CONDITIONAL EXPECTATION IN AN OPERATOR ALGEBRA

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1. Introduction. The concept of the conditional expectation in probability theory is very important, especially fundamental for the martingale theory. In a book [1][1)] of J. Doob, various properties of the conditional expectation in a probability space are described for the random variables having the expectations. While in a recent paper [2], Shuh-teh C. Moy has discussed the characteristic properties of the conditional expectation as a linear transformation of the space of all extended real-valued measurable functions on a probability space into itself.

The present paper deals with the conditional expectation as a mapping of a space of measurable operators belonging to a L^1-integrable class associated with a certain W^*-algebra into itself. This generalization seems to be a first attempt of a non-commutative probability theory. The non-commutative integration theory[2)] of I. E. Segal (cf. [3]) has its due application in the subject.

We shall show in §2, the existence of the conditional expectation for the space of measurable operators of the L^1-integrable class associated with a certain W^*-algebra, and in §3, the uniqueness in a certain sense of such a mapping which is a generalization of a characterization theorem of S. C. Moy.

2. Existence of conditional expectation. Let A be a W^*-algebra, acting on a Hilbert space H, with a complete (faithful) normal trace μ with $\mu(1) = 1$.

Let A_1 be an arbitrary (but fixed) W^*-subalgebra of A. In this section we shall introduce a conditional expectation in A relative to A_1.

First we shall prove in $L^1(A)$ the existence theorem of conditional expectation where $L^1(A)$ consists of all integrable operators on H with respect to the L^1-norm $\|x\|_1 = \mu(|x|)$ (cf. [3] Def. 3. 2, Cor. 10. 1 and Cor. 11. 3) which are associated with the W^*-algebra A. Similarly we denote the space $L^1(A_1)$ associated with the W^*-subalgebra A_1, then $L^1(A_1)$ can be considered as a closed subspace of $L^1(A)$.

THEOREM 1.[3)] *There exists a mapping $x \to x^e$ from $L^1(A)$ onto $L^1(A_1)$ satisfying the following conditions: for any $x, y \in L^1(A)$ and any complex numbers α, β*

(i) $$(\alpha x + \beta y)^e = \alpha x^e + \beta y^e,$$

1) Numbers in brackets refer to the reference at the end of the paper.

2) J. Dixmier has also described the similar theory under a different way (cf. [4]). In the present paper, we shall use the definitions and terminologies of I. E. Segal (cf. [3]). We shall denote the product, sum and difference of measurable operators x,y merely by $xy, x+y$ and $x-y$, e. g., ey implies $x\cdot y$ in the notations in [3]. When $x=y$ nearly everywhere, we shall denote merely $x=y$ (n. e.) or $x=y$.

3) After we had proved the Thm 1, we have been pointed out by M. Nakamura that the existence of mapping $x\to x^e$ from A to A_1 was proved by Dixmier using his operator method (cf. Thm. 8 of [4]). In this paper, we shall prove Thm. 1 by Radon-Nikodym Thm. of Segal (cf. [3]) and extend it onto $L^1(A)$.

(ii) $$x^{*e} = x^{e*},$$

(iii) $$x \geqq 0 \textit{ implies } x^e \geqq 0,$$

(iv) $$x \geqq 0 \textit{ and } x^e = 0 \textit{ imply } x = 0,$$

(v) $$z^e = z \qquad \textit{for any } z \textit{ in } L^1(A_1).$$

Moreover the mapping $x \to x^e$ *transforms* A *onto* A_1 *satisfying* $\|x^e\| \leqq \|x\|$, $x^{*e}x^e \leqq (x^*x)^e$ *and*

(vi) $$(x^e y)^e = (xy^e)^e = x^e y^e \qquad \textit{for } x \in L^1(A),\ y \in A \textit{ or } x \in A,\ y \in L^1(A),$$

(vii) $$x_\gamma \uparrow x \textit{ implies } x_\gamma^e \uparrow x^e \qquad \textit{for } x_\gamma,\ x \in A,$$

(vii)′ $x \to x^e$ *is strongly and weakly continuous on the unit sphere of* A,

(viii) $$(xy)^e = (yx)^e \qquad \textit{for } x \in L^1(A) \textit{ and } y \in A_1' \cap A,$$

(ix) $$\mu(|x^e|) \leqq \mu(|x|).$$

PROOF. For any $x \in A^+$, putting $\mu_x(z) = \mu(xz)$ for $z \in A_1$, μ_x is a positive linear functional on A_1 satisfying that $|\mu_x(z)| \leqq \|x\| \cdot \mu(|z|)$ for all $z \in A_1$. By a lemma of Segal (cf. Lem. 14.1 of [3]), there exists a unique positive operator x' in A_1 such that $\mu_x(z) = \mu(x'z)$ for all $z \in A_1$ where the operator x' is uniquely determined by x. Putting $x^e = x'$, $(\alpha x + \beta y)^e = \alpha x^e + \beta y^e$ for any $x, y \in A^+$ and numbers α, $\beta \geqq 0$. Since any $x (\in A) = x_1 - x_2 + ix_3 - ix_4$ (for some $x_k \in A^+$) and this expression is unique, putting $x^e = x_1^e - x_2^e + ix_3^e - ix_4^e$, $x \to x^e$ is defined for all $x \in A$ and it satisfies that

(1) $$\mu(xz) = \mu(x^e z) \qquad \text{for any } x \in A \text{ and } z \in A_1.$$

It is easily seen that $z_1 = z_2$ $(z_1, z_2 \in A_1)$ if and only if $\mu(z_1 z) = \mu(z_2 z)$ for all $z \in A_1$. This fact implies that the introduced mapping $x \to x^e$ is well defined on A, transforms A onto A_1 and satisfies the conditions i), ii) and v) for all $x, y \in A$ and $z \in A_1$. iii) is clear by the construction of x^e in A. iv) for $x \in A$ follows from (1), iii) and the completeness of μ. Moreover for any $x, y \in A$ and $z \in A_1$

$$\mu((x^e y)^e z) = \mu(x^e yz) = \mu(yzx^e) = \mu(x^e y^e z) = \mu((xy^e)^e z),$$

hence vi) holds for $x, y \in A$. The normal continuity of $x \to x^e$ (i.e., vii)) or more generally vii)′ follow from that iv), $x^{*e}x^e \leqq (x^*x)^e$ (as below) and the following fact: The trace μ is represented by a canonical trace, i.e., $\mu(x) = (x\zeta, \zeta)$ for some $\zeta \in H$, and for any $\xi \in H$ there exists a vector ξ' in $[A\zeta]$ such that $(x\xi, \xi) = (x\xi', \xi')$ for all $x \in A$ (by the Radon-Nikodym theorem of Segal, cf. Thm. 14 of [3]), and for any $y \in A$ there is $z \in A$, such that $\mu(x^e yy^*) = \mu(xzz^*)$. While for any $z \in A_1$, $x \in A$ and $y \in A_1' \cap A$

$$\mu((xy)^e z) = \mu(xyz) = \mu(xzy) = \mu(yxz) = \mu((yx)^e z),$$

hence viii) holds for such x, y in A. ix): For $x \in A$ there exists a partially isometric operator $w \in A_1$ such that $|x^e| = wx^e$, hence $\mu(|x^e|) = \mu(wx^e) \leqq \|w\| \cdot \mu(|x|) \leqq \mu(|x|)$ and we have ix) for $x \in A$. $\|x^e\| \leqq \|x\|$ is clear by the construction of $x \to x^e$. Since for any $x \in A$

$$0 \leqq ((x - x^e)^*(x - x^e))^e = (x^*x - x^{*e}x + x^{*e}x^e - x^*x^e)^e = (x^*x)^e - x^{*e}x^e,$$

we have $x^{*e}x^e \leqq (x^*x)^e$.

Since all $x \in A$ or A_1 are elementary operators respectively (cf. [3]), A and A_1 are dense in $L^1(A)$ and $L^1(A_1)$ respectively. By ix) for $x \in A$, $x \to x^e$ can be extended onto $L^1(A)$ and i), ix) hold for $x, y \in L^1(A)$. For any $x \in L^1(A)$ (or $L^1(A_1)$ resp.), taking $\{x_n\} \subset A$ (or A_1 resp.) such that

$$\mu(|x_n - x|) \to 0 \qquad (n \to \infty), \tag{2}$$

since $\mu(|y^*|) = \mu(|y|)$ for any $y \in L^1(A)$, we have ii) for $x \in L^1(A)$. (2) also implies that

$$\mu(x^e z) = \lim (\mu x_n^e z) = \lim \mu(x_n z) = \mu(xz) \qquad \text{for all } z \in A_1. \tag{3}$$

Since v) holds for A_1, by (2) it also holds for $L^1(A_1)$. If $x \geqq 0$ ($x \in L^1(A)$), there exists a $\{x_n\} \subset A^+$ satisfying (2) (by Cor. 12.1 of [3]). Whence $x^e = \lim x_n^e$ (in L^1-mean) and $x^e \geqq 0$ by Lem. 13.3 of [3]; and if $x \geqq 0$ and $x^e = 0$, then

$$\mu(x) = \mu(xI) = \mu(x^e) = 0$$

and $x = 0$ (n. e.), and we have iii) and iv) for $L^1(A)$. For any $x \in L^1(A)$ and $y \in A$ taking $\{x_n\} \subset A$ as (2),

$$(x^e y)^e = \lim (x_n^e y)^e = \lim x_n^e y^e = x^e y^e$$

and similarly $= (xy^e)^e$. We have also viii) by the similar way taking the sequence $\{x_n\}$ in A. The later part of vi) follows from the former and ii).

Q. E. D.

REMARK 1. Applying the proof of Thm. 1, it holds that

$$(xy)^e = (yx)^e \tag{viii)$'$}$$

for $x \in A$ and $y \in L^1(A_1' \cap A)$ which is the integrable operator associated with the W^*-subalgebra $A_1' \cap A$.

We shall call the mapping $x \to x^e$ from $L^1(A)$ to $L^1(A_1)$ the *conditional expectation relative to* A_1.

3. A characterization of conditional expectation. In this section, we shall prove a characterization theorem of the conditional expectation which is a generalization of a theorem of Shuh-teh Chen Moy (cf. Thm. 2.2 of [2]).

THEOREM 2. *Let $x \to x^\varepsilon$ be a mapping from A into itself satisfying* i), ii), vi), ix) *for $x, y \in A$, and*

$$I^\varepsilon = I. \tag{v)$'$}$$

Then the range A^ε of the mapping $x \to x^\varepsilon$ is a W^*-subalgebra of A and $x \to x^\varepsilon$ coincides with the conditional expectation relative to A^ε, that is,

$$x^\varepsilon = x^e \qquad \textit{for all } x \in A.$$

PROOF. Let A_0 be the collection of all $z \in A$ such that

$$(zx)^\varepsilon = zx^\varepsilon \text{ and } (xz)^\varepsilon = x^\varepsilon z \qquad \text{for all } x \in A.$$

Then A_0 is a self-adjoint subalgebra of A containing I. Indeed, for any z_1, z_2, $z \in A_0$ and $x \in A$,

$$((z_1 + z_2)x)^\varepsilon = (z_1 x)^\varepsilon + (z_2 x)^\varepsilon = z_1 x^\varepsilon + z_2 x^\varepsilon = (z_1 + z_2)x^\varepsilon,$$

$$(z^*x)^\epsilon = (x^*z)^{*\epsilon} = (x^*z)^{\epsilon *} = (x^{*\epsilon}z)^* = (x^{\epsilon *}z)^* = z^*x^\epsilon,$$
$$(z_1z_2x)^\epsilon = z_1(z_2x)^\epsilon = z_1z_2x^\epsilon,$$
$$(\alpha\, zx)^\epsilon = \alpha\, zx^\epsilon,$$

and similarly $(x(z_1+z_2))^\epsilon = x^\epsilon(z_1+z_2)$, ...etc. hold. Clearly $I \in A_0$. Let A_1 be the weak closure of A_0, and $x \to x^e$ be the conditional expectation relative to A_1. Putting $\mu(x) = \mu(x^\epsilon)$, since

$$|\mu_1(x)| \leqq \mu(|x^\epsilon|) \leqq \mu(|x|),$$

μ_1 is a bounded linear functoinal on $L^1(A)$. Hence there exists an operator r in A such that

$$\mu_1(x) = \mu(x^\epsilon) = \mu(xr) \qquad \text{for all } x \in A$$

(cf. Cor. 18.1 of [3] or Thm. 5 of [4]). Therefore for any $z \in A_0$,

$$\mu(zx^\epsilon) = \mu((zx)^\epsilon) = \mu(zxr) = \mu(z(xr)^e). \tag{4}$$

Since A_0 is strongly dense in A_1 and the both sides of the equation (4) are strongly continuous for $z \in A_0$,

$$\mu(zx^\epsilon) = \mu(z(xr)^e) \qquad \text{for all } z \in A_1.$$

Hence $x^\epsilon = (xr)^e$. Since

$$\mu(xr) = \mu(x^\epsilon) = \mu(x^{*\epsilon *}) = \overline{\mu(x^{*\epsilon})} = \overline{\mu(x^*r)} = \mu((x^*r)^*) = \mu(r^*x) = \mu(xr^*)$$

for all $x \in A$, $r^* = r$. By $|\mu(xr)| \leqq \mu(|x|)\|r\|$ and $\|r^2\| \leqq 1$, hence $r^2 \leqq I$. While

$$r^e = (Ir)^e = I^\epsilon = I$$

and by iii), iv)

$$0 \leqq ((I-r)(I-r))^e = (I - 2r^e + r^2)^e = I - 2r^e + (r^2)^e = (r^2)^e - I \leqq 0.$$

Therefore $((I-r)(I-r))^e = 0$ and $r = I$. This implies immediately $x^\epsilon = x^e$ and $A^\epsilon = A_0 = A_1$. Q. E. D.

Now we consider a normal continuous mapping $x \to x^\epsilon$ on A without the assumptions v)' and ix) (cf. Thm. 1.1 of [2]).

THEOREM 3. *Let* $x \to x^\epsilon$ *be a mapping from* A *into itself satisfying* i), iii), iv) *and* vii) *for* x, y *and* x_γ *in* A, *then there exists a unique positive operator* $r \in L^1(A)$ *such that*

$$I^\epsilon = r^e \text{ and } x^\epsilon = (xr)^e = (rx)^e \qquad \text{for all } x \in A, \tag{5}$$

where $x \to x^e$ *is a conditional expectation relative to a* W^**-subalgebra determined by the mapping* $x \to x^\epsilon$.

PROOF. Putting $\mu_1(x) = \mu(x^\epsilon)$ for all $x \in A$, μ_1 is a positive linear function and strongly continuous on the unit sphere S_0 of A. For, since $x \to x^\epsilon$ is weakly continuous on S_0 (cf. Dixmier's Cor. 1 of [4]), μ_1 is also weakly continuous on S_0, and hence strongly continuous on S_0 because μ_1 is a numerical function. By the Radon-Nikodym theorem of Segal (cf. Thm. 14 of [3]), there exists a positive operator $r \in L^1(A)$ such that

$$\mu_1(x) = \mu(xr) \qquad \text{for all } x \in A.$$

We now prove i) and iii) imply ii). Since any x in A can be expressed by

$x = x_1 - x_2 + ix_3 - ix_4$ for $x_k \in A$, we have $x^* = x_1 - x_2 - ix_3 + ix_4$. i) and iii) imply that $x_k^e \geqq 0$ $(k = 1, \ldots, 4)$ and $x^{*e} = x_1^e - x_2^e - ix_3^e + ix_4^e$, and the right side equals to x^{e*} hence we have $x^{*e} = x^{e*}$.

Taking A_0 as in the proof of Thm. 2, the weak closedness of A follows from the normal continuity of x^e and a theorem of Dixmier (cf. Nakamura-Turumaru [5]), that is, $A_0 = A_1$. Let $x \to x^e$ be the conditional expectation relative to A_1. Then by (3) in the proof of Thm. 1, for all $z \in A_1$,

$$\mu(zx^e) = \mu((zx)^e) = \mu(zxr) = \mu(z(xr)^e)$$

which implies that $x^e = (xr)^e$ and $I^e = r^e$. Moreover for any $x \in A$,

$$(xr)^e = x^e = x^{*e*} = (x^*r)^{e*} = (x^*r)^{*e} = (r^*x)^e = (rx)^e. \qquad \text{Q. E. D.}$$

REMARK 2. If the mapping $x \to x^e$ in Thm. 3 is L^1-continuous instead of the normal continuity, then we can find a positive operator r in A satisfying (5). More generally, for a mapping $x \to x^e$ from A into itself satisfying i), ii), vi) for all $x, y \in A$ and the L^1-continuity, (5) also holds for s. a. $r \in A$ where the conditional expectation $x \to x^e$ is taken relative to the W^*-subalgebra A_1 (cf. the proof of Thm. 2), and this fact results that the normal, strong and weak continuities (on the unit sphere of A) of the mapping $x \to x^e$ and $A^e = A_1$.

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