda R(\lambda\,;A)x - x\| = \|R(\lambda\,;A)Ax\| \leqq \frac{M}{\lambda}\|Ax\|$$

for $x \in D(A)$, so that we get by (ii) and (iii)

$$\lim_{\lambda\to\infty}\|\lambda R(\lambda\,;A)x - x\| = 0 \tag{3.8}$$

for all $x \in X$. It follows from (3.2) that $[\lambda R(\lambda\,;A) - 1]^2 x = [R(\lambda\,;A)]^2 A^2 x$ for $x \in D(A^2)$, where $D(A^2) \equiv \{x\,; x \in D(A) \text{ and } Ax \in D(A)\}$, and hence we obtain by (iii)

$$\|[\lambda R(\lambda\,;A) - 1]^2 x\| \leqq \frac{M^2}{\lambda^2}\|A^2 x\|. \tag{3.9}$$

Now, for each $\xi > 0$,

$$T_\lambda(\xi)x - T_\mu(\xi)x = \int_\mu^\lambda \frac{\partial}{\partial\nu}[T_\nu(\xi)x]d\nu$$

$$= -\xi\int_\mu^\lambda T_\nu(\xi)[1-\nu R(\nu; A)]^2x\,d\nu,$$

and therefore by (3.7) and (3.9)

$$\|T_\lambda(\xi)x - T_\mu(\xi)x\| \leqq \xi M^2 M_\varepsilon \|A^2x\| \int_\mu^\lambda \frac{d\nu}{\nu^2}$$

for $x\in D(A^2)$, where ε is a number such that $0<\varepsilon\leqq\xi\leqq 1/\varepsilon$. Thus the limit $\lim_{\lambda\to\infty} T_\lambda(\xi)x$ exists for $x\in D(A^2)$.

On the other hand it follows from (3.2), (3.8) and (ii) that $D(A^2)$ is dense in X. Since by (3.7)

$$\sup_{\lambda\geqq 1}\|T_\lambda(\xi)\|\leqq M_\varepsilon$$

for all ξ such that $0<\varepsilon\leqq\xi\leqq 1/\varepsilon$, the limit $\lim_{\lambda\to\infty} T_\lambda(\xi)x$ exists for all $x\in X$ and for all $\xi>0$, which we denote by $T(\xi)x$. Since $T_\lambda(\xi)$ is a semi-group of bounded linear operators strongly continuous on $(0, \infty)$, $T(\xi)$ is strongly measurable on $\xi>0$. We have by (3.7)

$$\begin{aligned}&\|T_\lambda(\xi)T_\lambda(\eta)x - T(\xi)T(\eta)x\|\\ &\leqq \|T_\lambda(\xi)\|\,\|T_\lambda(\eta)x - T(\eta)x\| + \|T(\eta)\|\,\|T_\lambda(\xi)x - T(\xi)x\|\\ &\leqq M_\varepsilon\{\|T_\lambda(\eta)x - T(\eta)x\| + \|T_\lambda(\xi)x - T(\xi)x\|\},\end{aligned}$$

where ε is a positive number such that $\varepsilon\leqq\eta\leqq 1/\varepsilon$ and $\varepsilon\leqq\xi\leqq 1/\varepsilon$. Thus we get for each $x\in X$

$$\lim_{\lambda\to\infty} T_\lambda(\xi)T_\lambda(\eta)x = T(\xi)T(\eta)x,$$

so that $\{T(\xi); 0\leqq\xi<\infty\}$ is a semi-group of bounded linear operators, where $T(0)=I$. Accordingly, $\{T(\xi); 0\leqq\xi<\infty\}$ is strongly continuous for $\xi>0$. By (3.6)

$$(3.10)\quad \left\{\begin{aligned}\int_0^\infty \|T_\lambda(\xi)x\|\,d\xi &\leqq \frac{1}{\lambda}\|x\| + \int_0^\infty e^{-\lambda\xi}\sum_{k=0}^\infty \frac{(\lambda^2\xi)^{k+1}}{k!\,(k+1)!}\int_0^\infty e^{-\lambda\eta}\eta^k f(\eta, x)d\eta\,d\xi\\ &= \frac{1}{\lambda}\|x\| + \int_0^\infty e^{-\lambda\eta}\sum_{k=0}^\infty \frac{(\lambda\eta)^k}{k!} f(\eta, x)\,d\eta\\ &= \frac{1}{\lambda}\|x\| + \int_0^\infty f(\eta, x)\,d\eta.\end{aligned}\right.$$

We have by the Fatou lemma

$$\int_0^\infty \|T(\xi)x\|\,d\xi \leqq \int_0^\infty f(\xi, x)d\xi < \infty$$

for each $x\in X$.

We now define $R(\lambda; \bar{A})$, for each λ with $\Re(\lambda) > 0$, by

$$R(\lambda; \bar{A})x = \int_0^\infty e^{-\lambda\xi} T(\xi)x d\xi \tag{3.11}$$

for each $x \in X$. From the definition of $T_\lambda(\xi)$, we have

$$\begin{aligned} T_\lambda(\xi_2)x - T_\lambda(\xi_1)x &= \int_{\xi_1}^{\xi_2} \frac{\partial}{\partial\xi} T_\lambda(\xi)x\, d\xi \\ &= \int_{\xi_1}^{\xi_2} T_\lambda(\xi)\, [\lambda^2 R(\lambda; A) - \lambda]x\, d\xi \\ &= \int_{\xi_1}^{\xi_2} T_\lambda(\xi)\, [\lambda R(\lambda; A)Ax]\, d\xi \end{aligned}$$

for $x \in D(A)$. Since, by (3.7) and (3.8), $T_\lambda(\xi)[\lambda R(\lambda; A)Ax] \to T(\xi)Ax$ boundedly in every finite interval $0 < \xi_1 \leqq \xi \leqq \xi_2 < \infty$, we obtain for $0 < \xi_1 < \xi_2 < \infty$

$$T(\xi_2)x - T(\xi_1)x = \int_{\xi_1}^{\xi_2} T(\xi)Ax\, d\xi \tag{3.12}$$

for $x \in D(A)$. Let $x \in D(A^2)$, then

$$T_\lambda(\xi)x = x + \xi[\lambda R(\lambda; A)Ax] + \int_0^\xi \int_0^\sigma T_\lambda(\tau)\, [\lambda R(\lambda; A)]^2 A^2 x\, d\tau\, d\sigma$$

and hence, by (3.10) and (iii),

$$\begin{aligned} \| T_\lambda(\xi)x - x - \xi[\lambda R(\lambda; A)Ax] \| &\leqq \| \lambda R(\lambda; A) \|^2 \int_0^\xi \left[\int_0^\infty \| T_\lambda(\tau)A^2x \|\, d\tau \right] d\sigma \\ &\leqq M^2\xi \left[\frac{1}{\lambda} \| A^2x \| + \int_0^\infty f(\eta, A^2x) d\eta \right]. \end{aligned}$$

Passing to the limit with λ we obtain

$$\| T(\xi)x - x - \xi Ax \| \leqq M^2\xi \int_0^\infty f(\eta, A^2x)\, d\eta$$

for $x \in D(A^2)$. It follows that $\lim_{\xi\to 0} T(\xi)x = x$ for $x \in D(A^2)$, and also by (3.12) $dT(\xi)x/d\xi = T(\xi)Ax$ for $\xi > 0$. Thus, by (3.11), we have for $x \in D(A^2)$ and for large $\lambda > 0$

$$\begin{aligned} R(\lambda; \bar{A})Ax &= \int_0^\infty e^{-\lambda\xi} T(\xi)Ax\, d\xi = \int_0^\infty e^{-\lambda\xi} \left(\frac{dT(\xi)x}{d\xi} \right) d\xi \\ &= [e^{-\lambda\xi} T(\xi)x]_0^\infty + \lambda \int_0^\infty e^{-\lambda\xi} T(\xi)x\, d\xi \\ &= -x + \lambda R(\lambda; \bar{A})x, \end{aligned}$$

so that by (3.2)

$$R(\lambda;\bar{A})(\lambda-A)x=R(\lambda;A)(\lambda-A)x$$

for $x\in D(A^2)$.

Now for $\lambda>0$, $(\lambda-A)D(A^2)=D(A)$ by (3.2). Hence $R(\lambda;\bar{A})x=R(\lambda;A)x$ on the set $D(A)$ dense in X and therefore $R(\lambda;A)=R(\lambda;\bar{A})$ for large $\lambda>0$. Thus we get $R(\lambda:A)\equiv R(\lambda;\bar{A})$ for $\lambda>0$ by the first resolvent equation. As we have already observed, $\lim_{\lambda\to\infty}\lambda R(\lambda;A)x=x$ for all $x\in X$, and hence it follows that $\{T(\xi);0\leqq\xi<\infty\}$ is a semi-group of class $(0,A)$ with $\int_0^\infty\|T(\xi)x\|d\xi<\infty$ for each $x\in X$. By Theorem 2.2 $R(\lambda;A)$ is the resolvent of the c.i.g. $\bar{A}$ of $T(\xi)$, and then we get

$$\text{(3.13)}\qquad \begin{cases}(\lambda-\bar{A})R(\lambda;A)x=x & \text{for } x\in X,\\ R(\lambda;A)(\lambda-\bar{A})x=x & \text{for } x\in D(\bar{A}).\end{cases}$$

Since $R(\lambda;A)[X]=D(A)$ by (3.2) and $R(\lambda;A)[X]=D(\bar{A})$ by (3.13), we have $D(A)=D(\bar{A})$. Again, by (3.2) and (3.13),

$$\lambda R(\lambda;A)Ax=\lambda R(\lambda;\bar{A})Ax=\lambda R(\lambda;A)\bar{A}x$$

for $x\in D(A)=D(\bar{A})$, and therefore $A=\bar{A}$ by (3.8).

Finally, applying Theorem 2.3, we have by (3.6)

$$\text{(3.14)}\qquad \|T(\xi)x\|\leqq f(\xi,x)$$

for each $x\in X$ and $\xi>0$. This concludes the proof of Theorem 3.1.

COROLLARY 3.1'. *If in Theorem 3.1 $f(\xi,x)=f(\xi)\|x\|$ for each $x\in X$, where $f(\xi)$ is continuous for $\xi>0$ and $\int_0^\infty f(\xi)d\eta<\infty$, then $\{T(\xi);0\leqq\xi<\infty\}$ is of class (1, A) with $\int_0^\infty\|T(\xi)\|d\xi<\infty$. In particular, for bounded $f(\xi)$, $\{T(\xi);0\leqq\xi<\infty\}$ is a semi-group of operators such that $T(\xi)$ is strongly continuous at $\xi=0$.*

PROOF. Since the first part is obvious by (3.14), it remains to prove the second part. If $f(\xi)\leqq M$, then by (3.14) we have $\|T(\xi)\|\leqq M$ for all $\xi>0$. By the definition of $R(\lambda;\bar{A})$, $\lim_{\xi\to0}T(\xi)R(\lambda;\bar{A})x=R(\lambda;\bar{A})x$ for all $x\in X$, and $R(\lambda;\bar{A})[X]=R(\lambda;A)[X]=D(A)$ is dense in X. It now follows from the Banach-Steinhaus theorem that $\lim_{\xi\to0}T(\xi)x=x$ for all $x\in X$.

THEOREM 3.2 *Let α be a positive integer. A necessary and sufficient condition that a semi-group of class $(0,A)$ is of class $(0,C_\alpha)$, is that there exist real numbers $M>0$ and $\omega\geqq0$ such that*

$$\text{(3.15)}\qquad \sup_{\lambda>0,k\geqq\alpha}\left\|\frac{\alpha}{k(k-1)\cdots(k-\alpha+1)}\sum_{i=1}^{k-\alpha+1}\frac{(k-i)!}{(k-\alpha+1-i)!}[\lambda R(\lambda+\omega;A)]^i\right\|\leqq M,$$

where $R(\lambda;A)$ *is defined by* (2.3).

In case of $\alpha=1$ this theorem is due to R. S. Phillips [9] and the present author [5], and the general case is due to the present author [6]. In particular, if $\int_0^\infty \|T(\xi)x\|d\xi<\infty$ for all $x\in X$ or $\|T(\xi)\|\leqq M'$ for sufficiently large ξ, then (3.15) may be replaced by

$$\sup_{\lambda>0,k\geqq\alpha}\left\|\frac{\alpha}{k(k-1)\cdots(k-\alpha+1)}\sum_{i=1}^{k-\alpha+1}\frac{(k-i)!}{(k-\alpha+1-i)!}[\lambda R(\lambda;A)]^i\right\|\leqq M.$$

From Theorems 3.1 and 3.2, we get the following

THEOREM 3.3 *Let* α *be a positive integer. A necessary and sufficient condition that a closed linear operator* A *is the* c. i. g. *of a semi-group* $\{T(\xi);0\leqq\xi<\infty\}$ *of class* $(0, C_\alpha)$ *with* $\int_0^\infty\|T(\xi)x\|d\xi<\infty$ *for each* $x\in X$, *is that*

(i′) *the spectrum of* A *is located in* $\Re(\lambda)\leqq 0$,

(ii′) $D(A)$ *is a dense linear subset in* X,

(iii′) *there exists a real number* $M>0$ *such that*

$$\left\|\frac{\alpha}{k(k-1)\cdots(k-\alpha+1)}\sum_{i=1}^{k-\alpha+1}\frac{(k-i)!}{(k-\alpha+1-i)!}[\lambda R(\lambda;A)]^i\right\|\leqq M$$

for all real $\lambda>0$ *and for all integers* $k\geqq\alpha$, *where* $R(\lambda;A)$ *is the resolvent of* A,

(iv′) *there exists a non-negative function* $f(\xi,x)$ *defined on the product space* $(0,\infty)\times X$ *having the following properties*:

(a″)) *for each* $x\in X$, $f(\xi,x)$ *is continuous for* $\xi>0$ *and is integrable on* $(0,\infty)$,

(b″) $$\|R^{(k)}(\lambda;A)x\|\leqq(-1)^kF^{(k)}(\lambda,x)$$

for each $x\in X$, *all real* $\lambda>0$ *and all integers* $k\geqq 0$, *where* $F(\lambda, x)$ *is defined by*

$$F(\lambda,x)=\int_0^\infty e^{-\lambda\xi}f(\xi,x)d\xi$$

for each $x\in X$ *and for all* $\lambda>0$, *and* $R^{(k)}(\lambda;A)$, $F^{(k)}(\lambda,x)$ *denote the k-th derivative of* $R(\lambda;A)$, $F(\lambda,x)$ *with respect to* λ, *respectively.*

If $\{T(\xi);0\leqq\xi<\infty\}$ is of class $(0,C_1)$, then the infinitesimal generator of $T(\xi)$ is closed (see [5] or [9]). Thus we have the following

COROLLARY 3.3′ *A necessary and sufficient condition that a closed linear operator* A *is the infinitesimal generator of a semi-group* $\{T(\xi);0\leqq\xi<\infty\}$ *of class* $(0, C_1)$ *with* $\int_0^\infty\|T(\xi)x\|d\xi<\infty$ *for each* $x\in X$, *is that the conditions* (i′), (ii′) *and* (iv′) *in Theorem* 3.3 *are satisfied and that there exists a real* *number* $M > 0$ *such that*

$$\left\| \frac{1}{k} \sum_{i=1}^{k} [\lambda R(\lambda; A)]^i \right\| \leqq M$$

for all real $\lambda > 0$ *and for all integers* $k \geqq 1$, *where* $R(\lambda; A)$ *is the resolvent of* A.

REMARK. The notion of the complete infinitesimal generator was introduced by R. S. Phillips [9]. If, instead of the c. i. g., we define the $(C, 1)$-continuity set Σ by $\Sigma = \left\{x; \lim_{\xi \to 0} \xi^{-1} \int_0^{\xi} T(\eta)x \, d\eta = x\right\}$ and the infinitesimal generator A by $Ax = \lim_{\xi \to 0} \xi^{-1} [T(\xi) - I]x$ whenever the limit on the right hand side exists and belongs to Σ, then we get the following theorem (see [6]).

THEOREM 3.4. *Let* $\{T(\xi); 0 \leqq \xi < \infty\}$ *be a semi-group of class* (0, A) *with* $\int_0^{\infty} \|T(\xi)x\| d\xi < \infty$ *for each* $x \in X$. *Then we get* (ii), (iii), (iv) *and furthermore*

(i″) *for each* λ *with* $\Re(\lambda) > 0$, *there exists a bounded linear operator* $R(\lambda; A)$ *from* X *into* Σ *such that*

$$(\lambda - A)R(\lambda; A)x = x \qquad \text{for each } x \in \Sigma,$$

$$R(\lambda; A)(\lambda - A)x = x \qquad \text{for each } x \in D(A),$$

(v) *if we define the new norm by*

$$N(x) = \sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^{\xi} T(\eta)x \, d\eta \right\| \qquad \text{for each } x \in X,$$

then Σ *is a Banach space with the norm* $N(x)$, $D(A)$ *is dense in* Σ *with the norm* $N(x)$ *and*

$$N(x) = \sup_{k \geqq 1, \lambda > 0} \left\| \frac{1}{k} \sum_{i=1}^{k} [\lambda R(\lambda; A)]^i x \right\| \tag{3.16}$$

for $x \in \Sigma$.

Conversely, let Σ *be a linear subset in* X *and* A *be a linear operator from* $D(A)$ *into* Σ *satisfying the conditions* (i″), (ii), (iii), (iv) *and* (v). *If* $N(x)$ *defined by* (3.16) *is finite and* Σ *is a Banach space with the norm* $N(x)$ *and further if* $D(A)$ *is dense in* Σ *with the norm* $N(x)$, *then there exists a semi-group* $\{T(\xi); 0 \leqq \xi < \infty\}$ *of class* $(0, A)$ *with* $\int_0^{\infty} \|T(\xi)x\| d\xi < \infty$ *for each* $x \in \Sigma$ *of which* A *is the infinitesimal generator,* Σ *is the* $(C, 1)$-*continuity set and for which*

$$N(x) = \sup_{\xi > 0} \left\| \frac{1}{\xi} \int_0^{\xi} T(\eta)x \, d\eta \right\| \qquad \text{for } x \in \Sigma.$$

PROOF. Suppose that $\{T(\xi); 0 \leqq \xi < \infty\}$ is of class (0, A) with $\int_0^{\infty} \|T(\xi)x\|$

$d\xi < \infty$ for each $x \in X$. If $R(\lambda; A)$ is defined by (2.3), the properties (i″) and (ii)-(v) are proved similarly as [5, Theorem 1].

Conversely, suppose that the conditions are satisfied. We obtain a semi-group $\{T(\xi); 0 \leqq \xi < \infty\}$ of class (0, A) with $\int_0^\infty \| T(\xi)x \| d\xi < \infty$ for each $x \in X$ similarly as in the proof of Theorem 3.1. We shall now prove that A is the infinitesimal generator of $\{T(\xi); 0 \leqq \xi < \infty\}$. If we define $R^*(\lambda; A^*)$, for λ with $\Re(\lambda) > 0$, by

$$R^*(\lambda; A^*)x = \int_0^\infty e^{-\lambda\xi} T(\xi)x\, d\xi \tag{3.17}$$

for all $x \in X$, and if we denote the $(C, 1)$-continuity set of $T(\xi)$ by Σ^* and its infinitesimal generator by A^*, then for each λ with $\Re(\lambda) > 0$ we have

$$\begin{cases} (\lambda - A^*)R^*(\lambda; A^*)x = x & \text{for } x \in \Sigma^*, \\ R^*(\lambda; A^*)(\lambda - A^*)x = x & \text{for } x \in D(A^*). \end{cases} \tag{3.18}$$

Then we have $R^*(\lambda; A^*) \equiv R(\lambda; A)$ for $\lambda > 0$ similarly as in the proof of Theorem 3.1. Since $\lim_{\xi \to 0} T(\xi)x = x$ for $x \in D(A^2)$, $D(A^2) \subset \Sigma^*$. Further, by (3.18), $D(A^*) = R^*(\lambda; A^*)[\Sigma^*] \subset R^*(\lambda; A^*)[X] = R(\lambda; A)[X] \subset \Sigma$.

We can see that Σ^* is a Banach space with the norm $N^*(x)$ defined by $N^*(x) = \sup_{\xi > 0} \left\| \xi^{-1} \int_0^\xi T(\eta)x\, d\eta \right\|$, $D(A^*)$ is dense in Σ^* with the norm $N^*(x)$ and that

$$N^*(x) = \sup_{k \geqq 1, \lambda > 0} \left\| \frac{1}{k} \sum_{i=1}^k [\lambda R^*(\lambda; A^*)]^i x \right\| \qquad \text{for } x \in \Sigma^*.$$

Accordingly, $N(x) = N^*(x)$ for $x \in \Sigma \cap \Sigma^*$ and $D(A^2) \subset \Sigma \cap \Sigma^*$, $D(A^*) \subset \Sigma \cap \Sigma^*$. Since $N(\lambda R(\lambda; A)x - x) = \sup_{k \geqq 1, \mu > 0} \left\| k^{-1} \sum_{i=1}^k [\mu R(\mu; A)]^i R(\lambda; A)Ax \right\| \leqq \frac{M}{\lambda} N(Ax)$ for $x \in D(A)$ and $R(\lambda; A)[D(A)] \subset D(A^2)$ is dense in Σ with the norm $N(x)$. Thus we get $\Sigma = \Sigma^*$. Finally we obtain from (3.18), (i″) and the strong Abel-ergodicity that

$$D(A^*) = D(A), \qquad A = A^*.$$

Theorem is now completely proved.

We note that Theorems 3.2 and 3.4 together give a necessary and sufficient condition that a linear operator is the infinitesimal generator of a semi-group of class $(0, C_\alpha)$. If $\{T(\xi); 0 \leqq \xi < \infty\}$ is a semi-group of class $(0, C_1)$, $\Sigma = X$ and the norm $N(x)$ is equivalent to the original one. Thus we get also Corollary 3.3′.

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