COHOMOLOGY THEORY AND DIFFERENT

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The relations between cohomology groups and different in the number theory were already treated by A. Weil [11], Y. Kawada [6], A. Kinohara [7] and M. Moriya [9] in cases of dimension 1 and 2. In the present paper we shall treat the same subjects for general dimensions under a slight modification.

In § 1 we shall explain the definitions and main results of this note. In § 2 we shall prove the equalities of the right-, left- and two sided homological differents. § 3 and § 4 are preliminaries for the following sections. In § 5 we shall prove, essentially, that the homological different is not zero, and in § 6 we shall treat the reduction to the local homological different. In § 7 we shall consider the local homological different and prove the different theorem, and in § 8 we shall show the equality between homological differents and the usual different.

1. Definitions and results. Let R be a Dedekind ring, K its quotient field, L a finite separable extension field over K and Λ the principal order (the unique maximal order) of L over R. We regard Λ as an algebra over R.[1] For any two sided Λ-module A, the homology groups $H_n(\Lambda, A)$ and the cohomology groups $H^n(\Lambda, A)$ are defined as usual [1] i.e.

$$
\begin{aligned}
H_n(\Lambda, A) &= \mathrm{Tor}_n^{\Lambda^e}(A, \Lambda), \\
H^n(\Lambda, A) &= \mathrm{Ext}_{\Lambda^e}^n(\Lambda, A).
\end{aligned} \tag{1.1}
$$

An element $\lambda^e = \sum \lambda \otimes \mu$ of Λ^e induces a Λ^e-endomorphism $\bar{\lambda^e}$ of A

$$
\bar{\lambda^e}: A \to A, \quad \bar{\lambda^e}(a) = \lambda^e a; \tag{1.2}
$$

$\bar{\lambda^e}$ induces an endomorphism $\widetilde{\lambda^e}$ of $H(\Lambda, A)$

$$
\begin{aligned}
\widetilde{\lambda^e}:\ & H_n(\Lambda, A) \to H_n(\Lambda, A), \\
& H_n(\Lambda, A) \to H_n(\Lambda, A).
\end{aligned} \tag{1.3}
$$

Therefore $H(\Lambda, A)$ may be considered as a Λ^e-module. Using these endomorphisms $\widetilde{\lambda^e}$, we define the n-homological (cohomological) different of Λ/R.

DEFINITION 1. Left n-homological and cohomological differents $D_n^l(\Lambda/R)$ and $D_l^n(\Lambda/R)$:

$$
\begin{aligned}
D_n^l(\Lambda/R) &= \{\lambda \in \Lambda \mid \widetilde{\lambda \otimes 1}\, H_n(\Lambda, A) = 0 \quad \text{for all } A\}, \\
D_l^n(\Lambda/R) &= \{\lambda \in \Lambda \mid \widetilde{\lambda \otimes 1}\, H^n(\Lambda, A) = 0 \quad \text{for all } A\}.
\end{aligned}
$$

1) In the following our main objects are these algebras, which we shall quote as "the number theoretical algebras" or "the number theoretical cases".

DEFINITION II. Right n-homological and cohomological differents $D_n^r(\Lambda/A)$ and $D_r^n(\Lambda/R)$:

$$D_n^r(\Lambda/R) = \{\lambda \in \Lambda \mid 1 \otimes \widetilde{\lambda} H_n(\Lambda, A) = 0 \qquad \text{for all } A\},$$

$$D_n(\Lambda/R) = \{\lambda \in \Lambda \mid 1 \otimes \widetilde{\lambda} H^n(\Lambda, A) = 0 \qquad \text{for all } A\}.$$

DEFINITION III. n-homological and cohomological differents $D_n(\Lambda/R)$ and $D^n(\Lambda/R)$:

$$D_n^e(\Lambda/R) = \{\Sigma\lambda \otimes \mu \in \Lambda^e \mid \Sigma\lambda \widetilde{\otimes} \mu H_n(\Lambda, A) = 0 \qquad \text{for all } A\},$$

$$D_n^e(\Lambda/R) = \{\Sigma\lambda \otimes \mu \in \Lambda^e \mid \Sigma\lambda \widetilde{\otimes} \mu H^n(\Lambda, A) = 0 \qquad \text{for all } A\},$$

$$D_n(\Lambda/R) = \rho(D_n^e(\Lambda/R)),$$

$$D^n(\Lambda/R) = \rho(D_e^n(\Lambda/R)),$$

where ρ is a Λ^e-homomorphism of Λ^e to Λ

$$\rho : \Lambda^e \to \Lambda, \qquad \rho(\lambda \otimes \mu) = \lambda\mu. \tag{1.4}$$

Since Λ is commutative, ρ is also a ring homomorphism of Λ^e to Λ.

DEFINITION IV. Commutative n-homological and cohomological differents $D_n^c(\Lambda/R)$ and $D_c^n(\Lambda/R)$. We denote by A_c the module in which $\lambda a = a \lambda$ for any $a \in A$ and $\lambda \in \Lambda$.

$$D_n^c(\Lambda/R) = \{\lambda \in \Lambda \mid \lambda \widetilde{\otimes} 1 \; H_n(\Lambda, A_c) = 0 \qquad \text{for all } A_c\},$$

$$D_c^n(\Lambda/R) = \{\lambda \in \Lambda \mid \lambda \widetilde{\otimes} 1 \; H^n(\Lambda, A_c) = 0 \qquad \text{for all } A_c\}.$$

Since $D^1{}_c(V/R)$ is the annulator of modules of derivations, this Def. IV corresponds to the definition in [6]. We may easily construct the different theory concerning $D^1{}_c(\Lambda/R)$.

Obviously these differents are ideals in Λ. Now, we explain the main results.

I (Cor. 2. 3)

$$D^n(\Lambda/R) = D_l^n(\Lambda/R) = D_r^n(\Lambda/R) = D_c^n(\Lambda/R),$$

$$D_n(\Lambda/R) = D_n^l(\Lambda/R) = D_n^r(\Lambda/R) = D_n^c(\Lambda/R).$$

II (Th. 6. 2)

$$D^n(\Lambda/R) \neq 0, \quad D_n(\Lambda/R) \neq 0.$$

III (Th. 7. 5)

Let $\mathfrak{P}$ be any prime in Λ, let n be a fixed integer $n \geqq 1$. Then $\mathfrak{P}$ divides $D^n(\Lambda/R)$ if and only if $\mathfrak{P}$ is ramified or inseparable. The result is also true for $D_n(\Lambda/R), n > 1$.

As a consequence of II and III, we know that, for any fixed n, $D^n(\Lambda/R)$ (or $D_n(\Lambda/R)$) plays the same rôle as the usual different.

IV (Th. 8, 6)

The homological and cohomological differents of any dimension are all equal to the usual different $\mathfrak{D}$ defined by $Sp_{L/K}$.

Though we may obtain II and III as an immediate consequence of IV, it is desirable to obtain them independent with the theory of the usual different $\mathfrak{D}$. In the present paper we shall prove them using D^n of given dimension n only, independent from the other D^m and $\mathfrak{D}$.

As for the chain theorem, we may prove it by using the local cohomological 0-different $D^0(\Lambda_\mathfrak{p}/R_\mathfrak{p})$. But, since the proof is essentially dependent with the theory of the usual different, we shall not state it here.

We also obtain a theorem similar to the theorem of Dedekind (Th. 8.7).

2. $\mathbf{D}(\Lambda/\mathbf{R}) = \mathbf{D}_l(\Lambda/\mathbf{R}) = \mathbf{D}_r(\Lambda/\mathbf{R})$. Let R be a commutative ring, Λ an algebra over R, A a Λ^e-module[2] and $\sum\lambda\otimes\mu^*$ an element in the center of Λ^e (we denote it briefly by λ^e). Similar to § 1, we have an induced endomorphism $\widetilde{\lambda^e}$ of $H(\Lambda, A)$,

$$(1.3)\qquad \widetilde{\lambda^e} : H(\Lambda, A) \to H(\Lambda, A).$$

On the other hand, $\widetilde{\lambda^e}$ is also considered as follows: Let

$$(2.1)\qquad \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} \Lambda \xrightarrow{\varepsilon} 0$$

be a Λ^e-projective resolution of Λ. Since $\lambda^e = \sum\lambda\otimes\mu^*$ induces a Λ^e-endomorphism $\overline{\lambda^e}$ of Λ

$$(2.2)\qquad \begin{aligned} &\overline{\lambda^e} : \Lambda \to \Lambda, \\ &\lambda^e(x) = \sum\lambda\gamma\mu \qquad \text{for } x \text{ in } \Lambda, \end{aligned}$$

there exists an extended Λ^e-endomorphism $\hat{\lambda^e}$ of X over $\overline{\lambda^e}$,

$$(2.3)\qquad \hat{\lambda^e} : X \to X,$$

and any two such maps are homotopic. Therefore, the map (2.2) induces a uniquely determined endomorphism of $H(\Lambda, A)$,

$$(2.4)\qquad \hat{\lambda^e} : H(\Lambda, A) \to H(\Lambda, A).$$

We may take the following map as one of the extended maps in (2.3):

$$(2.5)\qquad \begin{aligned} &\hat{\lambda^e} : X \to X \\ &\hat{\lambda^e}(x) = \sum\lambda x\mu, \end{aligned}$$

since $d_i(\lambda^e x_i) = \lambda^e d_i(x_i)$, $x_i \in X_i$. The induced map of (2.5) is

$$(2.6)\qquad \begin{aligned} &\hat{\lambda^e}(f(x)) = f(\hat{\lambda^e}(x)) = f(\lambda^e x) = \lambda^e f(x), && f(x) \in \operatorname{Hom}_{\Lambda^e}(X, A), \\ &\hat{\lambda^e}(a\otimes x) = a\otimes\hat{\lambda^e}x = a\otimes\sum\lambda x\mu = \sum a\mu\otimes x, && a\otimes x \in A\otimes_{\Lambda^e} X, \end{aligned}$$

which is the induced map (1.3). Thus we have

PROPOSITION 2.1. *The induced map* $\hat{\lambda^e}$ *of* (2.2) *is the same as the induced map* $\hat{\lambda^e}$ *of* (1.2)

COROLLARY 2.2 *If* λ *is an element in the center of* Λ, *the left operation induced by* λ *on* $H(\Lambda, A)$ *coincides with the right operation induced by* λ, *i.e.*

$$\widetilde{\lambda\otimes 1}\cdot u = \widetilde{1\otimes\lambda^*}\cdot u$$

2) It will always be assumed that R and Λ have the unity element in common, and the unity element acts also as the identity on all modules.

for any u in $H(\Lambda, A)$.

PROOF. Indeed, $\lambda\otimes 1 - 1\otimes\lambda^*$ induces the 0-endomorphism on Λ and the 0-endomorphism of X is one of its extended endomorphism.

COROLLARY 2.3. *In the number theoretical case we have*

$$D^n(\Lambda/R) = D^n_l(\Lambda/R) = D^n_r(\Lambda/R),$$
$$D_n(\Lambda/R) = D^l_n(\Lambda/R) = D^n_r(\Lambda/R)$$

for $n = 1, 2, \ldots$.

Next we consider the relations between D^n and D^n_c. Let Λ be a commutative algebra over R and A any Λ^e-module.

PROPOSITION 2.4.

$$\mathrm{Hom}_{\Lambda^e}(\Lambda, H^n(\Lambda, A)) \cong H^n(\Lambda, A) \tag{2.7}$$

$$\Lambda\otimes_{\Lambda^e} H_n(\Lambda, A) \cong H_n(\Lambda, A) \tag{2.8}$$

as Λ^e-modules.[2')]

PROOF. For each $u \in H^n(\Lambda, A)$ the map $1 \to u$ induces a Λ^e-homomorphism u_0 of Λ to $H^n(\Lambda, A)$ since $\lambda u = u\lambda$ for any λ in Λ. The mapping $u \to u_0$ is a Λ^e-epimorphism of $H^n(\Lambda, A)$ to $\mathrm{Hom}_{\Lambda^e}(\Lambda, H^n(\Lambda, A))$ which is also an isomorphism. Similariy, the mapping $u \to 1\otimes u$ is a Λ^e-isomorphism of $H_n(\Lambda, A)$ to $\Lambda\otimes_{\Lambda^e} H_n(\Lambda, A)$ since $\lambda\, u = u\, \lambda$ for any λ in Λ.

PROPOSITION 2.5. *We have the exact sequences*

$$0 \to \mathrm{Hom}_{\Lambda^e}(\Lambda, H^n(\Lambda, A)) \xrightarrow{i} H^n(\Lambda, \mathrm{Hom}_{\Lambda^e}(\Lambda, A)) \tag{2.9}$$

$$0 \to \Lambda\otimes_{\Lambda^e} H_n(\Lambda, A) \xrightarrow{i'} H_n(\Lambda, \Lambda\otimes_{\Lambda^e} A), \tag{2.10}$$

where i and i' are Λ^e-isomorphism.[2')]

PROOF. Let X be a Λ^e-projective resolution over Λ, then $X\otimes_{\Lambda^e}\Lambda = \Lambda\otimes_{\Lambda^e} X$ since Λ^e-left modules are two sided Λ module and also considered to be Λ^e-right modules. X is considered as Λ^e left-Λ^e right module since Λ is commutative, so we have

$$\mathrm{Hom}_{\Lambda^e}(\Lambda, \mathrm{Hom}_{\Lambda^e}(X, A)) \cong \mathrm{Hom}_{\Lambda^e}(X\otimes_{\Lambda^e}\Lambda, A) \cong \mathrm{Hom}_{\Lambda^e}(X, \mathrm{Hom}_{\Lambda^e}(\Lambda, A)).$$

From this isomorphism we have the first half of the assertion.

Similarly, we have the second part from the isomorphism

$$\Lambda\otimes_{\Lambda^e}(A\otimes_{\Lambda^e} X) \cong (\Lambda\otimes_{\Lambda^e} A)\otimes_{\Lambda^e} X$$

where A is considered as Λ^e-Λ^e two sided module.
The last part is obvious from the definition of the operations.

2') Let A and B be two sided Λ-modules. Since Λ is commutative, the operator λ^e of A induces an operator on $\mathrm{Hom}_{\Lambda^e}(B, A)$ and $B\otimes_{\Lambda^e} A$ as follows:

$$(\lambda\otimes\mu^*)\otimes f(b) = f(\mu b\lambda),\quad (\lambda\otimes\mu^*)(b\otimes a) = b\otimes\mu a\lambda = (b\otimes a)(\lambda\otimes\mu^*).$$

It also induces the operation $\widetilde{\lambda\otimes\mu}$ on $H^n(\Lambda, A)$ and $H_n(\Lambda, A)$ (cf (1.3)). Combining these process we have the operations on modules in (2.7)~(2.10).

COROLLARY 2.6.

$$D_c^n(\Lambda/R) = D^n(\Lambda/R), \quad D_n^c(\Lambda/R) = D_n(\Lambda/R)$$

PROOF. Obviously $D_c^n(\Lambda/R) \supset D^n(\Lambda/R)$. Conversely, by (2.9) and (2.7) we have $D^n(\Lambda/R) \supset D_c^n(\Lambda/R)$ since $\mathrm{Hom}_{\Lambda^e}(\Lambda, A)$ is one of the A_c.[2'] Similarly we have $D_n^c(\Lambda/R) = D_n(\Lambda/R)$ by (2.10) and (2.8)

3. Preliminaries about symmetric algebras. In this section we shall explain some properties about symmetric algebras. As for the details we refer [3] and [8].

Let R be a commutative ring and A an R-module, then we denote the dual R-module $\mathrm{Hom}_R(A, R)$ by A^0. If Λ is an algebra over R and A is a left Λ-module, then A^0 is a right Λ-module. If A is a two sided Λ-module, then A^0 is a two sided Λ-module; in particular, Λ^0 is also a two sided Λ-module.

Let Λ be an R-algebra, R-projective and finitely R-generated. Then Λ is called a Frobenius algebra when there exists an isomorphism Φ of Λ to Λ^0 as left Λ-modules. We say that Λ is a symmetric algebra when there exists an isomorphism Φ of Λ to Λ^0 as two sided Λ-modules.

If Λ is a Frobenius algebra over R, $\varphi = \Phi(1)$ is an R-homomorphism of Λ to R and

(3.3) $$[\Phi(r)](\lambda) = \varphi(\lambda r), \qquad \text{for any } r, \lambda \text{ in } \Lambda.$$

Conversely, starting from an R-homomorphism φ of Λ to R, we may define a left Λ-homomorphism Φ of Λ to Λ^0 by (3.3). Then the conditions that Φ is isomorphic and onto are equivalent respectively to the following conditions:

(l.1) if $\varphi(\lambda r) = 0$ for all λ in Λ then $r = 0$,

(l.2) for any f in Λ_0 there exists r in Λ such that

$$f(\lambda) = \varphi(\lambda r).$$

The condition that Φ is two sided Λ-homomorphism is reduced to

(s) $$\varphi(\lambda r) = \varphi(r\lambda), \qquad \text{for any } r, \lambda \text{ in } \Lambda.$$

We coonsider an R-free Frobenius algebra Λ over R. Let $u_1, \dots, u_n$ be a linearly independent basis of Λ over R, then there exists a linearly independent basis $v_1, \dots, v_n$ of Λ such that

(3.4) $$\varphi(u_i v_j) = \delta_{ij}.$$

The left regular representation of Λ by $u_1, \dots, u_n$ is the same as the right regular representation by $v_1, \dots, v_n$, i.e.

(3.5) $$\lambda(u_i) = (u_i)(a_{ij}), \qquad (v_i)\lambda = (a_{ij})(v_j).$$

PROPOSITION 3.1. *Let Λ be an R-free algebra over R and $u_1, \dots, u_n$ a linearly independent basis of Λ over R. If there exists R-homomorphism φ of Λ to R and a system of elements $v_1, \dots, v_n$ of Λ such that $\varphi(u_i v_j) = \varphi(v_j u_i) = \delta_{ij}$, then Λ is symmetric over R.*

PROOF. φ satisfies (l.1), (l.2) and (s). If $\varphi(r\lambda)=0$ for an element $\lambda=\sum a_i u_i$ and any r in Λ, then $a_i=\varphi(v_i\sum a_j u_j)=0$, so $\lambda=0$. For any f in $\mathrm{Hom}_R(\Lambda, R)$ we have $f(r)=\varphi(r\sum_i f(u_i)v_i)$.

Division algebras and full matric algebras over R are symmetric; tensor products over R of symmetric algebras over R are also symmetric.

Let Λ be an R-free symmetric algebra over R, $(u_1,\dots, u_n)$, $(v_1,\dots,v_n)$ dual basis of Λ over R and A a two sided Λ-module. We may consider the standard complete complex of Λ with augmentation [11]:

$$(3.6)\cdots\cdots\longrightarrow X_1\xrightarrow{d_1}X_0\xrightarrow{d_0}X_{-1}\xrightarrow{d_{-1}}X_{-2}\longrightarrow\cdots\cdots$$
$$\begin{array}{ccc} X_0 & & X_{-1}\\ \varepsilon\searrow & & \nearrow\varepsilon' \\ \Lambda & \xrightarrow[\Phi]{} & \Lambda^0\end{array}$$

We define as usual

(3.7) $$H^n(\Lambda, A)=H^n(\mathrm{Hom}_{\Lambda^e}(X, A)),\ n=\dots, -1, 0, 1, \dots.$$

The 0 and -1 dimensional cohomology groups are

(3.8) $$H^0(\Lambda, A)=A^\Delta/\big(\sum_i u_i\otimes v_i^*\big)A,$$
$$H^{-1}(\Lambda, A)=A_i^{\Sigma(u_i\otimes v_i^*)}/\Delta A,$$

where $A^\Delta=\{a\in A\,|\,\lambda\, a=a\,\lambda \text{ for all } \lambda\in\Lambda\}$,

(3.9) $$\big(\sum_i u_i\otimes v_i^*\big)A=\big\{\sum_i u_i a v_i\,\big|\,a\in A\big\},$$
$$A_i^{(\Sigma u_i\otimes v_i^*)}=\big\{a\in A\,\big|\,\sum_i u_i a v_i=0\big\}$$
$$\Delta A=\text{submodule of } A \text{ generated by } \lambda a-a\lambda,\ a\in A,\ \lambda\in\Lambda.$$

The other negative dimensional cohomology groups coincide with the homology groups of Λ over A, i.e. there exists an isomorphism

(3.10) $$\sigma: H^{-n}(\Lambda, A)\approx H_{n-1}(\Lambda, A),\qquad n=2, 3, \dots.$$

If τ is a Λ^e-homomorphism of A into B,
$$\tau: A\to B,$$
then the diagram

(3.11) $$\begin{array}{ccc} H^{-n}(\Lambda, A) & \xrightarrow{\tau} & H^{-n}(\Lambda, B)\\ \sigma\downarrow & & \sigma\downarrow\\ H_{n-1}(\Lambda, A) & \xrightarrow{\tau} & H_{n-1}(\Lambda, B)\end{array}$$

is commutative for $n=2, 3, \dots$.

If A', A and A'' are Λ^e-modules and
$$0\to A'\to A\to A''\to 0$$
is an exact sequence of Λ^e-homomorphisms, then the sequence

(3.12) $$\cdots\cdots\to H^n(\Lambda, A')\to H^n(\Lambda, A)\to H^n(\Lambda, A'')\to H^{n+1}(\Lambda, A')\to H^{n+1}(\Lambda, A)\to\cdots\cdots$$

is exact.

From the definition we have

PROPOSITION 3.2. *If I is a Λ^e-injective module then*

$$H^n(\Lambda, I) = 0$$

for any integer n.

PROOF. $\mathrm{Hom}_{\Lambda^e}(\ , I)$ is an exact functor.

PROPOSITION 3.3. *If P is Λ^e-projective then $H^n(\Lambda, P) = 0$ ($n \neq 0, 1$).*

PROOF. It is sufficient to prove the prop. for Λ^e-free F.

For $n < -1$: We have $H^n(\Lambda, P) \cong \mathrm{Tor}^{\Lambda^e}_{1-n}(P, \Lambda) = 0$.

For $n > 0$: $F = \Lambda^e \otimes H$ where H is an R-free R-module.
Then [3, Prop. 7]

$$H^n(\Lambda, \Lambda^e \otimes_R H) \cong \mathrm{Ext}^n_{\Lambda^e}(\Lambda, \Lambda^e \otimes_R H) \cong \mathrm{Ext}^n_R(\Lambda, H),$$

where $\mathrm{Ext}^n_R(\Lambda, H) = 0$ because Λ is R-projective.

4. The element $\sum u_i v_i$. As we have explained above, the element $\sum u_i \otimes v_i^*$ of symmetric algebras plays the same rôle as the norm of groups. If $A = A^\Lambda$ i.e. $\lambda\, a = a\, \lambda$ for all $a \in A$ and $\lambda \in \Lambda$, then it reduces to $\sum u_i v_i$. In this section we prepare some propositions about $\sum u_i v_i$.

Let Λ be an R-free commutative symmetric algebra over R, let φ be a defining R-homomorphism of Λ to R and let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be the dual basis of Λ over R.

PROPOSITION 4.1. *If $u_1', \ldots, u_n'$ is another (linearly independent) basis of Λ over R and $v_1', \ldots, v_n'$ is its dual basis with respect to φ, then*

$$\sum u_i v_i = \sum u_i' v_i' \tag{4.1}$$

PROOF. Let (a_{ij}) be the matrix in R such that $u_i' = \sum a_{ij} u_j$ and (b_{ij}) the inverse matrix of (a_{ij}) then $v_1', \ldots, v_n'$, $v_k' = \sum b_{lk} v_l$ is the dual basis of $(u_1', \ldots, u_n')$: For

$$\varphi(u_i' v_k') = \sum_{j,l} a_{ij} b_{lk} \varphi(u_j v_l) = \sum_j a_{ij} b_{jk} = \delta_{ik}.$$

Hence

$$\sum u_i' v_i' = \sum_{j,li} a_{ij} u_j b_{li} v_l = \sum_{j,l} \Big(\sum_i b_{li} a_{ij}\Big) u_j v_l = \sum_{j,l} \delta_{jl} u_j v_l = \sum_j u_j v_j.$$

PROPOSITION 4.2. *Let ψ be another R-homomorphism of Λ to R satisfying* (1.1) and (1.2), *and let $v_1', \ldots, v_n'$ be the dual basis of $u_1, \ldots, u_n$ with respect o ψ. Then we have*

(4.2) $$\sum u_i v_i' = \left(\sum u_i v_i\right)\lambda,$$

where λ *is a regular element in* Λ.

PROOF. In this case, there exist λ and λ' in Λ such that $\varphi(x) = \psi(x\lambda)$, $\psi(x) = \varphi(x\lambda')$. Therefore $\varphi(x) = \psi(x\lambda) = \varphi(x\lambda\lambda')$ for all x in Λ, so $\varphi(x) - \varphi(x\lambda\lambda') = \varphi(x(1-\lambda\lambda') = 0$. From $(l.1)$ we conclude $1-\lambda\lambda' = 0$; λ is regular in Λ. Now, if we put $v_i' = v_i\ \lambda$ then $\psi(u_i v_t') = \varphi(u_i v_j \lambda\lambda') = \delta_{ij}$. This shows that $v_1', \dots, v_n'$ is the dual basis of $u_1, \dots u_n$ with respect to ψ. $\sum u_i v_i = \left(\sum u_i v_i\right)\lambda$ is obvious.

PROPOSITION 4.2'. *Let* Λ *be a commutative R-free symmetric algebra over R, let* φ *be the defining R-homomorphism of* Λ *to R, let* $(u_1, \dots, u_n)$ *and* $(v_1, \dots v_n)$, *be the dual basis of* Λ *with respect to* φ. *Assume, further, that R and* Λ *are both integral domains and the quotient field L of* Λ *is separable over the quotient field K of R. Let* ψ *be any non zero K-homomorphism of L to K; let* $(u_1', \dots u_n')$ *be a basis of L over K in* Λ *and let* $(v_1, \dots, v_u')$ *be the dual basis of* (u_i') *of L with respect to* ψ. *If* $(v_1', \dots v_n')$ *is also contained in* Λ, *then*

$$\sum u_i' v' \in \left(\sum u_i v_i\right) \Lambda.$$

PROOF. In this case, $L = \Lambda \otimes_R K$, R-homomorphism φ can be extended naturaly to a K-homomorphism $\bar{\varphi}$ of L to K. The dual basis of $(u_1, \dots, u_n)$ with respect to $\bar{\varphi}$ is also $(v_1, \dots, v_n)$. By definition of symmetric algebra L/K, there exists an element α in L such that $\bar{\varphi}(x) = \psi(x\alpha)$. Then the dual basis of $(u_1, \dots, u_n)$ with respect to ψ is $(v_1\alpha, \dots, v_n\alpha)$. If we put $u_i' = \sum a_{ij} u_j$, $a_{ij} \in R$ then $v_j' = \sum b_{ij} v_i\ \alpha$ where (b_{ij}) is the inverse matrix of (a_{ij}). Moreover, if we put $\alpha = \sum c_i u_i$, $c_i \in K$ then $\bar{\varphi}(\alpha v_i) = c_i$, so $\bar{\varphi}(v_j') = \sum_i b_{ij}\ \bar{\varphi}(v_i\alpha) = \sum_i b_{ij} c_i$. Therefore $\sum {}_j\bar{\varphi}(v_j')a_{ji} = \sum_{i,j} c_i b_{ij} a_{ji} = c_i$, where a_{ji}, $\bar{\varphi}(v_i') \in R$. This shows that $c_i \in R$ and $\alpha \in \Lambda$; We have, by the same argument in Prop. 4.2, that $\sum u_i' v' = \left(\sum u_i v_i\right)\alpha$.

PROPOSITION 4.3. *Let* Λ, R, φ, u_i *and* v_i *be as above,* Λ, R *integral domains and L and K their quotient fields, respectively. Then,* $\sum u_i v_i \neq 0$ *if and only if L is separable over K.*

REMARK. It is already known [10] that for any Frobenius algebra L over a field K the ideal $\left\{\sum u_i \lambda v_i \mid \lambda \in L\right\}$ of the center C of L is equal to C if and only if L is separable over K. But $\sum u_i v_i$ may be zero even if L is a total matrix algebra over a field K. For example, if characteristic of K is $p > 0$ and $L = (K)_p$, then $\sum u_i v_i = 0$.

PROOF OF PROP. 4.3. $u_1, \ldots, u_n$ is also a linearly independent basis of L/K. R-homomophism φ of to R is extended to a K-homomorphism $\bar{\varphi}$ of L to K. L is a symmetric algebra over K and $(u_1, \ldots, u_n)$, $(v_1, \ldots, v_n)$ are also dual bases of L over K. So, we may consider $\sum u_i v_i$ in L. Then the property $\sum u_i v_i \neq 0$ is unaltered when we take another basis u_i' or another K-homomorphism $\bar{\psi}$ (Prop. 4.1, 4.2).

Case 1. $L = K(\theta)$, where $\theta^n + a_1\theta^{n-1} + \ldots + a_n = 0$ is the defining equation of θ in K. We take $1, \theta, \theta^2, \ldots, \theta^{n-1}$ as a basis of L/K and a map $\bar{\psi} : \bar{\psi}(\theta^{n-1}) = 1$, $\bar{\psi}(\theta^i) = 0 (i \neq n-1)$ as a defining K-homomorphism of L to K. Then

$$v_i = \theta^{i-1} + a_1\theta^{i-2} + \ldots + a_{i-1}$$

is the dual basis of $u_i (= \theta^{i-1})$ and

$$\sum u_i v_i = n\,\theta^{n-1} + (n-1)a_1\theta^{n-2} + \ldots + a_{n-1} = f'(\theta).$$

So we proved the proposition for Case 1.

Case 2. If L is not simple over K, we take a chain of fields as follows: $L = L_r \supset \ldots \supset L_0 = K$, L_i/L_{i-1} simple, and prove it by induction. $r = 1$ is Case 1. Assume that it is proved for $r-1$. We consider two steps L/L_1 and L_1/K. Let φ_1, φ_0 be L_1-and L_0-homomorphisms of L to L_1 and L_1 to K, respectively, and $(U_1, \ldots, U_N)$, $(u_1, \ldots, u_n)$ are their bases and $(V_1, \ldots, V_N)$, $(v_1, \ldots, v_n)$ are corresponding dual bases concerning to ψ_1, ψ_0, respectively. Then $\tilde{\varphi} = \varphi_0 \circ \varphi_1$ is a L_0-homomorphism of L to K and $v_i V_j$ is the corresponding dual basis of $u_i U_j$, which is a basis of L/K. So $\tilde{\varphi}$ is a defining map of the symmetric algebra L over K. Therefore,

$$\sum_{ij} u_i U_j v_i V_i = \Big(\sum_j u_i v_i\Big)\Big(\sum_i U_j V_j\Big)$$

is a considering element of L/K. This proves the proposition for r.

PROPOSITION 4.4. *Let K be a field, L a finite separable extension of K and $u_1, \ldots, u_n$ a basis of L over K. It is a symmetric algebra. If we take $Sp_{L/K}$ as the defining map φ, then the corresponding element, $\sum u_i v_i = 1$, where $v_1, \ldots, v_n$ is the dual basis af $u_1, \ldots, u_n$ with respect to $Sp_{L/K}$.*

PROOF. We take a normal closure $\bar{L}$ of L over K and consider an algebra $L \otimes_K \bar{L}$ over $\bar{L}$ which is contained in the full matrix ring of degree n over $\bar{L}$. The $Sp_{L/K}$ of an element in L coincides with the trace of the corresponding element in $L \otimes_K \bar{L}$ regarding as a matrix over $\bar{L}$. So $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are also dual bases of $L \otimes \bar{L}$ over $\bar{L}$ with respect to Sp. Since $\sum u_i v_i$ is independent to the choice of $u_1, \ldots, u_n$ (Prop. 3.1), we may choose the most suitable one. We decompose 1 of $L \otimes \bar{L}$ in the direct components of $L \otimes \bar{L} \cong$

$\bar{L}+\ldots\ldots+\bar{L}$,

$$1=e_1+\ldots\ldots+e_n,$$

$e_1,\ldots\ldots e_n$ is a basis of $L\otimes\bar{L}$ over $\bar{L}$ and their dual basis with respect to Sp is also $e_1,\ldots\ldots e_n$. Therefore,

$$\sum u_iv_i=\sum e_i^2=1.$$

In the preceding section we prove Prop. 3.3 for $n\neq 0,\ -1$. Here, we prove it for principal orders of fields, which is sufficient for our purpose.

PROPOSITION 4.5. *Let Λ be an R-free symmetric algebra over R, both Λ and R be integral domains. Assume that the quotient field of Λ is separable over the quotient field R. Then for any Λ^e-projective module P we have*

$$H^0(\Lambda, P)=0, \qquad H^{-1}(\Lambda,\ P)=0,$$

PROOF. It is suficient to prove it for Λ^e-free modules, especially for Λ^e. We devide the proof in three lemmas.

LEMMA 1.

$$(\Lambda^e)^\Delta=\left(\sum u_i\otimes v_i^*\right)\Lambda^e. \tag{4.3}$$

PROOF. Let λ be any element in Λ and let (a_{ij}) be its right regular representation by $u_1,\ldots\ldots u_n$. Since $(\lambda\otimes 1)\left(\sum u_i\otimes v_i^*\right)(\mu\otimes\mu')=\left(\sum_{i,j} u_ja_{ji}\otimes v_i^*\right)$ $(\mu\otimes\mu')=\left(\sum u_j\otimes\sum_i a_{ij}v_i^*\right)(\mu\otimes\mu')=\left(\sum_i u_j\otimes\lambda^*v_j^*\right)(\mu\otimes\mu')=(1\otimes\lambda^*)\sum(u_j\otimes v_j^*)$ $(\mu\otimes\mu')$, the right hand side of (4.3) is contained in the left hand side.

Let $\sum b_{ij}u_i\otimes v_j^*$ be an element of $(\Lambda^e)^\Delta$, i.e.

$$\begin{aligned}(\lambda\otimes 1)\sum_{ij} b_{ij}u_i\otimes v_i^* &= \sum_{ik}\left(\sum_j a_{ki}b_{ij}\right)u_k\otimes v_j^*\\ &=(1\otimes\lambda^*)\left(\sum_j b_{ij}u_i\otimes v_j^*\right)=\sum_{il}\left(\sum_j b_{ij}a_{jl}\right)u_i\otimes v_l^*;\end{aligned}$$

so we have $\sum_j b_{ij}a_{jl}=\sum_j a_{ij}b_{jl}$, for $u_i\otimes v_l^*$ is a linearly independent basis of Λ^e over R. In other words, the square matrix (b_{ij}) commutes with any matrix (a_{ij}) which is the right regular representation of an element in Λ by the basis $u_1,\ldots\ldots u_n$; therefore, (b_{ij}) commutes with any matrix which is the right regular representation of an element of the quotient field $Q(\Lambda)$ of Λ, and it belongs to fhe same set of matrices of the representation. So there exists an element μ in $Q(\Lambda)$ such that $\mu(u_i)=(u_i)(b_{ij})$. Put $\mu=\sum c_i v_i$, $c_i\in Q(R)$, then $\varphi(\mu u_i)=c_i$. On the other hand, since $\mu u_i=\sum_j u_jb_{ij}$ belongs to Λ, $\varphi(\mu u_i)$ is in R; so $\mu\in\Lambda$. Thus we have

$$\sum_{ij} b_{ij}u_i\otimes v_j^*=(\mu\otimes 1)\sum_i u_i\otimes v_i^*,$$

which proves the Lemma.

LEMMA 2. $\Delta\Lambda^e$ in (3.8) is the kernel of the map $\rho:\Lambda^e\to\Lambda$ in (1.4).

PROOF. Obviously, the kernel of ρ contains $\Delta\Lambda^e$. On the other hand, we decompose the map ρ in two parts

$$\rho:\Lambda^e\xrightarrow{\rho_1}\Lambda^e/\Delta\Lambda^e\xrightarrow{\rho_2}\Lambda\,; \tag{4.4}$$

though each part of (4.4) is Λ^e-homomorphism, it suffices to consider them as homomorphisms without operators. We also consider modules as additive groups without operators. Then, $\Lambda^e=(\Lambda\otimes 1,\ \Delta\Lambda^e)$, $\Lambda^e/\Delta\Lambda^e\cong\Lambda\otimes 1/\Lambda\otimes 1\cap\Delta\Lambda^e$; ρ maps the subgroup $\Lambda\otimes 1$ of $\Lambda\otimes\Lambda$ isomorphically onto Λ; so we have $\Lambda\otimes 1\cap\Delta\Lambda^e=0$ and ρ_2 is isomorphic.
This shows that kern. $\rho=\Delta\Lambda^e$.

LEMMA 3. $(\Lambda^e)^{\Sigma u_i\otimes u_i*}=\Delta\Lambda^e$

PROOF. Since $\left(\sum u_i\otimes v_i^*\right)\Delta\Lambda^e=0$, $(\Lambda^e)^{\Sigma u_i\otimes v_i*}\supset\Delta\Lambda^e$.

Conversely, if $\left(\sum u_i\otimes v_i^*\right)\left(\sum\mu\otimes\nu^*\right)=0$, we map each term of this by the homomorphism ρ. Since Λ is commutative, ρ is a ring homomorphism. Therefore

$$\rho\left(\left(\sum u_i\otimes v_i^*\right)\left(\sum\mu\otimes\nu^*\right)\right)=\left(\sum u_i v_i\right)\left(\sum\mu\nu\right)=0.$$

On the other hand, by Prop. 4.3, $\sum u_i v_i\neq 0$ in Λ. So we have $\rho\left(\sum\mu\otimes\nu^*\right)=\sum\mu\nu=0$.

5. **An annulator of $H(\Lambda, A)$.** In this section we show that there exists a non trivial annulator of $H(\Lambda, A)$ in our number theoretical case. Our homological and cohomological n-differents are, consequently, non zero ideals in Λ.

THEOREM 5.1. *Let R be an integral domain, K its quotient field, Λ an R-projective algebra over R. If $L=\Lambda\otimes_R K$ is a Frobenius algebra over K with finite dimension, then there exists an element $\sum\lambda\otimes\mu^*$ in the center of Λ^e such that*

$$\begin{aligned}&\textstyle\sum\lambda\otimes\mu^*\ H^n(\Lambda, A)=0,\\ &\textstyle\sum\lambda\otimes\mu^*\ H_n(\Lambda, A)=0\end{aligned} \tag{5.1}$$

for any Λ^e-module A and any $n\geqq 1$.

More precisely, if we take dual bases $(u_1,\dots.u_n)$, $(v_1,\dots.v_n)$ of L/K from Λ, then

$$\sum_{i,i}u_i v_i\otimes v_j^*u_j^*=\left(\sum_i u_i v_i\right)\otimes\left(\sum_j u_i v_i\right)^*$$

is one of the elements.

PROOF. In the present case any element of L is the form λ/r, $\lambda\in\Lambda$, $r\in R$, and we may take a basis $(u_1, \ldots, u_n)$ of L/K from Λ. Let φ be a defining K homomorphism of the Frobenius algebra Λ/R and $(v_1/s, \ldots, v_n/s)$, $v_i \in \Lambda$, $s \in R$ be the dual basis of $(u_1, \ldots, u_n)$ with respect to φ. Then the K-homomorphism $\varphi_0(x) = \varphi(xs^{-1})$ satisfies the defining conditions $(l.1)$, $(l.2)$ in §1. The dual basis of (u) with respect to φ_0 is $(v_1, \ldots, v_n)$; so we may always take dual bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ of L/K from Λ. In the following proof we use u and v in this sense.

For any Λ^e-module A we consider the following sequence of homomorphism:

$$(5.2)\qquad A \xrightarrow{i} \mathrm{Hom}_{\Lambda^e}(\Lambda^e, A) \xrightarrow{\eta} \mathrm{Hom}_R(\Lambda^e, A) \xrightarrow{(*)} \Lambda^e \otimes_R A \xrightarrow{\xi} \Lambda^e \otimes_{\Lambda^e} A \xrightarrow{j} A,$$

where i, j are connonical Λ^e-isomorphism, η is the cannonical Λ^e-monomorphism, ξ is the cannonical Λ^e-epimorphism and the map $(*)$:$\mathrm{Hom}_R(\Lambda^e, A) \to \Lambda^e \otimes_R A$ is defined as follows;

$$(5.3)\qquad (*)(f) = \sum_{i,.} (u_i \otimes v_j^*) \otimes_R f(v_i \otimes u_j^*).$$

Obviously, $(*)$ is an R-homomorphism, Moreover, we have

LEMMA. $(*)$ *is a* Λ^e-*homomorphism.*

PROOF. Case 1: A is R-free. Let $v_i\lambda' = \sum_k b'_{ik} v_k$ and $\lambda u_j = \sum_l u_l b_{lj}$ be the regular representation of λ' and λ for any $\lambda' \otimes \lambda^*$ in Λ^e. There exists an element d in R such that db'_{ik} and db_{lj} are all in $R(i, j, k, l = 1, \ldots, n)$. Then

$$(5.4)\qquad \begin{aligned} d^2(*)[(\lambda' \otimes \lambda^*)f] &= \sum_{i,j} (u_i \otimes v_j^*) \otimes f((v_i \otimes u_j^*)(\lambda' \otimes \lambda^*)d^2) \\ &= \sum_{i,j} (u_i \otimes v_j^*) \otimes \sum_{k,l} f(db'_{ik} v_k \otimes d\, b_{lj} u_l^*) \\ &= \sum_{i,j}\sum_{k,l} [u_i\, db'_{ik} \otimes v_j^* db_{lj}] \otimes f(v_k \otimes u_l^*) \\ &= \sum_{k,l} (d\lambda' u_k \otimes d\lambda^* v_l^*) \otimes f(v_k \otimes u_l^*) \\ &= d^2 \sum_{k,l} (\lambda' \otimes \lambda^*)(u_k \otimes v_l^*) \otimes f(v_k \otimes u_l^*) \\ &= d^2(\lambda' \otimes \lambda^*)\,[(*)(f)]. \end{aligned}$$

Since A is R-free and Λ^e is R-projective, $\Lambda^e \otimes_R A$ is also R-projective; so it is torsion free over R. We have, therefore, from (5.4),

$$(*)[(\lambda' \otimes \lambda^*)f] - (\lambda' \otimes \lambda^*)[(*)(f)] = 0.$$

Case 2: A is not R-free. We consider A as an R-homomorphic image of R-free module F,

$$(5.5)\qquad 0 \to C \to F \to A \to 0 \qquad \text{(exact)}.$$

From this, using the fact that Λ^e is R-projective, we have the following commutative diagram:

$$(5.6)\qquad \begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_R(\Lambda^e, C) & \to & \mathrm{Hom}_R(\Lambda^e, F) & \to & \mathrm{Hom}_R(\Lambda^e, A) & \to 0 & \text{(exact)} \\ & & & \downarrow (*)_F & & \downarrow (*)_A & & \\ 0 \longrightarrow & \Lambda^e \otimes_R C & \to & \Lambda^e \otimes_R F & \longrightarrow & \Lambda \otimes^e{}_R A & \to 0 & \text{(exact),} \end{array}$$

where all horizontal maps and $(*)_F$ are Λ^e-homomorphism. This shows that the mapping $(*)_A$ is also Λ^e-homomorphism, which is the conclusion of the lemma.

Now we continue the proof of Th. 5.1. Operating $i, \eta, (*), \xi,$ and j successively, we have an endomorphism of $H(\Lambda, A)$:

$$(5.7)\qquad H(\Lambda, A) \underset{\eta\circ i}{\rightarrow} H(\Lambda, \mathrm{Hom}_R(\Lambda^e, A)) \underset{(*)}{\rightarrow} H(\Lambda, \Lambda^e \otimes_R A) \underset{j\circ\xi}{\rightarrow} H(\Lambda, A).$$

Let X be a Λ^e-projective resolution of Λ. It may be also considered as an R-projective resolution of Λ. Then, we have [1, Ch. II, Prop. 5.2]

$$(5.8)\qquad \begin{aligned} &H^n(\Lambda, \mathrm{Hom}_R(\Lambda^e, A)) = H^n(\mathrm{Hom}_{\Lambda^e}(X, \mathrm{Hom}_R(\Lambda^e, A))) \\ &\qquad \cong H^n(\mathrm{Hom}_R(X, A)) = \mathrm{Ext}_R{}^n(\Lambda, A) = 0, \\ &H_n(\Lambda, \Lambda^e \otimes_R A) = H_n((\Lambda^e \otimes_R A) \otimes_{\Lambda^e} X) = H_n(A \otimes_R X) = \mathrm{Tor}_n^R(A, \Lambda) = 0, \end{aligned}$$

since Λ is P-projective. In both case, therefore, the endomorphism (5.7) is the 0 endomorphism.

On the other hand the explicit from of the map $j\circ\xi\circ(*)\circ\eta\circ i$ is

$$(5.9)\qquad j\circ\xi\circ(*)\circ\eta\circ i(a) = \Big[\sum_{i,j} u_i v_i \otimes v_j^* u_j^*\Big] a.$$

Since $\sum_{i,j} u_i v_i \otimes v_j^* u_j^*$ belongs to the center of Λ^e, it induces an endomorphism of $H(\Lambda, A)$, (§ 1), which is, by (5.8), the zero endomorphism.

REMARK: since $\sum u_i v_i$ belongs to the center of Λ, the operations on $H(\Lambda, A)$ induced by its left-and right mutiplication to A are the same one (Cor. 2.2), so we may take

$$(5.10)\qquad \big(\sum u_i v_i\big)^2 \otimes 1$$

as the seeking element in Prop. 5.1. This may be zero even if L is a separable algebra over K. But in our number theoretical case, L is a separable extension field over K; so we have $\sum u_i v_i \neq 0$ in L (Prop. 4.3). Thus (5.10) is a non trivial annulator of $H(\Lambda, A)$.

6. **The homological and cohomological differents.** Let R be a Dedekind ring, K its quotient field, L a finite separable extension over K and Λ the principal order of L over R. We have already defined homological differents $D_n(\Lambda/R)$, $D_n^l(\Lambda/R)$, $D_n^r(\Lambda/R)$ etc. and proved that $D_n(\Lambda/R) = D_n^l(\Lambda/R) = D_r^n(\Lambda/R)$.

PROPOSITION 6.1. *In the above case Λ is R-projective.*

PROOF. Since R is a Dedekind ring, R is hereditary [1; Ch. VII, Prop. 3.2].

On the other hand Λ is an R-submodule of an R-free module, so Λ is R-projective [1; I, Th. 5.4].

THEOREM 6.2. *For any* $n \geqq 1$, $D^n(\Lambda/R) \neq 0$, $D_n(\Lambda/R) \neq 0$.

PROOF. It follows immediately from Prop. 6.1, Th. 5.1 and the remark to Th. 5.1.

Next we consider the local factors of $D(\Lambda/R)$. Let $\mathfrak{p}$ be any prime in R, $R_\mathfrak{p}$ and $\Lambda_\mathfrak{p}$ be the quotient ring of R and Λ by $\mathfrak{p}$ respectively. $\Lambda_\mathfrak{p}$ is the principel order of L over $R_\mathfrak{p}$. For any ideal D of Λ the ideal $D_\mathfrak{p} = D\Lambda_\mathfrak{p}$ may identify with the $\mathfrak{p}$-component of D. Since this case is also the number theoretical case, we may consider $D(\Lambda_\mathfrak{p}/R_\mathfrak{p})$ etc. We shall prove

THEOREM 6.3. $D^n(\Lambda/R)_\mathfrak{p} = D^n(\Lambda_\mathfrak{p}/R_\mathfrak{p})$, $D_n(\Lambda/R)_\mathfrak{p} = D_n(\Lambda_\mathfrak{p}/R_\mathfrak{p})$ *for* $n \geqq 1$.

To prove the theorem we prepare several lemmas. At first, for any Λ^e-module A we denote $A_\mathfrak{p}$ the quotient module of A by $\mathfrak{p}$. It is also $\Lambda_\mathfrak{p} \otimes_{R_\mathfrak{p}} \Lambda_\mathfrak{p}^*$-module.[3]

LEMMA 1. *For any R-module A, we have*

$$A \otimes_R R_\mathfrak{p} \cong A_\mathfrak{p}.$$

Moreover, if A is a Λ-module then the above is a $\Lambda_R \otimes R_\mathfrak{p}$-isomorphism. (*So $\Lambda_\mathfrak{p}$-isomorphism by lemma* 3)

PROOF. We consider the mappings

$$\varphi : A \otimes_R R_\mathfrak{p} \to A_\mathfrak{p}, \quad \varphi(a \otimes (r, s)) = (ar, s)^{4)}$$
$$\psi : A_\mathfrak{p} \to A \otimes_R R_\mathfrak{p}, \quad \psi(a, s) = a \otimes (1, s),$$

which are both $R_\mathfrak{p}$-homomorphism and are inverse maps each other. The second part of the lemma is obvious.

LEMMA 2. *For any R-modules A and B, we have*

$$(A \otimes_R B) \otimes_R R_\mathfrak{p} \cong (A \otimes_R R_\mathfrak{p}) \otimes R_\mathfrak{p} (B \otimes_R R_\mathfrak{p}).$$

Moreover, if A and B are Λ-modules then the above is a $(\Lambda \otimes_R \Lambda) \otimes_R R_\mathfrak{p}$-isomorphism (so $(\Lambda_\mathfrak{p})^e$-isomorphism[3] *by lemma* 3).

PROOF. In general, for any commutative rings S and R, $S \supset R$ which have the unity element 1 in common, we have an canonical isomorphism

$$(A \otimes_R S) \otimes_S (B \otimes_R S) \cong A \otimes_R (S \otimes_S S) \otimes_R B \cong A \otimes_R S \otimes_R B \cong (A \otimes_R B) \otimes_R S.$$

The second part of the lemma is obvious.

LEMMA 3. *The isomorphism in lemmas* 1 *and* 2 *are ring isomorphisms if A, B are both Λ, i.e.*

$$\Lambda \otimes_R R_\mathfrak{p} \cong \Lambda_\mathfrak{p} \qquad \text{(ring isomorphism)}.$$
$$\Lambda^e \otimes_R R_\mathfrak{p} \cong \Lambda_\mathfrak{p} \otimes_{R_\mathfrak{p}} \Lambda_\mathfrak{p}^* \qquad \text{(ring isomorphism)}.$$

---

3) We shall denote $\Lambda_\mathfrak{p} \otimes_{R_\mathfrak{p}} \Lambda_\mathfrak{p}^*$ briefly by $(\Lambda_\mathfrak{p})^e$.

4) The element of $A_\mathfrak{p}$ is represented by a pair (a, s), where $a \in A$ and $s \in R_\mathfrak{p}$. In the following we use these representations of elements of $A_\mathfrak{p}$ and $R_\mathfrak{p}$.

PROOF. In lemma 1, φ is a ring homomorphism if A is Λ. The isomorphism in Lemma 2 is also a ring isomorphism, so we have the later part of the lemma using the first part of it,

LEMMA 4. *If*

$$\dots\dots \to X_1 \to X_0 \to \Lambda \to 0$$

is the standard complex of the algebra Λ *over* R, *then*

$$\dots\dots \to X_1 \otimes_R R_{\mathfrak{p}} \to X_0 \otimes_R R_{\mathfrak{p}} \to \Lambda \otimes_R R_{\mathfrak{p}} \to 0$$

is the standard complex of the algebra $\Lambda_{\mathfrak{p}}$ *over* $R_{\mathfrak{p}}$.

PROOF. From lemmas 1, 2 and 3 we see that the second modules are identical with the standard complex of $\Lambda_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ as $(\Lambda_{\mathfrak{p}})^e$-modules. And induced differen ial operators coincide with those of standard complex, too.

Now, if $A_{\mathfrak{p}}$ is a $(\Lambda_{\mathfrak{p}})^e$-module, we may also consider $A_{\mathfrak{p}}$ as a Λ^e-module. So we may consider $H(\Lambda/R, A_{\mathfrak{p}})$ as well as $H(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}, A_{\mathfrak{p}})$. Since $\Lambda_{\mathfrak{p}} \supset \Lambda$, Λ^e operates on both $H(\Lambda/R, A_{\mathfrak{p}})$ and $H(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}, A_{\mathfrak{p}})$.

LEMMA 5. *For any* $(\Lambda_{\mathfrak{p}})^e$*-module* $A_{\mathfrak{p}}$, *we have a* Λ^e*-isomorphism*

$$H^n(\Lambda_{\mathfrak{p}},\ A_{\mathfrak{p}}) \cong H^n(\Lambda,\ A_{\mathfrak{p}}),$$
$$H_n(\Lambda_{\mathfrak{p}}, A_{\mathfrak{p}}) \cong H_n(\Lambda, A_{\mathfrak{p}}).$$

PROOF. We have [1, Ch II. Prop. 5.2] a Λ^e-isomorphism

$$\mathrm{Hom}_{\Lambda^e\otimes R_{\mathfrak{p}}}(X \otimes_{\Lambda^e} \Lambda^e \otimes R_{\mathfrak{p}},\ A_{\mathfrak{p}}) \cong \mathrm{Hom}_{\Lambda^e}(X, \mathrm{Hom}_{\Lambda^e\otimes R_{\mathfrak{p}}}(\Lambda^e \otimes R_{\mathfrak{p}}, A_{\mathfrak{p}}))$$

where $\otimes$ means the tensor product over R. The left hand side is $\mathrm{Hom}_{(\Lambda_{\mathfrak{p}})^e}(X \otimes R_{\mathfrak{p}}, A_{\mathfrak{p}})$ (lemma 3); the right hand side is isomorphic to $\mathrm{Hom}_{\Lambda^e}(X, A_{\mathfrak{p}})$, since $\mathrm{Hom}\ \Lambda^e \otimes_{R_{\mathfrak{p}}} (\Lambda^e \otimes R_{\mathfrak{p}}, A_{\mathfrak{p}}) \cong A_{\mathfrak{p}}$. From this and lemma 4 the first half of the lemma 5follows immediately.

Similarly, the Λ^e-isomorphism

$$(X \otimes R_{\mathfrak{p}}) \otimes_{\Lambda^e\otimes R_{\mathfrak{p}}} A_{\mathfrak{p}} \cong (X \otimes_{\Lambda^e} \Lambda^e) \otimes_R R_{\mathfrak{p}} \otimes_{\Lambda^e\otimes R_{\mathfrak{p}}} A_{\mathfrak{p}}$$
$$\cong X \otimes_{\Lambda^e} (\Lambda^e \otimes_R R_{\mathfrak{p}} (\otimes_{\Lambda^e\otimes R_{\mathfrak{p}}} A \cong X \otimes_{\Lambda^e} A_{\mathfrak{p}}$$

gives the second part of the lemma.

PROPOSITION 6.4. *We have*

$$D^n(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}) \supset D^n(\Lambda/R),\quad D^n(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}) \supset D^n(\Lambda/R)$$

for $n \geqq 1$.

PROOF. We may consider the left differents only. So we consider the left operations of Λ on $H(\Lambda, A_{\mathfrak{p}})$. Then the proposition follows immediately from lemma 5.

LEMMA 6. *For any* Λ^e*-module* A, *we have a* Λ^e*-isomorphism*

$$H^n(\Lambda, A) \otimes_R R_{\mathfrak{p}} \cong H^n(\Lambda, A \otimes_R R_{\mathfrak{p}}),$$
$$H^n(\Lambda, A) \otimes_R R_{\mathfrak{p}} \cong H^n(\Lambda, A \otimes_R R_{\mathfrak{p}}) \qquad \textit{for } n \geqq 0.$$

PROOF. Let X be the standard complex of Λ over R. We consider a

Λ^e-homomorphism

$$\varphi : \mathrm{Hom}_{\Lambda^e}(X, A) \otimes R_\mathfrak{p} \to \mathrm{Hom}_{\Lambda^e}(X, A \otimes R_\mathfrak{p})$$
$$\varphi[f \otimes (r, s)](x) = f(x) \otimes (r, s).$$

Conversely, any homogeneous element in $\mathrm{Hom}_{\Lambda^e}(X, A \otimes R_\mathfrak{p})$ we may take an element f in $\mathrm{Hom}_{\Lambda^e}(X, A)$ such that $g(x) = f(x) \otimes (1, s)$ for all x in X, since X_n is finitely generated over Λ^e. We put, then,

$$\psi : g \to f \otimes (1, s),$$

and have a Λ^e-homomorphism ψ of $\mathrm{Hom}_{\Lambda^e}(X, A \otimes R_\mathfrak{p})$ to $\mathrm{Hom}_{\Lambda^e}(X, A) \otimes R_\mathfrak{p}$. Since φ and ψ are inverse mapping each other, they are both Λ^e-isomorphism. Obviously, both φ and ψ commute with the differential operator d of X. This shows the first half of the theorem.

As for the second part, it is obvious since we have

$$(A \otimes_{\Lambda^e} X) \otimes R_\mathfrak{p} \cong (A \otimes R_\mathfrak{p}) \otimes_{\Lambda^e} X.$$

LEMMA 7. *For any Λ^e-module A, we have a Λ^e-isomorphism*

$$H^n(\Lambda, A) \otimes_R R_\mathfrak{p} \cong H^n(\Lambda_\mathfrak{p}, A \otimes_R R_\mathfrak{p})$$
$$H_n(\Lambda, A) \otimes_R R_\mathfrak{p} \cong H_n(\Lambda_\mathfrak{p}, A \otimes_R R_\mathfrak{p})$$

for $n \geqq 0$.

PROOF. It is obvious from lemmas 5 and 6.

LEMMA 8. *Let A be any R-module. For any element a of A we take an element of $\Pi_\mathfrak{p} A_\mathfrak{p}$, the each component of which is $(a, 1)$ in $A_\mathfrak{p}$. Then the mapping*

$$a \to \{\ldots, (a, 1), \ldots\}$$

is an R-isomorphism of into $\Pi_\mathfrak{p} A_\mathfrak{p}$.
Moreover if A is a Λ^e-module, the above mapping is a Λ^e-isomorphism.

PROOF. Let a be a non zero element in A. The annulators of a forms an ideal $\mathfrak{a}$ of R. If $\mathfrak{p}$ is any prime divisor of $\mathfrak{a}$, then $(a, 1)$ in $A_\mathfrak{p}$; for, $(a, 1) = 0$ in $A_\mathfrak{p}$ if and only if there exists s in R such that $sa = 0$ and $s \notin \mathfrak{p}$, that is, $\mathfrak{p}$ does not divide $\mathfrak{a}$.

REMARK. If A has a non trivial annulator, then $\Pi_\mathfrak{p} A_\mathfrak{p}$ is reduced to a direct sum of finite number of factors and into isomorphism is reduced to onto.

PROPOSITION 6.5. *If an element λ of Λ is contained in $D(\Lambda_\mathfrak{p}/R_\mathfrak{p})$ for all $\mathfrak{p}$, then λ is contained in $D(\Lambda/R)$*

PROOF. We consider λ as a left operator of $H(\Lambda, A)$. Then the proof follows immediately from Lemmas 7 and 8.

PROOF OF THEOREM 6.3. Prop. 6.4. and 6.5 constitute the proof.

7. The different theorem. At first we prove two lemmas, the proofs of which are almost obvious.

LEMMA 1. *Let Λ be a commutative ring, R a subring of Λ and π any element* *in R. Then we have a ring isomorphism*

$$\Lambda^e/\pi\Lambda^e \cong (\Lambda/\pi\Lambda)\otimes_{R/\pi R}(\Lambda/\pi\Lambda).$$

LEMMA 2. *Let R be a commutative ring, π any element in R and A, B R-modules. Then we have an isomorphism*

$$(A\otimes_R B)/\pi(A\otimes_R B) \cong (A/\pi A)\otimes_{R/\pi R}(B/\pi B).$$

PROOF OF LEMMAS 1 AND 2. Since the lemma 2 is proved by the same method as lemma 1 we prove lemma 1 only. We have the desired isomorphism from the following ring homomorphism:

$$\text{(7.1)}\qquad \begin{aligned}&\varphi : \Lambda\otimes_R\Lambda \to (\Lambda/\pi\Lambda)\otimes_{R/\pi R}(\Lambda/\pi\Lambda)\\ &\varphi(\lambda_1\otimes\lambda_2) = (\lambda_1 \bmod \pi)\otimes(\lambda_2 \bmod \pi),\end{aligned}$$

the kernel of which is $\pi(\Lambda\otimes_R\Lambda)$.

PROPOSITION 7.1. *Let Λ be an R-projective commutative R-algebre, π any element in R and X the standard complex of Λ with the differential operator d and the homotopy map s. We denote residue rings $R/\pi R$ and $\Lambda/\pi\Lambda$ by $\bar{R}$ and $\bar{\Lambda}$, respec tively. If $\bar{\Lambda}$ is $\bar{R}$-projective then $\bar{X}=X/\pi X$ is the standard complex of the algebra $\bar{\Lambda}$ over $\bar{R}$ with the differential operator $\bar{d}=d$ mod π and the homotopy map $\bar{s} = s$ mod π.*

PROOF. From lemma 2 we have

$$\text{(7.2)}\qquad \begin{aligned}\bar{X}_n = X_n/\pi X_n &= (\Lambda\otimes_R\cdots\otimes_R\Lambda)/\pi(\Lambda\otimes_R\cdots\otimes_R\Lambda)\\ &\cong \bar{\Lambda}\otimes_{\bar{R}}\cdots\otimes_{\bar{R}}\bar{\Lambda}\end{aligned}$$

where
$$\begin{aligned}d_n \bmod \pi : \bar{\lambda}_0\otimes\cdots\otimes\bar{\lambda}_{n+1} &\to \sum(-1)^i\,\bar{\lambda}_0\otimes\cdots\otimes\overline{\lambda_i\lambda_{i+1}}\otimes\cdots\otimes\bar{\lambda}_{n+1}\\ &= \sum(-1)^i\,\bar{\lambda}_0\otimes\cdots\otimes\overline{\lambda_i\lambda_{i+1}}\otimes\cdots\otimes\bar{\lambda}_{n+1},\end{aligned}$$

$$S_n \bmod \pi : \bar{\lambda}_0\otimes\cdots\otimes\bar{\lambda}_{n+1} \to 1\otimes\bar{\lambda}_0\otimes\cdots\otimes\bar{\lambda}_{n+1}$$

which are the original differential and homotopy maps of the standard complex of $\bar{\Lambda}$ over $\bar{R}$.

PROPOSITION 7.2. *Let Λ be an R-projective commutative R-algebra, π any element in R, and let $\bar{\Lambda}=\Lambda/\pi\Lambda$ and $\bar{R}=R/\pi R$ be residue rings. If $\bar{\Lambda}$ is $\bar{R}$-projective, then for any $\bar{\Lambda}^e$-moduel $\bar{A}$ we have*

$$H^n(\Lambda,\bar{A}) \cong H^n(\bar{\Lambda},\bar{A}),$$
$$H^n(\Lambda,\bar{A}) \cong H_n(\bar{\Lambda},\bar{A}).$$

PROOF. If we take the standard complex X of Λ over R, then we have

$$\text{(7.3)}\qquad \begin{aligned}&\mathrm{Hom}_{\Lambda^e}(X,\bar{A}) = \mathrm{Hom}_{\Lambda^e}(\bar{X},\bar{A}) = \mathrm{Hom}_{\Lambda^e/\pi\Lambda^e}(\bar{X},A) = \mathrm{Hom}_{\bar{\Lambda}^e}(\bar{X},\bar{A}),\\ &\bar{A}\otimes_{\Lambda^e}X = \bar{A}\otimes_{\Lambda^e}\bar{X} = \bar{A}\otimes_{\Lambda^e/\pi\Lambda^e}\bar{X} = \bar{A}\otimes_{\bar{\Lambda}^e}\bar{X}\end{aligned}$$

which proves the proposition.

Now we consider the number theoretical algebras: at first, the local case. If we take $\Lambda_\mathfrak{p}$, $R_\mathfrak{p}$ and a prime element π of $\mathfrak{p}$ in R as Λ, R and π in Prop.

7.2, respectively, then $\bar{R}=R/\pi R$ is a field and $\bar{\Lambda}$ is $\bar{R}$-projective; so the assumption of prop. 7.2 is satisfied in this case.

PROPOSITION 7.3 *If* $\mathfrak{p}$ *is unramified and separable in* $\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}$, *then*

$$\dim \bar{\Lambda}_{\mathfrak{p}} = 0, \qquad w.\ \dim_{\bar{\Lambda}^e_{\mathfrak{p}}} \bar{\Lambda}_{\mathfrak{p}} = 0.$$

If $\mathfrak{p}$ *is ramified or inseparable in* $\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}$, *then*

$$\dim \bar{\Lambda}_{\mathfrak{p}} = \infty, \qquad v.\ \dim_{\bar{\Lambda}^e_{\mathfrak{p}}} \bar{\Lambda}_{\mathfrak{p}} = \infty,$$

i.e. for any integer $n \geqq 1$ *there exist* $\bar{\Lambda}^e_{\mathfrak{p}}$*-modules* $\bar{A}$ *and* $\bar{A}'$ *such that*

$$H^n(\bar{\Lambda}, \bar{A}) \neq 0, \qquad H_n(\bar{\Lambda}, \bar{A}') \neq 0.$$

PROOF. The first assertion is obvious.

Let $\mathfrak{p}=\mathfrak{P}_1^{e_1}\dots\mathfrak{P}_r^{e_r}$ be the decomposition of $\mathfrak{p}$ in $\Lambda_{\mathfrak{p}}$ (For the simplicity we omit the suffix $\mathfrak{p}$). Then we have the direct decomposition of $\bar{\Lambda}$

$$\bar{\Lambda} \cong \Lambda/\mathfrak{P}_1^{e_1} + \dots + \Lambda/\mathfrak{P}_r^{e_r},$$

So we have [1; Ch. IX, Th. 5.3]

$$H(\bar{\Lambda}, \bar{A}) \cong H(\Lambda/\mathfrak{P}_1^{e_1}, \bar{A}_1) + \dots + H(\Lambda/\mathfrak{P}_r^{e_r}, \bar{A}_r), \tag{7.4}$$

where $A_i = \bar{A}_i\bar{A}\bar{A}_i, \bar{A}_i = \Lambda/\mathfrak{P}_1^{e_1} + \dots + \Lambda/\mathfrak{P}_{i-1}^{e_{i-1}} + \Lambda/\mathfrak{P}_{i+1}^{e_{i+1}} + \dots + \Lambda/\mathfrak{P}_r^{e_r}$.

Hence the proof is sufficient to do with $\Lambda/\mathfrak{P}_i^{e_i}$. (We shall also omit the suffix i).

If $\mathfrak{P}$ is inseparable, then the algebra $\Lambda/\mathfrak{P}^e$ over a field $R/\mathfrak{p}$ has the radical $\mathfrak{P}/\mathfrak{P}^e$. Moreover if $\dim \Lambda/\mathfrak{P}^e < \infty$ then$(\Lambda/\mathfrak{P}^e)/(\mathfrak{P}/\mathfrak{P}^e)(\cong\Lambda/\mathfrak{P})$is separable over $R/\mathfrak{A}$[2], which is not the present case.

When $\mathfrak{P}$ is separable and ramified, we assume that $\dim \Lambda/\mathfrak{P}^e < \infty$ and deduce a contradiction. Under such assumption we have [2]

$$\dim \Lambda/\mathfrak{P}^e = l\cdot\dim_{\Lambda/\mathfrak{P}^e} \Lambda/\mathfrak{P}. \tag{7.5}$$

We may construct a suitable $\Lambda/\mathfrak{P}^e$-resolution of ($\Lambda/\mathfrak{P}^e$-left module) $\Lambda/\mathfrak{P}$:

$$\to X_3 \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\varepsilon} \Lambda/\mathfrak{P} \to 0 \tag{7.6}$$

where $$X_n=\Lambda/\mathfrak{P}^e,$$

ε: natural homomorphism

$$d_{2m+1}: \lambda \to \pi\lambda$$
$$d_{2m}: \quad \lambda \to \pi^{e-1} \qquad\qquad \lambda, \lambda \in \Lambda/\mathfrak{P}^e,$$

(π is a prime element of $\mathfrak{P}$ in Λ). Then we have

$$\mathrm{Ext}^n{}_{\Lambda/\mathfrak{P}^e}(\Lambda/\mathfrak{P}, \bar{A}) = H^n(\mathrm{Hom}_{\Lambda,\mathfrak{P}^e}(X, \bar{A}))$$
$$= \begin{cases} \bar{A}^{\Pi^{e-1}}/\Pi\bar{A} & n=2m+1 \\ \bar{A}^{\Pi}/\Pi^{e-1}\bar{A} & n=2m \neq 0, \end{cases}$$

where
$$\bar{A}^{\Pi^{e-1}} = \{a \in \bar{A} \mid \Pi^{e-1} a = 0\}$$
$$\Pi A = \{\Pi a \mid a \in \bar{A}\}$$
$$\bar{A}^{\Pi} = \{a \in \bar{A} \mid \Pi a=0\}$$
$$\Pi^{e-1}A = \{\Pi^{e-1}a \mid a \in \bar{A}\}.$$

Since $e > 1$, we may choose a suitable $\bar{A}$ such that

$$\mathrm{Ext}^n_{\Lambda/\mathfrak{P}^e}(\Lambda/\mathfrak{P}, \bar{A}) = 0$$

for any n (for example $A = \Lambda/\mathfrak{P}^e$). So we have

$$l\text{-}\dim_{\Lambda/\mathfrak{P}^e} \Lambda/\mathfrak{P} = \infty$$

which contradicts (7.5) and the assumption.

Since $\bar{\Lambda}$ is finite rank over $R/\pi R$ and $\bar{\Lambda}^e$ is Noetherian so we have [1; Ch. VI, p. 122] $w.\ \dim_{\Lambda^e}\Lambda = \dim \bar{\Lambda} = \infty$ in both cases.

THEOREM 7.4. *Let n be a fixed integer $\geqq 1$. Then, $H^n(\Lambda, A) = 0$ for any $\Lambda^e_{\mathfrak{p}}$-module A if and only if $\mathfrak{p}$ is unramified and separable in $\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}$.*

The similar theorem holds for $H_n(\Lambda, A)$, $n > 1$.

PROOF. If $\mathfrak{p}$ is ramified or inseparable in Λ/R (we shall omit the suffix $\mathfrak{P}$), there exist by Prop. 7.3, two sided $\bar{\Lambda}$-module $\bar{A}$ and $\bar{A}'$ over $\bar{R}$ such that

$$H^n(\bar{\Lambda}, \bar{A}) \neq 0, \qquad H_n(\bar{\Lambda}, \bar{A}') \neq 0,$$

where $\bar{\Lambda}, \bar{R}$ are residue rings $\Lambda/\pi\Lambda$, $R/\pi R$ respectively, and π is a prime element of $\mathfrak{p}$ in R. So we have, from Prop. 7.2,

$$H^n(\Lambda, \bar{A}) = H^n(\bar{\Lambda}, \bar{A}) \neq 0$$
$$H_n(\Lambda, \bar{A}') = H_n(\bar{\Lambda}, \bar{A}') \neq 0.$$

When $\mathfrak{p}$ is unramified and separable in Λ/R, for any Λ^e-module A we consider two exact sequences:

$$(7.7) \qquad 0 \to \pi A \xrightarrow{i} A \to A/\pi A \to 0 \qquad \text{(exact)},$$

$$(7.8) \qquad 0 \to A' \to A \xrightarrow{\pi} \pi A \to 0 \qquad \text{(exact)}$$

where $A' = \{a \in A \mid \pi a = 0\}$. From these sequences we have

$$H^n(\Lambda, \pi A) \xrightarrow{i} H^n(\Lambda, A) \to H^n(\Lambda, A/\pi A) \qquad \text{(exact)}$$
$$H^n(\Lambda, A) \xrightarrow{\pi} H^n(\Lambda, \pi A) \to H^{n+1}(\Lambda, A') \qquad \text{(exact)}$$

where, by Prop. 7.3 and Prop. 7.2, the third modules in both sequences are 0. Hence the product

$$(7.9) \qquad i \circ \pi : H^n(\Lambda, A) \to H^n(\Lambda, A)$$

is a homomorphism onto. Since this is the same map as $\tilde{\pi}$ in (1.3), we have

$$\tilde{\pi} H^n(\Lambda, A) = H^n(\Lambda, A).$$

Therefore, for any positive integer s,

$$H^n(\Lambda, A) = \tilde{\pi}^s H^n(\Lambda, A),$$

and the right hand side is 0, by Prop. 6.2, for sufficiently large s.

From (7.7) and (7.8) we have also

$$H_n(\Lambda, \pi A) \xrightarrow{i} H_n(\Lambda, A) \to H_n(\Lambda, A/\pi A) \qquad \text{(exact)}$$
$$H_n(\Lambda, A) \xrightarrow{\pi} H_n(\Lambda, \pi A) \to H_{n-1}(A, A') \qquad \text{(exact)},$$

and for $n > 1$ the third modules in both sequences are 0.

The remaining part of the proof goes similarly as above.

Now we consider the global case and prove the following main theorem.

THEOREM 7.5. *Let n be any fixed integer ≥ 1. Then a prime ideal $\mathfrak{P}$ in Λ divides the n-cohomological different $D^n(\Lambda/R)$ if and only if $\mathfrak{P}$ is ramified or inseparable in Λ/R.*

The similar reults hold for the n-homological different $D_n(\Lambda/R)$, $n > 1$.

PROOF. Th. 6.3 shows that it suffices to prove the theorem for the local case. So we only consider the local case and write Λ and R instead of $\Lambda_{\mathfrak{p}}$ and $R_{\mathfrak{p}}$ throughout the proof.

Sufficiency: Let $\mathfrak{P}$ be ramified or inseparable in Λ/R and let $\mathfrak{p}=\mathfrak{P}^e\mathfrak{A}$, $(\mathfrak{P}, \mathfrak{A}) = 1$ be the decomposition of $\mathfrak{p}$ in Λ. We take a prime element π of $\mathfrak{p}$ in R, then we have

$$(7.10) \qquad \Lambda/\pi\Lambda = \mathfrak{P}^e/\pi\Lambda + \mathfrak{A}/\pi\Lambda \qquad \text{(direct)},$$

$$\mathfrak{P}^e/\pi\Lambda \cong \overline{\Lambda}/\overline{\mathfrak{A}}, \quad \mathfrak{A}/\pi\Lambda \cong \overline{\Lambda}/\overline{\mathfrak{P}}^e \qquad \text{(ring isomorphism)}$$

where $\overline{\Lambda}, \overline{\mathfrak{P}}, \overline{\mathfrak{A}}$ are $\Lambda/\pi\Lambda, \mathfrak{P}/\pi\Lambda, \mathfrak{A}/\pi\Lambda$ respectively. Since $e > 1$ or $\overline{\Lambda}/\overline{\mathfrak{P}}$ is inseparable over R, the proof of Prop. 7.4 shows that there exists two sided $\overline{\Lambda}/\overline{\mathfrak{P}}^e$-module $\overline{A}$ such that $H^n(\overline{\Lambda}/\overline{\mathfrak{P}}^e, \overline{A}) \neq 0$. In the decomposition (7.10) we define the operations of $\overline{\Lambda}/\overline{\mathfrak{A}}$ on $\overline{A}$ as 0 operator, then $\overline{A}$ is a two sided $\overline{\Lambda}$-module. Thus [1, Ch. IX, Th. 5.3]

$$H^n(\overline{\Lambda}, \overline{A}) = H^n(\overline{\Lambda}/\overline{\mathfrak{A}}, 0) + H^n(\overline{\Lambda}/\overline{\mathfrak{P}}^e, \overline{A})$$

$$= H^n(\overline{\Lambda}/\overline{\mathfrak{P}}^e, \overline{A}) \neq 0$$

and, by Prop. 7.2,

$$H^n(\Lambda, \overline{A}) = H^n(\overline{\Lambda}, \overline{A}).$$

So the annulator D' of $H^n(\Lambda, \overline{A})$ does not contain 1. Since, by Prop. 6.2, $D' \supset \mathfrak{P}^e$ and, by the definition $D' \supset D^n(\Lambda/R)$, we have $D' \supset (\mathfrak{P}^e, D^n(\Lambda/R))$. Thus we have $\mathfrak{P} \supset D' \supset (\mathfrak{P}^e, D^n(\Lambda/R)) \supset D^n(\Lambda/R)$.

Necessity: Let $\mathfrak{P}$ be unramified and separable in Λ/R and let $\mathfrak{P} = \mathfrak{P}\mathfrak{A}$, $(\mathfrak{A}, \mathfrak{P}) = 1$ be the decomposition of $\mathfrak{p}$ in Λ. If we take sufficiently large power π^d of the prime element π of $\mathfrak{p}$, then, by Prop. 6.2, $\widetilde{\pi^d} H^n(\Lambda, A)=0$ for any Λ^e-module Λ. We prove that $\mathfrak{A}^{2d}H^n(\Lambda, A)=0$ for any A, which implies the necessity, since $D^n(\Lambda/R) \supset \mathfrak{A}^{2d}$, so $(D^n(\Lambda/R), \mathfrak{P})=1$.

Case 1. For modules A such that $\pi A=0$, we have $\mathfrak{A}H^n(\Lambda, A)=0$.

Since $\overline{\Lambda} = \overline{\mathfrak{A}} + \overline{\mathfrak{P}}$ we have

$$H^n(\overline{\Lambda}, A) = H^n(\overline{\mathfrak{A}}, \overline{\mathfrak{A}}A\overline{\mathfrak{A}}) + H^n(\overline{\mathfrak{P}}, \overline{\mathfrak{P}}A\overline{\mathfrak{P}})$$

where $\overline{\mathfrak{A}}$ is separable as an $R/(\pi)$ algebra, for $\overline{\mathfrak{A}} = \mathfrak{A}/\pi\Lambda \cong \overline{\Lambda}/\overline{\mathfrak{P}}$. So we have $H^n(\overline{\mathfrak{A}}, \overline{\mathfrak{A}}A\overline{\mathfrak{A}}) = 0$ and $H^n(\Lambda, A)=H^n(\overline{\Lambda}, A)=H^n(\overline{\mathfrak{P}}, \overline{\mathfrak{P}}A\overline{\mathfrak{P}})$. Thus $\mathfrak{A}H^n(\Lambda, A) = 0$ because $(\overline{\mathfrak{P}}A\overline{\mathfrak{P}})\mathfrak{A} = 0$.

Case 2. If A has an annulator π^i, then $\mathfrak{A}^i H^n(\Lambda, A) = 0$.

We prove it by induction. $i = 1$ is the Case 1.

Assume it for $i-1$. From the exact sequence of homomorphisms

$$0 \to \pi A \to A \to A/\pi A \to 0,$$

we have the exact sequence

$$H^n(\Lambda, \pi A) \xrightarrow{\psi} H^n(\Lambda, A) \xrightarrow{\varphi} H^n(\Lambda, A/\pi A).$$

For any u in $H^n(\Lambda, A)$ and any α in A, $\varphi(\alpha u) = \alpha\varphi(u) = 0$ in $H^n(\Lambda, A/\pi A)$; so there exists u' in $H^n(\Lambda, \pi A)$ such that $\psi(u') = \alpha u$. From the assumption of the induction, we have $\alpha' u' = 0$ for any α' in $\mathfrak{A}^{i-1}$. Thus we have

$$\alpha'\alpha\, u = \alpha'\psi(u') = \psi(\alpha' u') = 0$$

where α' and α are any elements in $\mathfrak{A}^{s-1}$ and $\mathfrak{A}$, respectively, and $\sum \alpha'\alpha$ runs over $\mathfrak{A}^i$.

Case 3. For general A, consider the exact sequences

$$0 \longrightarrow \pi^d A \xrightarrow{i} A \longrightarrow A/\pi^d A \longrightarrow 0, \tag{7.11}$$

$$0 \longrightarrow A' \longrightarrow A \xrightarrow{\pi d} \pi^d A \longrightarrow 0, \tag{7.12}$$

where A' is the module of all elements in A such that $\pi^d a = 0$ Then we have

$$H^n(\Lambda, \pi^d A) \xrightarrow{i} H^n(\Lambda, A) \longrightarrow H^n(\Lambda, A/\pi^d A) = 0 \qquad \text{(exact)} \tag{7.13}$$

$$H^n(\Lambda, A) \xrightarrow{\pi d} H^n(\Lambda, \pi^d A) \longrightarrow H^{n+1}(\Lambda, A') = 0 \qquad \text{(exact),} \tag{7.14}$$

Let α, α' be arbitrary elements in $\mathfrak{A}^d$ and let u be any class in $H^n(\Lambda, A)$ Since $\alpha u = 0$ in $H^n(\Lambda, A/\pi^d A)$ in (7.13), there exists u' in $H^n(\Lambda, \pi^d A)$ such that $i(u') = \alpha u$. Since $\alpha' u' = 0$ in $H^{n+1}(\Lambda, A')$ in (7.14), there exists u'' in $H^n(\Lambda, A)$ such that $\pi^d(u'') = \alpha' u'$. Operating i and π^d successively, we have

$$\pi^d \circ i\,(u'') = \alpha'\alpha\, u.$$

On the other hand, as an endomorphism of $H^n(\Lambda, A)$, the mapping $i \circ \pi^d$ is the same as $\widetilde{\pi^d}$ in (1.3), which is the zero endomorphism. Therefore, $\alpha'\alpha\, u = 0$ for any α, α' in $\mathfrak{A}^d$ and u in $H^n(\Lambda, A)$, where $\sum \alpha\alpha'$ runs over $\mathfrak{A}^{2d}$.

The similar proof holds for $H_n(\Lambda, A)$, except for $n = 1$.

Summalizing the above arguments, we have

THEOREM 7.6. *Let n be any fixed positive integer. Then, using $D^n(\Lambda/R)$ only, we have the finiteness of the ramification. The theorem also holds for $D_n(\Lambda/R)$ $n \geqq 1$.*

8. Relations between various differents D^n, D^n_l, D^n_r etc. and the usual different. $\mathfrak{D}$ Let R, Λ, L and K be the same as in § 7. We have already proved that

$$D^n_l(\Lambda/R) = D^n_r(\Lambda/A) = D^n(\Lambda/R), \; = D^n_c(\Lambda/R),$$

$$D_n^l(\Lambda/R) = D_n^r(\Lambda/R) = D_n(\Lambda/R) = D_n^c(\Lambda/R)$$

for $n > 0$. We consider the relations between differents of various dimensions.

THEOREM 8.1.

$$D^1 \subset D^2 \subset \ldots, \quad D_1 \subset D_2 \subset \ldots.$$

PROOF. For any Λ^e-module A we take a Λ^e-injective module I containing A,

$$0 \to A \to I \to A' \to 0 \qquad \text{(exact).} \tag{8.1}$$

Then, for any element $\lambda \in \Lambda$ we have the commutaive diagram

$$\begin{tikzcd}
0 = H^n(\Lambda, I) \arrow[r] \arrow[d, "\lambda \tilde{\otimes} 1"'] & H^n(\Lambda, A') \arrow[r] \arrow[d, "\lambda \tilde{\otimes} 1"'] & H^{n+1}(\Lambda, A) \arrow[r] \arrow[d, "\lambda \tilde{\otimes} 1"'] & H^{n+1}(\Lambda, I) = 0 \arrow[d, "\lambda \tilde{\otimes} 1"'] & \text{(exact)} \\
0 = H^n(\Lambda, I) \arrow[r] & H^n(\Lambda, A') \arrow[r] & H^{n+1}(\Lambda, A) \arrow[r] & H^{n+1}(\Lambda, I) = 0 & \text{(exact).}
\end{tikzcd}$$

Therefore, if $\lambda \in D^n = D^n$ then $\lambda \tilde{\otimes} 1 \; H^{n+1}(\Lambda, A) = 0$ for any A, i.e. $\lambda \in D_l^{n+1} = D^{n+1}$.

As for D_n, we consider A as a homomorphic image of a Λ^e-projective module P,

$$0 \to A'' \to P \to A \to 0 \qquad \text{(exact).} \tag{8.2}$$

Then we have, instead of (8.1),

$$\begin{tikzcd}
H_n(\Lambda, P)(=0) \arrow[r] \arrow[d, "\lambda \tilde{\otimes} 1"'] & H_n(\Lambda, A) \arrow[r] \arrow[d, "\lambda \tilde{\otimes} 1"'] & H_{n-1}(\Lambda, A'') \arrow[r] \arrow[d, "\lambda \tilde{\otimes} 1"'] & H_{n-1}(\Lambda, P)(=0) \arrow[d, "\lambda \tilde{\otimes} 1"'] & \text{(exact)} \\
H_n(\Lambda, P)(=0) \arrow[r] & H_n(\wedge, A \arrow[r] & H_{n-1}(\Lambda, A'') \arrow[r] & H_{n-1}(\Lambda, P)(=0) & \text{(exact).}
\end{tikzcd}$$

So if $\lambda \in D_{n-1}^l$ then $\lambda \tilde{\otimes} 1 \; H_n(\Lambda, A) = 0$ for any A.

Now we consider local theory. Let $\mathfrak{p}$ be a prime ideal in R, $\Lambda_\mathfrak{p}$ and $R_\mathfrak{p}$ be the quotient rings of Λ and R by $\mathfrak{p}$, respectively (as § 6). This is also our number theoretical case; so $\Lambda_\mathfrak{p}$ is $R_\mathfrak{p}$-projective. Moreover we have

PROPOSITION 8.2. *The algebra $\Lambda_\mathfrak{p}$ over $R_\mathfrak{p}$ is a symmetric algebra* (§ 2).

PROOF. Since $\Lambda_\mathfrak{p}$ is the principal order of L over $R_\mathfrak{p}$ and L is separable over the quotient field K of $R_\mathfrak{p}$, $\Lambda_\mathfrak{p}$ is $R_\mathfrak{p}$-free and $R_\mathfrak{p}$-finitely generated. We take a non zero K-homomorphism φ' of L to K. Let $(u_1, \ldots, u_n)$ be a linearly independent basis of $\Lambda_\mathfrak{p}$ over $R_\mathfrak{p}$, $(v_1, \ldots, v_n)$ be the dual basis of $(u_1, \ldots, u_n)$ with respect to φ', as a basis of L/K. Then the $\Lambda_\mathfrak{p}$-module $\{x \in L \mid \varphi'(\Lambda_\mathfrak{p} x) \in R_\mathfrak{p}\}$ is generated by $v_1, \ldots, v_n$ over $R_\mathfrak{p}$. So it is a fractional ideal of $\Lambda_\mathfrak{p}$, it is, therefore, a principal ideal (d'). If we put $\varphi(x) = \varphi'(xd')$, then φ is also a non zero K-homomorphism of L to K, and the dual basis of u with respect to φ is v_i/d', which belongs to $\Lambda_\mathfrak{p}$. So φ is considered an $R_\mathfrak{p}$-homomorphism of $\Lambda_\mathfrak{p}$ to $R_\mathfrak{p}$ and satisfies all the assumption of Prop. 3.1. Therefore, $\Lambda_\mathfrak{p}$ is a symmetric algebra over $R_\mathfrak{p}$.

Prop. 8.2 shows that we may apply the results of §3 to $\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}$. In particular, if we define D_l^{-n} as

(8.3) $D_l^{-n}(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}) = \{\lambda \in \Lambda_{\mathfrak{p}} \mid \lambda \widetilde{\otimes} 1 \; H^{-n}(\Lambda_{\mathfrak{p}}, A) = 0$ for all $\Lambda_{\mathfrak{p}}^e$-module $A\}$

for $n > 1$, then

(8.4) $$D'_n(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}) = D_l^{-n-1}(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}})$$

by virtue of (3.11). We may also define, analogously, $D_l^0(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}})$ and $D_l^{-1}(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}})$ by using (3.8). D_l^0 and D_l^{-1} are not zero ideals, since $\sum u_i v_i$ is a non trivial annulator of H^0 and H^{-1} by (3.8).

PROPOSITION 8.3. *In the local case* $\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}$, *we have*
$$D_l^n = D_l^{n+1}$$
for all integer n.

PROOF. Let A be any $\Lambda_{\mathfrak{p}}$-two sided modul. We take a $\Lambda_{\mathfrak{p}}^e$-injective module I and consider the exact sequence (8.1), then we have, by (3.11) and Prop. 3.2, $D_l^n \subset D_l^{n+1}$ entirely same as the proof of Th. 8.1.

Conversely, if we consider the exact sequence (8.2) and use Prop. 3.3 and Prop 4.5, then we have $D_l^n \supset D_l^{n+1}$. The proof is also the same as in Th. 8.1.

COROLLARY 8.4. *In the local case* $\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}$, *we have*
$$D^n(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}) = (\Sigma u_i v_i)\Lambda_{\mathfrak{p}}$$
where $(u_1, \dots, u_s)$ *is a linearly independent basis of* $\Lambda_{\mathfrak{p}}$ *over* $R_{\mathfrak{p}}$ *and* $(v_1, \dots, v_s)$ *is a dual basis of* $(u_1, \dots, u_s)$.

PROOF. From (3.8) it is obvious that $D^0(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}) \ni \Sigma u_i v_i$. Conversely, if λ belongs to $D^0(\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}})$, then for the $\Lambda_{\mathfrak{p}}^e$-module $\Lambda_{\mathfrak{p}}$ we have $\lambda \Lambda_{\mathfrak{p}} \subset (\Sigma u_i v_i)\Lambda_{\mathfrak{p}}$; in particular $\lambda \cdot 1 \in (\Sigma u_i v_i)\Lambda_{\mathfrak{p}}$.

THEOREM 8.5. *The homological and cohomological differents of various dimensions are all equal each other.*

PROOF. It follows immediately from Prop. 8.3 and Th. 6.3.

Now we consider the relations between the different $\mathfrak{D}$ in ordinary sense and our homological differents. It is sufficient, by Th. 6.3, to compare the $\mathfrak{p}$-component of two differents.

THEOREM 8.6. *The homological (cohomological) different is equal to the usual different.*

PROOF. It is sufficient to prove for the $\mathfrak{p}$-component. In the local case $\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}$, let (δ) be the inverse different defined by $Sp_{L/K}$. Then the proof of Prop. 8.2 shows that $\varphi(x) = Sp(x\delta)$ is the defining homomorphism of the symmetric algebra $\Lambda_{\mathfrak{p}}/R_{\mathfrak{p}}$. Let (u) be a basis of $\Lambda_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$ and (v), (v') be the dual bases of (u) with respect to Sp and φ, respectively. From Prop 4.2

we have $\sum uv' = (\sum uv)\delta^{-1}$. But by Prop. 4.4, $\sum uv = 1$. Thus we have

$$D^n(\Lambda_\mathfrak{p}/R_\mathfrak{p}) = D^0(\Lambda_\mathfrak{p}/R_\mathfrak{p}) = (\Sigma uv')\Lambda_\mathfrak{p} = (\delta^{-1})\Lambda_\mathfrak{p} = \mathfrak{D}(\Lambda_\mathfrak{p}/R_\mathfrak{p}).$$

As for the connections between the elements $\sum u_i v_i$ and the usual different, we have the following theorem something like to that of Dedekind.

THEOREM 8.7. *The different $\mathfrak{D}(\Lambda/R)$ is the greatest common divisor of all the elements*

$$\sum u_i' v_i'$$

where $(u_1', \ldots, u_n')$ is a basis of L/K contained in Λ and $(v_1', \ldots, v_n')$ is the dual basis of $(u_1', \ldots u_n')$ with respect to some K-homomorphism of L to K and also belong to Λ.

PROOF. It follows from Prop. 4.2' that $\sum u_i' v' \in D^0(\Lambda_\mathfrak{p}/R_\mathfrak{p}) = \mathfrak{D}(\Lambda_\mathfrak{p}/R_\mathfrak{p})$. It is sufficient, therefore, to prove that there exists one of above elements $\sum u_i' v_i'$ such that $\mathfrak{p}$-component of the principal ideal $(\sum u_i v_i')$ is $D^0(\Lambda_\mathfrak{p}/D_\mathfrak{p})$.

Let φ be the defining $R_\mathfrak{p}$-homomorphism of the symmetric algebra $\Lambda_\mathfrak{p}/R_\mathfrak{p}$, $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ a dual bases of $\Lambda_\mathfrak{p}$ over $R_\mathfrak{p}$ with respect to φ. We extend φ to a K-homomorphism $\bar{\varphi}$ of L to K. Since $u_1, \ldots, u_n$ are $\mathfrak{p}$-integral, their denominators are prime to $\mathfrak{p}$, even if they do not belong to Λ. So we may take $a_1, \ldots, a_n$ in R, all prime to $\mathfrak{p}$, and $(u_1a_1, u_2a_2, \ldots, u_na_n)$ is a (linearly independent) basis of $\Lambda_\mathfrak{p}/R_\mathfrak{p}$ contained in Λ. The dual basis of (u_ia_i) is $(v_ia_i^{-1})$. Since $(a_i, \mathfrak{p}) = 1$, $v_ia_i^{-1}$ are all $\mathfrak{p}$-integral. There exists, therefore, an element b in R, $(b, \mathfrak{p}) = 1$, such that $v_ia_i^{-1}b$ are all in Λ. Now we take K-homomorphism $\bar{\varphi}'$ defined by $\bar{\varphi}'(x) = \varphi(xb^{-1})$. Then the dual basis of $(u_i a_i)$ with respect to $\bar{\varphi}'$ is $(v_ia_i^{-1} b)$; this is the basis in the present proposition. On the other hand, as b is a $\mathfrak{p}$-unit, $\bar{\varphi}'$ induces an $R_\mathfrak{p}$-homomorphism φ' of $\Lambda_\mathfrak{p}$ to $R_\mathfrak{p}$ which is also a defining map of the symmetric algebra $\Lambda_\mathfrak{p}/R_\mathfrak{p}$. Thus we have

$$\sum_i (u_i a_i)(v_i a_i^{-2} b)\Lambda_\mathfrak{p} = D^0(\Lambda_\mathfrak{p}/R_\mathfrak{p}) = \mathfrak{D}(\Lambda_\mathfrak{p}/R_\mathfrak{p}),$$

which proves the proposition.

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