ALMOST ANALYTIC VECTORS IN ALMOST COMPLEX SPACES

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0. Introduction. In a complex space covered by a system of complex coordinate neighborhoods $(z^{\kappa}, z^{\bar{\kappa}})$[1], the fact that the components $(v^{\kappa}, v^{\bar{\kappa}})$ of a self-conjugate contravariant vector field v^h are analytic functions of complex coordinates of the form

$$v^{\kappa} = v^{\kappa}(z^{\lambda}), \qquad v^{\bar{\kappa}} = v^{\bar{\kappa}}(z^{\bar{\lambda}}) \tag{0.1}$$

and the fact that the components $(w_{\lambda}, w_{\bar{\lambda}})$ of a self-conjugate covariant vector field w_i are analytic functions of complex coordinates of the form

$$w_{\lambda} = w_{\lambda}(z^{\kappa}), \qquad w_{\bar{\lambda}} = w_{\bar{\lambda}}(z^{\bar{\kappa}}) \tag{0.2}$$

have both a meaning which is independent of the choice of the local complex coordinates. We call such vector fields a *contravariant analytic vector* and a *covariant analytic vector* respectively.

In the case where the complex space admits a Kähler metric $ds^2 = 2g_{\mu\bar{\lambda}}dz^{\mu}dz^{\bar{\lambda}}$, equations (0. 1) and (0. 2) can be written in the form

$$\nabla_{\bar{\mu}}v^{\kappa} = 0, \qquad \nabla_{\mu}v^{\bar{\kappa}} = 0 \tag{0.3}$$

and

$$\nabla_{\bar{\mu}}w_{\lambda} = 0, \qquad \nabla_{\mu}w_{\bar{\lambda}} = 0 \tag{0.4}$$

respectively, where ∇_i denotes the covariant differentiation with respect to the Riemannian connection $\{{}_j{}^h{}_i\}$ defined by the Kähler metric g_{ji}.

Using the tensor

$$F_i{}^h = \begin{pmatrix} \sqrt{-1}\,\delta_{\lambda}^{\mu} & 0 \\ 0 & -\sqrt{-1}\,\delta_{\bar{\lambda}}^{\bar{\mu}} \end{pmatrix}, \tag{0.5}$$

we can write (0. 3) and (0. 4) in the form

$$F_i{}^a\nabla_a v^h - F_a{}^h\nabla_i v^a = 0 \tag{0.6}$$

and

$$F_j{}^a\nabla_a w_i - F_i{}^a\nabla_j w_a = 0 \tag{0.7}$$

1) In the sequel, the Latin indices $h, i, j, \ldots\ldots$ run over the range $1, 2, \ldots\ldots, n; \bar{1}, \bar{2}, \ldots\ldots, \bar{n}$, and the Greek indices $\kappa, \lambda, \mu, \ldots\ldots$ over the range $1, 2, \ldots\ldots, n$.

respectively.

In the case where the Kähler space is compact, one of the present authors has proved (K. Yano [6], [7])[2]: *A necessary and sufficient condition for a vector v^h in a compact Kähler space to be contravariant analytic is*

$$g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i = 0, \tag{0.8}$$

where K_{ji} is the Ricci tensor of the space.

Since a necessary and sufficient condition for a vector field v^h in a compact orientable Riemannian space to be a Killing vector is that

$$g^{ji}\nabla_j\nabla_i v^h + K_i{}^h v^i = 0, \quad \nabla_i v^i = 0, \tag{0.9}$$

(K. Yano [7], [8]), we can see from (0.8) and (0.9) that *a contravariant analytic vector v^h satisfying $\nabla_i v^i = 0$ in a compact Kähler space is a Killing vector and that a Killing vector in a compact Kähler space is contravariant analytic.*

One of the present authors has also proved (K. Yano [6]): *A necessary and sufficient condition for a vector w_i in a compact Kähler space to be covariant analytic is*

$$g^{ji}\nabla_j\nabla_i w_h - K_h{}^i w_i = 0. \tag{0.10}$$

Since a necessary and sufficient condition for a vector w_i in a compact orientable Riemannian space to be harmonic is given just by (0.10) (K. Yano, [7], [8]), we see that *a necessary and sufficient condition for a vector w_i in a compact Kähler space to be covariant analytic is that w_i be harmonic.*

The purpose of the present paper is to generalize these results to the case of the most general almost Hermitian space, that is, to the case of spaces in which a mixed tensor $F_i{}^h$ satisfying

$$F_j{}^i F_i{}^h = -A_j^h, \tag{0.11}$$

A_j^h being the unit tensor, and a Riemannian metric g_{ji} satisfying

$$g_{ji} = F_j{}^c F_i{}^b g_{cb} \tag{0.12}$$

are given.

The Kähler space is characterized by the equation

$$\nabla_j F_i{}^h = 0. \tag{0.13}$$

An almost Hermitian space in which we have

$$\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0, \tag{0.14}$$

where $F_{ji} = F_j{}^a g_{ai}$ is called an almost Kähler space (K. Yano [7]). The results

2) See the Bibliography at the end of the paper.

mentioned above have been already generalized to the case of almost Kähler spaces by S. Tachibana [4]. S. Tachibana [5] has also generalized these results to the case where

$$\nabla_j F_i^{\ h} + \nabla_i F_j^{\ h} = 0 \tag{0.15}$$

is satisfied. We shall call such an almost Hermitian space an *almost Tachibana space*.

In § 1 we give some important formulas in the differential geometry of almost complex spaces and of almost Hermitian spaces. In § 2 we give several formulas which are valid in almost Kähler and almost Tachibana spaces. The § 3 is devoted to the discussions of curvature tensors of these spaces. In § 4 and § 5 we discuss contravariant and covariant almost analytic vectors in the most general almost Hermitian space and deduce as corollaries the theorems on these vectors in almost Kähler and almost Tachibana spaces.

1. Preliminaries. We consider a $2n$-dimensional real differentiable manifold M of class C^∞ covered by a system of coordinate neighborhoods (ξ^h). We can introduce in this manifold M a system of complex coordinate neighborhoods $(z^\kappa, z^{\bar\kappa})$ defined by

$$z^\kappa = \xi^\kappa + \sqrt{-1}\,\xi^{\bar\kappa}, \quad z^{\bar\kappa} = \xi^\kappa - \sqrt{-1}\,\xi^{\bar\kappa}. \tag{1.1}$$

If we can cover the manifold M by a system of complex coordinate neighborhoods $(z^\kappa, z^{\bar\kappa})$ in such a way that in the intersection of two complex coordinate neighborhoods $(z^\kappa, z^{\bar\kappa})$ and $(z^{\kappa'}, z^{\bar\kappa'})$ we have

$$z^{\kappa'} = f^{\kappa'}(z^\lambda), \qquad z^{\bar\kappa'} = f^{\bar\kappa'}(z^{\bar\lambda}) \tag{1.2}$$

$$\left| \frac{\partial f^{\kappa'}}{\partial z^\kappa} \right| \neq 0, \tag{1.3}$$

where $f^{\bar\kappa'}$ are complex conjugate functions of $f^{\kappa'}$, we say that the manifold M admits a complex structure and call M a complex manifold.

When we write above equations in the form

$$\xi^{h'} = \xi^{h'}(\xi), \tag{1.4}$$

$\xi^{h'}(\xi)$ are real analytic functions of ξ^h and

$$\left| \frac{\partial \xi^{h'}}{\partial \xi^h} \right| = \left| \frac{\partial z^{\kappa'}}{\partial z^\kappa} \right| \cdot \left| \frac{\partial z^{\bar\kappa'}}{\partial z^{\bar\kappa}} \right| > 0. \tag{1.5}$$

Thus a complex manifold is of class C^ω and orientable.

The complex structure is also characterized by the existence of a mixed tensor $F_i^{\ h}$ which has numerical components

$$F_i^{\ h} = \begin{pmatrix} \sqrt{-1}\,\delta_\lambda^\kappa & 0 \\ 0 & -\sqrt{-1}\,\delta_{\bar\lambda}^{\bar\kappa} \end{pmatrix} \tag{1.6}$$

in all complex coordinate neighborhoods $(z^\kappa, \bar{z}^\kappa)$ and consequently satisfies

$$(1.7) \qquad F_j^{\ t} F_t^{\ h} = -A_j^h,$$

where A_j^h is the unit tensor.

Indeed the components (1.6) of the tensor in the coordinate neighborhoods $(z^\kappa, \bar{z}^\kappa)$ and the components

$$(1.8) \qquad F_{i'}^{\ h'} = \begin{pmatrix} \sqrt{-1}\,\delta_{\lambda'}^{\kappa'} & 0 \\ 0 & -\sqrt{-1}\,\delta_{\bar{\lambda}'}^{\bar{\kappa}'} \end{pmatrix}$$

of the tensor in the coordinate neighborhoods $(z^{\kappa'}, \bar{z}^{\kappa'})$ should be related by equations of the form

$$F_{i'}^{\ h'} = \frac{\partial z^{h'}}{\partial z^h} \frac{\partial z^t}{\partial z^{i'}} F_t^{\ h}$$

from which we find

$$\frac{\partial z^{\kappa'}}{\partial \bar{z}^\lambda} = 0, \qquad \frac{\partial \bar{z}^{\kappa'}}{\partial z^\lambda} = 0$$

and consequently we get equations of the form (1.2).

We next consider a $2n$-dimensional real differentiable manifold M of class C^∞ covered by a system of coordinate neighborhoods (ξ^h). If there exists a mixed tensor $F_i^{\ h}$ of class C^∞ which satisfies (1.7), we say that the manifold M admits an almost complex structure and call such a manifold an almost complex manifold. An almost complex manifold is orientable. If there exists a system of complex coordinate neighborhoods $(z^\kappa, \bar{z}^\kappa)$ with respect to which the tensor $F_i^{\ h}$ has always numerical components (1.6), then we say that the almost complex structure induces a complex structure.

It is now a well known fact (A. Newlander and L. Nirenberg [1]) that an almost complex structure $F_i^{\ h}$ induces a complex structure if and only if the so-called Nijenhuis tensor

$$(1.9) \qquad N_{ji}^{\ \ h} = F_j^{\ a}(\partial_a F_i^{\ h} - \partial_i F_a^{\ h}) - F_i^{\ a}(\partial_a F_j^{\ h} - \partial_j F_a^{\ h})$$

vanishes identically, where ∂_a denotes partial differentiation with respect to the coordinate ξ^a.

The Nijenhuis tensor $N_{ji}^{\ \ h}$ satisfies the following identities (K. Yano [7]):

$$(1.10) \qquad N_{ja}^{\ \ a} = 0,$$

$$(1.11) \qquad N_{ji}^{\ \ h} + N_{ij}^{\ \ h} = 0,$$

$$(1.12) \qquad N_{ja}^{\ \ h} F_i^{\ a} = -N_{ji}^{\ \ a} F_a^{\ h} = -N_{ia}^{\ \ h} F_j^{\ a},$$

$$(1.13) \qquad N_{ji}^{\ \ h} + F_j^{\ c} F_i^{\ b} N_{cb}^{\ \ h} = 0, \quad N_{ji}^{\ \ h} - F_i^{\ b} F_a^{\ h} N_{jb}^{\ \ a} = 0.$$

We now introduce the following tensors (K. Yano [7]):

$$O_{ia}^{bh} = \frac{1}{2}(A_i^b A_a^h - F_i^{\ b} F_a^{\ h}), \tag{1.14}$$

$$*O_{ia}^{bh} = \frac{1}{2}(A_i^b A_a^h + F_i^{\ b} F_a^{\ h}). \tag{1.15}$$

For a mixed tensor $T_i^{\ h}$ for example we form

$$O_{ia}^{bh} T_b^{\ a} \tag{1.16}$$

and call it the *pure part* of the tensor $T_i^{\ h}$. In the case where the almost complex structure $F_i^{\ h}$ induces a complex structure and $F_i^{\ h}$ has numerical components (1.6), putting

$$T_i^{\ h} = \begin{pmatrix} T_\lambda^{\ \kappa} & T_\lambda^{\ \bar{\kappa}} \\ T_{\bar{\lambda}}^{\ \kappa} & T_{\bar{\lambda}}^{\ \bar{\kappa}} \end{pmatrix},$$

we have

$$O_{ia}^{bh} T_b^{\ a} = \begin{pmatrix} T_\lambda^{\ \kappa} & 0 \\ 0 & T_{\bar{\lambda}}^{\ \bar{\kappa}} \end{pmatrix}.$$

For a mixed tensor $T_i^{\ h}$, we form also

$$*O_{ia}^{bh} T_b^{\ a} \tag{1.17}$$

and call it the *hybrid part* of the tensor $T_i^{\ h}$. In the complex case, we have

$$*O_{ia}^{bh} T_b^{\ a} = \begin{pmatrix} 0 & T_\lambda^{\ \bar{\kappa}} \\ T_{\bar{\lambda}}^{\ \kappa} & 0 \end{pmatrix}.$$

Similarly $O_{ji}^{cb} T_{cb}$ is the pure part of the tensor T_{ji} and $*O_{ji}^{cb} T_{cb}$ is the hybrid part of the tensor T_{ji}.

Take a general tensor $T_{\cdot\cdot i\cdot\cdot}^{\cdot\cdot h\cdot\cdot}$. If we have

$$O_{ia}^{bh} T_{\cdot\cdot b\cdot\cdot}^{\cdot\cdot a\cdot\cdot} = 0, \tag{1.18}$$

that is, if the pure part of the tensor with respect to the indices h and i vanishes, we say that the tensor $T_{\cdot\cdot i\cdot\cdot}^{\cdot\cdot h\cdot\cdot}$ is hybrid in h and i. If we have

$$*O_{ia}^{bh} T_{\cdot\cdot b\cdot\cdot}^{\cdot\cdot a\cdot\cdot} = 0, \tag{1.19}$$

that is, if the hybrid part of the tensor with respect to the indices h and i vanishes, we say that the tensor $T_{\cdot\cdot i\cdot\cdot}^{\cdot\cdot h\cdot\cdot}$ is pure in h and i.

Similary if we have

$$O_{ji}^{cb} T_{\cdot\cdot c\cdot\cdot b\cdot\cdot}^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot} = 0, \tag{1.20}$$

we say that the tensor $T_{\cdot\cdot j\cdot\cdot i\cdot\cdot}^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}$ is hybrid in j and i and if we have

$$*O_{ji}^{cb}T_{\cdot\cdot c\cdot\cdot b\cdot\cdot}^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot} = 0, \tag{1.21}$$

we say that the tensor $T_{\cdot\cdot j\cdot\cdot i\cdot\cdot}^{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}$ is pure in j and i.

The almost complex structure $F_i^{\ h}$ satisfying (1.7), we have

$$*O_{ia}^{bh} F_b^{\ a} = \frac{1}{2}(A_i^b A_a^h + F_i^{\ b}F_a^{\ h})F_b^{\ a} = 0, \tag{1.22}$$

which shows that $F_i^{\ h}$ is pure in h and i.

The equations (1.13) may respectively be written as

$$*O_{ji}^{cb} N_{cb}^{\ \ h} = 0, \quad O_{ia}^{bh} N_{jb}^{\ \ a} = 0, \tag{1.23}$$

which shows that $N_{ji}^{\ \ h}$ is pure in j and i and is hybrid in h and i.

Now we can introduce in the differentiable manifold M of class C^∞ a positive definite Riemannian metric a_{ji} of class C^∞. From this we form

$$g_{ji} = \frac{1}{2}(a_{ji} + F_j^{\ c}F_i^{\ b}a_{cb}) = *O_{ji}^{cb} a_{cb}, \tag{1.24}$$

then the g_{ji} thus defined is also positive definite and satisfies

$$g_{ji} = F_j^{\ c}F_i^{\ b}g_{cb}, \tag{1.25}$$

that is

$$O_{ji}^{cb} g_{cb} = 0, \tag{1.26}$$

which shows that the covariant tensor g_{ji} is hybrid in j and i.

When an almost complex manifold admits a hybrid positive definite Riemannian metric g_{ji}, we call such a metric a *Hermitian metric*. We call an almost complex manifold admitting a Hermitian metric an *almost Hermitian space*. When the almost complex manifold reduces to a complex manifold an almost Hermitian space is called a *Hermitian space*.

In an almost Hermitian space, we put

$$F_{ji} = F_j^{\ a}g_{ai}, \tag{1.27}$$

then (1.7), (1.25) and (1.27) give

$$F_{ji} = -F_{ij}, \tag{1.28}$$

$$O_{ji}^{cb} F_{cb} = 0. \tag{1.29}$$

These equations show that F_{ji} is skew-symmetric and is hybrid.

Raising and lowering indices by use of the fundamental metric tensor g_{ji}, we can define F^{ih} and N_{jih}. These tensors satisfy

$$O_{ba}^{ih} F^{ba} = 0, \tag{1.30}$$

$$*O_{ih}^{ba} N_{jba} = 0 \tag{1.31}$$

respectively. Thus F^{ih} is hybrid and N_{jih} is pure in i and h.

Following two lemmas will be very useful in the sequel.

LEMMA 1. *The operators O_{ia}^{bh} and $^*O_{ia}^{bh}$ satisfy*

$$O + {}^*O = A, \qquad O\cdot O = O, \qquad O\cdot{}^*O = 0, \tag{1.32}$$
$$^*O\cdot O = 0, \qquad {}^*O\cdot{}^*O = {}^*O,$$

A being the identity operator.

This will be proved by a straightforward calculation.

LEMMA 2. *Let R_{ji} be pure in j and i and S^{ji} be hybrid in j and i, then we have*

$$R_{ji}\, S^{ji} = 0. \tag{1.33}$$

Indeed, under the assumption, we have

$$^*O\cdot R = 0 \quad \text{and} \quad O\cdot S = 0.$$

From the first of the identities in Lemma 1 and the first of these equations we find

$$R = O\cdot R \quad \text{or} \quad R_{ji} = O_{ji}^{cb}\, R_{cb},$$

thus

$$R_{ji}\, S^{ji} = (O_{ji}^{cb}\, R_{cb})S^{ji} = R_{cb}(O_{ji}^{cb}\, S^{ji}) = 0.$$

Applying these lemmas, we have for example

$$F^{ji}\, N_{ji}{}^{h} = 0, \qquad F^{ih}\, N_{jih} = 0, \tag{1.34}$$

F^{ji} being hybrid in j and i and N_{jih} being pure in j and i and also in i and h.

If we denote by ∇_j the covariant differentiation with respect to Riemannian connection defined by the almost Hermitian metric g_{ji}, the Nijenhuis tensor $N_{ji}{}^{h}$ can be written as

$$N_{ji}{}^{h} = F_j{}^{a}(\nabla_a F_i{}^{h} - \nabla_i F_a{}^{h}) - F_i{}^{a}(\nabla_a F_j{}^{h} - \nabla_j F_a{}^{h}). \tag{1.35}$$

We now define following tensors:

$$F_{jih} = \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji}, \tag{1.36}$$

$$F_i = \nabla^a F_{ai}, \tag{1.37}$$

$$G_{ji}{}^{h} = \nabla_j F_i{}^{h} + \nabla_i F_j{}^{h}, \tag{1.38}$$

where

$$\nabla^a = g^{ai}\nabla_i. \tag{1.39}$$

We call an *almost Kähler* (*Tachibana*)[3] space an almost Hermitian space in which $F_{jih} = 0$ ($G_{ji}{}^h = 0$) is satisfied. We call a *Kähler* (*Tachibana*) *space* an almost Kähler (Tachibana) space in which $N_{ji}{}^h = 0$ is satisfied.

2. Almost Kähler and almost Tachibana spaces. The covariant components of the Nijenhuis tensor may be written in the form

$$N_{jih} = F_j{}^a F_{aih} - F_i{}^a F_{ajh} + 2F_j{}^a(\nabla_h F_{ia}), \tag{2.1}$$

from which we have

THEOREM 2.1. *In an almost Kähler space, we have*

$$N_{jih} = 2F_j{}^a(\nabla_h F_{ia}) \tag{2.2}$$

and consequently $\nabla_j F_i{}^h$ *is pure in j and i.*

THEOREM 2.2. *In a Kähler space, we have*

$$\nabla_j F_i{}^h = 0 \tag{2.3}$$

and conversely, if (2.3) *is satisfied in a Hermitian space, it is a Kähler space.*

On the other hand, the Nijenhuis tensor is also written in the form

$$N_{ji}{}^h = -4(\nabla_j F_i{}^a)F_a{}^h + 2G_{ji}{}^a F_a{}^h + F_j{}^a G_{ai}{}^h - F_i{}^a G_{aj}{}^h, \tag{2.4}$$

from which we get

THEOREM 2.3. *In an almost Tachibana space, we have*

$$N_{ji}{}^h = -4(\nabla_j F_i{}^a)F_a{}^h \tag{2.5}$$

and consequently $\nabla_j F_i{}^h$ *is pure in j and i.*

THEOREM 2.4. *In a Tachibana space, we have* (2.3) *and consequently a Tachibana space is a Kähler space.*

Contracting F^{ih} to equation (2.1) we find, by virtue of (1.10) and (1.34),

$$0 = N_{jih}F^{ih} = F_j{}^a F_{aih}F^{ih} + 2F_j,$$

from which

$$F_{jih}F^{ih} = 2F_a F_j{}^a. \tag{2.6}$$

Equation (2.6) proves

THEOREM 2.5. *In an almost Kähler space, we have*

$$F_j = 0 \tag{2.7}$$

3) See, S. Tachibana [5].

and consequently F_{ji} is a harmonic tensor.

A skew-symmetric tensor $T_{i_1 i_2 \cdots i_p}$ is called a *Killing tensor* (K. Yano and S. Bochner [8]) when it satisfies

$$\nabla_j T_{i i_2 \cdots i_p} + \nabla_i T_{j i_2 \cdots i_p} = 0$$

and consequently

$$\nabla^a T_{a i_2 \cdots i_p} = 0.$$

Thus in an almost Tachibana space, the tensor F_{ji} is a Killing tensor and we have $F_j = 0$.

By a straightforward calculation, we find

$$3\nabla_j F_{ih} - F_{jih} = G_{jih} - G_{ijh}, \tag{2.8}$$

from which

THEOREM 2.6. *In an almost Tachibana space we have*

$$3\nabla_j F_{ih} = F_{jih} \tag{2.9}$$

and consequently $\nabla_j F_{ih}$ is skew-symmetric in all its indices.

In general the Nijenhuis tensor N_{jih} satisfies

$$N_{jih} + N_{jhi} = - F_{jia} F_h{}^a - F_{jha} F_i{}^a + 2F_j{}^a G_{iha}. \tag{2.10}$$

Thus if, in an almost Kähler space, we have $N_{jih} + N_{jhi} = 0$, then we deduce from it $G_{jih} = 0$ and the space is an almost Tachibana space. Thus from Theorem 2.6 we find $3\nabla_j F_{ih} = F_{jih} = 0$ and consequently the space is a Kähler space. It is evident that in a Kähler space we have $N_{jih} + N_{jhi} = 0$. Thus we have

THEOREM 2.7. *An almost Kähler space is a Kähler space if and only if*

$$N_{jih} + N_{jhi} = 0, \tag{2.11}$$

that is, if and only if N_{jih} is skew-symmetric in all its indices (S. Sawaki and S. Kotō [3]).

Also the tensor $G_{ji}{}^h$ satisfies

$$2{}^*O_{ji}^{cb} G_{cbh} = - (F_j{}^c F_{cbi} + F_i{}^c F_{cbj}) F_h{}^b, \tag{2.12}$$

from which

THEOREM 2.8. *In an almost Kähler space, the tensor G_{jih} is pure in j and i.*

From (2.10) and (2.12), we have in general

$$N_{jih} + N_{jhi} = 2F_j^{\ c}\, O_{ih}^{ba}\, G_{bac} \tag{2.13}$$

and we get

THEOREM 2.9. *In an almost Tachibana space, the Nijenhuis tensor N_{jih} is skew-symmetric in all its indices.*

For an almost Tachibana space, we have from Theorems 2.3 and 2.6

$$N_{jih} = \frac{4}{3} F_{jia} F_h^{\ a}, \tag{2.14}$$

or equivalently

$$F_{jih} = -\frac{3}{4} N_{jia} F_h^{\ a}. \tag{2.15}$$

Thus we have

THEOREM 2.10. *In an almost Tachibana space, the tensor $N_{jia} F_h^{\ a}$ is skew-symmetric in all its indices.*

3. Curvature tensors. We denote the curvature tensor of the Hermitian metric g_{ji} by

$$K_{kji}^{\ \ \ h} = \partial_k \left\{ {h \atop j\,i} \right\} - \partial_j \left\{ {h \atop k\,i} \right\} + \left\{ {h \atop k\,a} \right\} \left\{ {a \atop j\,i} \right\} - \left\{ {h \atop j\,a} \right\} \left\{ {a \atop k\,i} \right\}, \tag{3.1}$$

its covariant components by

$$K_{kjih} = K_{kji}^{\ \ \ a} g_{ah}, \tag{3.2}$$

and the Ricci tensor and the curvature scalar by

$$K_{ji} = K_{aji}^{\ \ \ a} \quad \text{and} \quad K = g^{ji} K_{ji} \tag{3.3}$$

respectively. Moreover we put

$$H_{kj} = \frac{1}{2} K_{kjih} F^{ih} \tag{3.4}$$

and

$$K^*_{ji} = -H_{ja} F_i^{\ a} \quad \text{or} \quad K^*_{ja} F_i^{\ a} = H_{ji}. \tag{3.5}$$

Thus H_{ji} and $K^*_{ja} F_i^{\ a}$ are both skew-symmetric. From (3.5) we find $K^*_{ja} F_i^{\ a} + K^*_{ia} F_j^{\ a} = 0$, from which

$$K^*_{ji} = F_i^{\ c} F_j^{\ b} K^*_{cb}. \tag{3.6}$$

Thus if K^*_{ji} is symmetric then it is hybrid.

Applying the Ricci identity to the tensor $F_i^{\ h}$, we find

$$\nabla_k \nabla_j F_i^{\ h} - \nabla_j \nabla_k F_i^{\ h} = K_{kja}^{\ \ \ h} F_i^{\ a} - K_{kji}^{\ \ \ a} F_a^{\ h}. \tag{3.7}$$

Contraction with respect to k and h in this equation gives

$$\nabla_a \nabla_j F_i^{\ a} = K_{ja} F_i^{\ a} - H_{ji} - \nabla_j F_i, \tag{3.8}$$

from which

$$\nabla_a G_{ji}^{\ \ a} = K_{ja} F_i^{\ a} + K_{ia} F_j^{\ a} - (\nabla_j F_i + \nabla_i F_j). \tag{3.9}$$

Thus

THEOREM 3.1. *In an almost Tachibana space, the tensor* $K_{ja} F_i^{\ a}$ *is skew-symmetric and consequently* K_{ji} *is hybrid.*

Equation (3.8) can be written as

$$\nabla_a \nabla_j F_i^{\ a} + \nabla_j F_i = (K_{ja} - K^*_{ja}) F_i^{\ a}. \tag{3.10}$$

Thus we have

THEOREM 3.2. *In order that* $K_{ji} = K^*_{ji}$ *in an almost Hermitian space, it is necessary and sufficient that*

$$\nabla_a \nabla_j F_i^{\ a} + \nabla_j F_i = 0. \tag{3.11}$$

COROLLARY 1. *In order that* $K_{ji} = K^*_{ji}$ *in an almost Hermitian space with* $F_i = 0$, *it is necessary and sufficient that*

$$\nabla_a \nabla_j F_i^{\ a} = 0. \tag{3.12}$$

COROLLARY 2. *In order that* $K_{ji} = K^*_{ji}$ *in an almost Tachibana space, it is necessary and sufficient that*

$$\nabla^a F_{aih} = 0 \quad or \quad \nabla^a \nabla_a F_{ih} = 0. \tag{3.13}$$

COROLLARY 3. *In a Kähler space, we have*

$$K_{ji} = K^*_{ji}. \tag{3.14}$$

4. Contravariant almost analytic vectors. Let us consider a self-conjugate contravariant vector field $(v^\kappa, v^{\bar\kappa})$ in a complex manifold. If the components $v^\kappa(v^{\bar\kappa})$ are functions of $z^\lambda(z^{\bar\lambda})$ only, then the vector is called a *contravariant analytic vector* (S. Sasaki and K. Yano [2], K. Yano [7]). The condition for v^h to be a contravariant analytic vector is given by

$$\partial_{\bar\lambda} v^\kappa = 0, \qquad \partial_\lambda v^{\bar\kappa} = 0, \tag{4.1}$$

where

$$\partial_{\bar\lambda} = \partial / \partial z^{\bar\lambda}, \qquad \partial_\lambda = \partial / \partial z^\lambda.$$

The condition (4.1) is equivalent to

$$^*O_{ia}^{bh}\, \partial_b v^a = 0, \tag{4.2}$$

that is to the fact that $\partial_b v^a$ is pure. The condition (4. 2) is also written as

$$\underset{v}{\pounds} F_i^{\ h} = v^a \partial_a F_i^{\ h} - F_i^{\ a} \partial_a v^h + F_a^{\ h} \partial_i v^a = 0, \tag{4. 3}$$

which is a tensor equation, where $\underset{v}{\pounds}$ denotes the Lie differentiation with respect to the vector field v^h (K. Yano [7]).

Thus in an almost complex space, we define a *contravariant almost analytic vector field* as a vector which satisfies (4. 3), that is, as a vector v^h, an infinitesimal transformation with respect to which does not change the almost complex structure.

In an almost Hermitian space, the equation (4. 3) may be written as

$$\underset{v}{\pounds} F_i^{\ h} = v^a \nabla_a F_i^{\ h} - F_i^{\ a} \nabla_a v^h + F_a^{\ h} \nabla_i v^a = 0, \tag{4. 4}$$

from which

$$v^a \nabla_a F_{ih} - F_i^{\ a} \nabla_a v_h - F_h^{\ a} \nabla_i v_a = 0,$$

and taking the symmetric part of this with respect to i and h

$$O_{ji}^{cb} (\nabla_c v_b + \nabla_b v_c) = 0 \quad \text{or} \quad O_{ji}^{cb} (\underset{v}{\pounds} g_{cb}) = 0, \tag{4. 5}$$

and also

$$O_{cb}^{ji} (\nabla^c v^b + \nabla^b v^c) = 0 \quad \text{or} \quad O_{cb}^{ji} (\underset{v}{\pounds} g^{cb}) = 0, \tag{4. 6}$$

where

$$\underset{v}{\pounds} g_{ji} = v^a \nabla_a g_{ji} + g_{ai} \nabla_j v^a + g_{ja} \nabla_i v^a = \nabla_j v_i + \nabla_i v_j$$

and

$$\underset{v}{\pounds} g^{ji} = v^a \nabla_a g^{ji} - g^{ai} \nabla_a v^j - g^{ja} \nabla_a v^i = - \nabla^j v^i - \nabla^i v^j.$$

Equation (4. 5) and (4. 6) show that $\underset{v}{\pounds} g_{ji}$ and $\underset{v}{\pounds} g^{ji}$ are both hybrid for a contravariant almost analytic vector v^h in an almost Hermitian space.

Now by a straightforward calculation we can prove

$$\frac{1}{2}(F_j^{\ a} F_{ai}^{\ \ h} + F_i^{\ a} F_{aj}^{\ \ h}) - G_{ji}^{\ \ a} F_a^{\ h}$$
$$+ F_j^{\ c} F_i^{\ b} \left[\frac{1}{2}(F_c^{\ a} F_{ab}^{\ \ h} + F_b^{\ a} F_{ac}^{\ \ h}) - G_{cb}^{\ \ a} F_a^{\ h} \right] = 0,$$

which shows that the tensor

$$\frac{1}{2}(F_j^{\ a} F_{ai}^{\ \ h} + F_i^{\ a} F_{aj}^{\ \ h}) - G_{ji}^{\ \ a} F_a^{\ h},$$

symmetric in j and i, is pure in j and i. Thus from Lemma 2 and (4. 6) which shows that $\underset{v}{\pounds} g^{ji}$ is hybrid, we obtain

$$\frac{1}{2}(F_j{}^a F_{ai}{}^h + F_i{}^a F_{aj}{}^h)(\underset{v}{\pounds}\, g^{ji}) - G_{ji}{}^a F_a{}^h(\underset{v}{\pounds}\, g^{it}) = 0$$

or

$$\frac{1}{2}F_{ji}{}^h(\underset{v}{\pounds}\, F^{ji}) = G_{ji}{}^a F_a{}^h(\nabla^j v^i) \tag{4.7}$$

by virtue of $\underset{v}{\pounds}\, F_i{}^h = 0$ for a contravariant almost analytic vector field v^h in an almost Hermitian space.

Now applying the operator ∇^t to (4. 4), we find

$$F_a{}^h[\nabla^t\nabla_i v^a + K_i{}^a v^t - F_i{}^a \underset{v}{\pounds} F^t - G_{ji}{}^b F_b{}^a(\nabla^j v^t)] = 0,$$

or

$$\nabla^t\nabla_i v^h + K_i{}^h v^t - F_i{}^h \underset{v}{\pounds} F^t - G_{ji}{}^a F_a{}^h(\nabla^j v^t) = 0, \tag{4.8}$$

or equivalently

$$\nabla^t\nabla_i v^h + K_i{}^h v^t - F_i{}^h \underset{v}{\pounds} F^t - \frac{1}{2}F_{ji}{}^h(\underset{v}{\pounds} F^{jt}) = 0 \tag{4.9}$$

by virtue of (4. 7).

Equations (4. 7) and (4. 8) or (4. 7) and (4. 9) are necessary conditions for a vector v^h in a general almost Hermitian space to be contravariant almost analytic. Thus a necessary condition for a vector v^h in an almost Kähler space to be contravariant almost analytic is

$$\nabla^t\nabla_i v^h + K_i{}^h v^t = 0, \qquad G_{ji}{}^h(\nabla^j v^i) = 0,$$

and a necessary condition for a vector v^h in an almost Tachibana space to be contravariant almost analytic is

$$\nabla^t\nabla_i v^h + K_i{}^h v^t = 0, \qquad F_{ji}{}^h(\underset{v}{\pounds} F^{ji}) = 0.$$

If we put

$$T^{ji} = g^{ja}(\underset{v}{\pounds} F_a{}^i) \tag{4.10}$$

we have

$$\begin{aligned} \frac{1}{2}T^{ji}T_{ji} = {} & \frac{1}{2}v^c v^b(\nabla_c F^{ji})(\nabla_b F_{ji}) - v^c(\nabla_c F^{ji})F_j{}^b(\nabla_b v_i) \\ & + v^b(\nabla_b F^{ji})F_{ai}(\nabla_j v^a) + (\nabla^j v^i)(\nabla_j v_i) \\ & - F^{jc}F_{bi}(\nabla_c v^i)(\nabla_j v^b) \end{aligned} \tag{4.11}$$

and

$$\nabla^j\{(\underset{v}{\pounds} F_j{}^a)F_a{}^t v_t\} + \left[\nabla^t\nabla_t v^h + K_t{}^h v^t - F_t{}^h(\underset{v}{\pounds} F^t) - \frac{1}{2}F_{ji}{}^h(\underset{v}{\pounds} F^{ji})\right]v_h \tag{4. 12}$$
$$+\frac{1}{2}T^{ji}T_{ji} = 0.$$

Assuming that the Hermitian space is compact, we have from (4. 12)

$$\int_M\left[\{\nabla^t\nabla_t v^h + K_t{}^h v^t - F_t{}^h(\underset{v}{\pounds} F^t) - \frac{1}{2}F_{ji}{}^h(\underset{v}{\pounds} F^{ji})\}v_h\right. \tag{4. 13}$$
$$\left.+\frac{1}{2}T^{ji}T_{ji}\right]d\sigma = 0,$$

where $d\sigma$ denotes the volume element of the space.

From (4. 9) and (4. 13), we obtain

THEOREM 4. 1. *A necessary and sufficient condition for a vector field v^h in a compact almost Hermitian space to be contravariant almost analytic is*

$$\nabla^t\nabla_t v^h + K_t{}^h v^t - F_t{}^h(\underset{v}{\pounds} F^t) - \frac{1}{2}F_{ji}{}^h(\underset{v}{\pounds} F^{ji}) = 0.$$

COROLLARY 1. *A necessary and sufficient condition for a vector field v^h in a compact almost Kähler space to be contravariant almost analytic is*

$$\nabla^t\nabla_t v^h + K_t{}^h v^t = 0 \tag{4. 14}$$

(S. Tachibana [4]).

Since a necessary and sufficient condition for a vector field v^h in a compact orientable Riemannian space to be a Killing vector field is

$$\nabla^t\nabla_t v^h + K_t{}^h v^t = 0, \qquad \nabla_t v^t = 0$$

we have

COROLLARY 2. *A Killing vector field in a compact almost Kähler space is contravariant almost analytic.*

This corollary may be proved also in the following way. Since v^h is a Killing vector, we have

$$\underset{v}{\pounds}\, g_{ji} = 0 \quad \text{and} \quad \underset{v}{\pounds}\, g^{ji} = 0.$$

On the other hand, it is well known that the Lie derivative of a harmonic tensor with respect to a Killing vector vanishes in a compact orientable Riemannian space (K. Yano [7]). Since F_{ji} is harmonic in a compact almost Kähler space, we have

$$\underset{v}{\pounds} F_{ji} = 0$$

from which

$$\underset{v}{\pounds} F_i{}^h = \underset{v}{\pounds}(F_{ia}g^{ah}) = 0,$$

which shows that v^h is contravariant almost analytic.

COROLLARY 3. *A contravariant almost analytic vector field v^h satisfying $\nabla_i v^i = 0$ in a compact almost Kähler space is a Killing vector.*

From (4. 7), (4. 8) and (4. 13), we have

COROLLARY 4. *A necessary condition for a vector field v^h in an almost Tachibana space to be contravariant almost analytic is that*

$$\nabla^t\nabla_t v^h + K_t{}^h v^t = 0, \qquad F_{ji}{}^h(\underset{v}{\pounds} F^{ji}) = 0$$

and a sufficient condition for v^h in a compact almost Tachibana space to be contravariant almost analytic is

$$\nabla^t\nabla_t v^h + K_t{}^h v^t - \frac{1}{2} F_{ji}{}^h(\underset{v}{\pounds} F^{ji}) = 0.$$

5. Covariant almost analytic vectors. In a complex manifold, a self-conjugate covariant vector field $(w_\lambda, w_{\bar\lambda})$ is said to be *covariant analytic* when its components $w_\lambda(w_{\bar\lambda})$ are functions of $z^\kappa(z^{\bar\kappa})$ only. The condition for $(w_\lambda, w_{\bar\lambda})$ to be covariant analytic is given by

(5. 1) $$\partial_{\bar\mu} w_\lambda = 0, \qquad \partial_\mu w_{\bar\lambda} = 0$$

or

(5. 2) $$^*O_{ji}^{cb}\, \partial_c\, w_b = 0$$

or

(5. 3) $$[(\partial_j F_i{}^a) - (\partial_i F_j{}^a)]w_a - F_j{}^a \partial_a w_i + F_i{}^a \partial_j w_a = 0,$$

which is easily verified to be a tensor equation.

Thus in an almost complex space we define a covariant almost analytic vector as a vector field w_i which satisfies (5. 3).

In an almost Hermitian space, the equation (5. 3) may be written as

(5. 4) $$[(\nabla_j F_i{}^a) - (\nabla_i F_j{}^a)]w_a - F_j{}^a \nabla_a w_i + F_i{}^a \nabla_j w_a = 0,$$

from which, taking the symmetric part with respect to j and i, we find

(5. 5) $$^*O_{ji}^{cb}\,(\nabla_c w_b - \nabla_b w_c) = 0,$$

which shows that $\nabla_j w_i - \nabla_i w_j$ is pure for a covariant almost analytic vector w_i in an almost Hermitian space.

Contracting $\nabla_k F^{ji}$ to the equation (5. 4), we find

$$(5.6) \qquad (\nabla_k F^{ji})(\nabla_j F_i^{\ h})w_h = 0$$

for a covariant almost analytic vector field w_i.

Now we define tensors P_{ji} and Q_{ji} by

$$P_{ji} = (\nabla_j F_i^{\ a} - \nabla_i F_j^{\ a})w_a, \qquad Q_{ji} = F_j^{\ a}\nabla_a w_i - F_i^{\ a}\nabla_j w_a$$

respectively. Then for a covariant almost analytic vector field w_i, we have

$$P_{ji} = Q_{ji},$$

$$P_{ji}P^{ji} = 2(F_{jia} - \nabla_a F_{ji})(\nabla^j F^{ih})w^a w_h$$

or

$$P_{ji}P^{ji} = 2F_{ji}^{\ \ b}(\nabla^j F^{ia})w_b w_a$$

by virtue of (5. 6) and

$$P_{ji}Q^{ji} = F_j^{\ b}(\nabla_b w_i + \nabla_i w_b)(G^{jia} - 2\nabla^i F^{ja})w_a.$$

Thus in an almost Kähler space we have $P_{ji}P^{ji} = 0$ for a covariant almost analytic vector w_i from which we obtain

$$P_{ji} = 0, \qquad Q_{ji} = 0.$$

But in an almost Kähler space, we have

$$\nabla_j F_i^{\ a} - \nabla_i F_j^{\ a} = - \nabla^a F_{ji}.$$

Thus $P_{ji} = 0$ gives

$$w^a \nabla_a F_{ji} = 0.$$

On the other hand, in an almost Tachibana space, we have, following Theorem 2. 6,

$$\nabla_j F_{ih} = \frac{1}{3} F_{jih}$$

and consequently (5. 6) may be written as

$$F_{jib}F^{jia}w_a = 0,$$

from which

$$(F_{ji}^{\ \ b}w_b)(F^{jia}w_a) = 0,$$

and consequently $F_{ji}^{\ \ a}w_a = 0$, that is,

$$(\nabla_j F_i^{\ a})w_a = 0 \quad \text{or} \quad w^a \nabla_a F_{ji} = 0,$$

by virtue of $\nabla_j F_{ia} = \nabla_a F_{ji}$.

Thus in an almost Tachibana space, we have $P_{ji}Q^{ji} = 0$ and the equations $P_{ji} = Q_{ji}$ and $P_{ji}Q^{ji} = 0$ give $P_{ji} = 0$, $Q_{ji} = 0$, that is,

$$w^a \nabla_a F_{ji} = 0, \qquad Q_{ji} = 0$$

for a covariant almost analytic vector in an almost Tachibana space. Thus we have

THEOREM 5. 1. *A necessary and sufficient condition for a vector field w_i in an almost Kähler or an almost Tachibana space to be covariant almost analytic is that*

$$w^a \nabla_a F_{ji} = 0, \tag{5.7}$$

$$F_j{}^a \nabla_a w_i - F_i{}^a \nabla_j w_a = 0. \tag{5.8}$$

For a covariant almost analytic vector field w_i in an almost Hermitian space, we have

$$\begin{aligned}
N_{ji}{}^h w_h &= [F_j{}^a(\nabla_a F_i{}^h - \nabla_i F_a{}^h) - F_i{}^a(\nabla_a F_j{}^h - \nabla_j F_a{}^h)]w_h \\
&= F_j{}^a(F_a{}^t \nabla_t w_i - F_i{}^t \nabla_a w_t) - F_i{}^a(F_a{}^t \nabla_t w_j - F_j{}^t \nabla_a w_t) \\
&= -(\nabla_j w_i - \nabla_i w_j) - F_j{}^c F_i{}^b(\nabla_c w_b - \nabla_b w_c) \\
&= -2{}^*O_{ji}^{cb}(\nabla_c w_b - \nabla_b w_c),
\end{aligned}$$

that is,

$$N_{ji}{}^h w_h = 0 \tag{5.9}$$

by virtue of (5. 4) and (5. 5).

For such a vector, we have also

$$\begin{aligned}
&(\nabla_j F_i{}^a - \nabla_i F_j{}^a)(\nabla^j w^t)F_a{}^h \\
&= \frac{1}{2}(\nabla_j F_i{}^a - \nabla_i F_j{}^a)(\nabla^j w^t - \nabla^t w^j)F_a{}^h \\
&= \frac{1}{2}(\nabla_j F_i{}^a - \nabla_i F_j{}^a)O_{cb}^{ji}(\nabla^c w^b - \nabla^b w^c)F_a{}^h \\
&= \frac{1}{2}[(\nabla_c F_b{}^a - \nabla_b F_c{}^a) - F_c{}^j F_b{}^t(\nabla_j F_t{}^a - \nabla_t F_j{}^a)](\nabla^c w^b)F_a{}^h
\end{aligned}$$

by virtue of

$$\nabla^j w^t - \nabla^t w^j = O_{cb}^{jt}(\nabla^c w^b - \nabla^b w^c)$$

derived from (5. 5). From this we have

$$(\nabla_j F_i^{\ a} - \nabla_i F_j^{\ a})(\nabla^j w^t) F_a^{\ h} = -\frac{1}{2} N_{ji}^{\ \ h}(\nabla^j w^t)$$

and consequently

$$(5.10) \qquad (\nabla_j F_i^{\ a} - \nabla_i F_j^{\ a})(\nabla^j w^t) F_a^{\ h} w_h = 0,$$

by virtue of (5. 9), for a covariant almost analytic vector field w_i in an almost Hermitian space.

If we suppose that w_i is a contravariant and at the same time covariant almost analytic vector field in an almost Hermitian space, then adding

$$w^a \nabla_a F_j^{\ h} - F_j^{\ a} \nabla_a w^h + F_a^{\ h} \nabla_j w^a = 0$$

or

$$w^a \nabla_a F_{ji} - F_j^{\ a} \nabla_a w_i - F_i^{\ a} \nabla_j w_a = 0$$

and

$$(\nabla_j F_{ia} - \nabla_i F_{ja}) w^a - F_j^{\ a} \nabla_a w_i + F_i^{\ a} \nabla_j w_a = 0,$$

we find

$$(5.11) \qquad F_{jia} w^a - 2 F_j^{\ a} \nabla_a w_i = 0.$$

In an almost Kähler space, equation (5. 11) reduces to

$$F_j^{\ a} \nabla_a w_i = 0.$$

In an almost Tachibana space, (5. 11) is also written as

$$3 w^a \nabla_a F_{ji} - 2 F_j^{\ a} \nabla_a w_i = 0$$

or

$$F_j^{\ a} \nabla_a w_i = 0$$

by virtue of $w^a \nabla_a F_{ji} = 0$ in Theorem 5. 1. Thus

THEOREM 5.2. *If, in an almost Kähler or almost Tachibana space, w_i is a contravariant and at the same time covariant almost analytic vector field, then it is covariantly constant.*

The equation (5. 4) is written as

$$(5.12) \qquad \nabla_j \tilde{w}_i - \nabla_i \tilde{w}_j = F_j^{\ a}(\nabla_a w_i - \nabla_i w_a),$$

where

$$(5.13) \qquad \tilde{w}_i = F_i^{\ a} w_a.$$

The equation (5. 12) may also be written as

$$(5.14) \qquad -(\nabla_j w_i - \nabla_i w_j) = F_j^{\ a}(\nabla_a \tilde{w}_i - \nabla_i \tilde{w}_a).$$

The equations (5. 12) and (5. 14) give

THEOREM 5. 3. *If a vector field* w_i *in an almost Hermitian space is covariant almost analytic, then the vector field* $\tilde{w}_i = F_i^{\ a} w_a$ *is also covariant almost analytic.*

If vectors w_i and $\tilde{w}_i$ are both closed, then equation (5. 12) is satisfied. Thus we have

THEOREM 5. 4. *If vectors* w_i *and* $\tilde{w}_i = F_i^{\ a} w_a$ *in an almost Hermitian space are both closed, then they are both covariant almost analytic vectors.*

From (5. 12) we have by contractions of F^{ji} and of g^{ji}

(5. 15) $$F^{ji}\nabla_j \tilde{w}_i = 0 \quad \text{and} \quad F^{ji}(\nabla_j w_i - \nabla_i w_j) = 0$$

respectively. Thus applying $g^{ji}\nabla_j$ to $\tilde{w}_i = F_i^{\ a} w_a$, we find

(5. 16) $$g^{ji}\nabla_j \tilde{w}_i - F^i w_i = 0.$$

If a covariant almost analytic vector field w_i is closed, we have from (5. 12)

(5. 17) $$\nabla_j \tilde{w}_i - \nabla_i \tilde{w}_j = 0.$$

Thus from (5. 16) and (5. 17) we have

THEOREM 5. 5. *If, in an almost Hermitian space with* $F^i = 0$, *a covariant almost analytic vector* w_i *is closed, then* $\tilde{w}_i$ *is harmonic* (S. Tachibana [4], [5]).

Thus $\tilde{w}_i$ is covariant almost analytic (Theorem 5. 3) and is closed and we have

THEOREM 5. 6. *If, in an almost Hermitian space with* $F^i = 0$, *a covariant almost analytic vector field* w_i *is closed, then it is harmonic.*

Now applying $F_h^{\ j}\nabla^i$ to (5. 4) and changing indices, we obtain

(5. 18) $$\nabla^a \nabla_a w_i - (2K^*_{ji} - K_{ji}) w^j + F_i^{\ c}\nabla^b (F_{cba} w^a) + (\nabla^c w^b) G_{cba} F_i^{\ a} + F_i^{\ a}(w^b \nabla_b F_a + F_b \nabla^b w_a) = 0.$$

For the tensor T_{ji} defined by

(5. 19) $$T_{ji} = (\nabla_j F_i^{\ a} - \nabla_i F_j^{\ a}) w_a - F_j^{\ a}\nabla_a w_i + F_i^{\ a}\nabla_j w_a,$$

we have the identity

(5. 20) $$\nabla^j (T_{ji} F_a^{\ t} w^a) + [\nabla^a \nabla_a w_i - (2K^*_{ji} - K_{ji}) w^j + F_i^{\ c}\nabla^b (F_{cba} w^a) + (\nabla^c w^b) G_{cba} F_i^{\ a}$$

$$+ F_i^{\ a}(w^b \nabla_b F_a + F_b \nabla^b w_a)$$
$$- (\nabla_c F_{ba} - \nabla_b F_{ca})(\nabla^c w^b) F_i^{\ a}] w^i$$
$$+ \frac{1}{2} T^{ji} T_{ji} = 0.$$

Thus, in a compact almost Hermitian space, we have

$$\begin{aligned}(5.21)\qquad \int_M \Big[\{ \nabla^a \nabla_a w_i &- (2K^*_{ji} - K_{ji}) w^j + F_i^{\ c} \nabla^b (F_{cba} w^a) \\ &+ (\nabla^c w^b) G_{cba} F_i^{\ a} + F_i^{\ a}(w^b \nabla_b F_a + F_b \nabla^b w_a) \\ &- (\nabla_c F_{ba} - \nabla_b F_{ca})(\nabla^c w^b) F_i^{\ a} \} w^i \\ &+ \frac{1}{2} T^{ji} T_{ji} \Big] d\sigma = 0,\end{aligned}$$

and consequently

THEOREM 5.7. *A necessary condition for a vector field w_i in an almost Hermitian space to be covariant almost analytic is that* (5.10) *and* (5.18) *are satisfied and a sufficient condition for w_i in a compact almost Hermitian space to be covariant almost analytic is*

$$\begin{aligned}(5.22)\qquad \nabla^a \nabla_a w_i &- (2K^*_{ji} - K_{ji}) w^j + F_i^{\ c} \nabla^b (F_{cba} w^a) \\ &+ (\nabla^c w^b) G_{cba} F_i^{\ a} + F_i^{\ a}(w^b \nabla_b F_a + F_b \nabla^b w_a) \\ &- (\nabla_c F_{ba} - \nabla_b F_{ca})(\nabla^c w^b) F_i^{\ a} = 0.\end{aligned}$$

COROLLARY 1. *A necessary condition for a vector field w_i in an almost Kähler space to be covariant almost analytic is that*

$$(5.23)\qquad (\nabla_a F_{ji})(\nabla^j w^i) F_h^{\ a} w^h = 0$$

and

$$(5.24)\qquad \nabla^a \nabla_a w_i - (2K^*_{ji} - K_{ji}) w^j + (\nabla^c w^b) G_{cba} F_i^{\ a} = 0$$

are satisfied and a sufficient condition for w_i in a compact almost Kähler space to be covariant almost analytic is

$$\begin{aligned}(5.25)\qquad \nabla^a \nabla_a w_i &- (2K^*_{ji} - K_{ji}) w^j \\ &+ (\nabla^c w^b) G_{cba} F_i^{\ a} + (\nabla_a F_{cb})(\nabla^c w^b) F_i^{\ a} = 0.\end{aligned}$$

The equation (5.24) can be written as

$$\nabla^a \nabla_a w_i - K_{ji} w^j - 2(K^*_{ji} - K_{ji}) w^j + (\nabla^c w^b) G_{cba} F_i^{\ a} = 0.$$

On the other hand, taking account of (3.10) and (5.7), we have

$$- 2(K^*_{ji} - K_{ji})w^j + (\nabla^c w^b)G_{cba}F_i^{\ a}$$
$$= - 2F_i^{\ l}(\nabla_a \nabla_j F_l^{\ a})w^j + F_i^{\ l}(\nabla^a w^j)G_{ajl}$$
$$= 2F_i^{\ l}(\nabla_j F_{la})(\nabla^a w^j) + F_i^{\ l}(\nabla^a w^j)(\nabla_a F_{jl} + \nabla_j F_{al})$$
$$= F_i^{\ l}(\nabla^a w^j)(\nabla_a F_{jl} - \nabla_j F_{al})$$

and consequently

$$(\nabla^a \nabla_a w_i - K_{ji}w^j)w^i = - F_i^{\ l}(\nabla^a w^j)(\nabla_a F_{jl} - \nabla_j F_{al})w^i = 0$$

by virtue of (5. 10). Thus the integral formula (K. Yano [7])

$$\int_M \Big[(\nabla^a \nabla_a w_i - K_{ji}w^j)w^i + \frac{1}{2}(\nabla^j w^i - \nabla^i w^j)(\nabla_j w_i - \nabla_i w_j) + (\nabla_j w^j)(\nabla_i w^i)\Big] d\sigma = 0$$

shows that

$$\nabla_j w_i - \nabla_i w_j = 0, \ \nabla_i w^i = 0,$$

that is, w_i is harmonic. Thus we have

COROLLARY 2. *A covariant almost analytic vector in a compact almost Kähler space is harmonic.*

For a covariant almost analytic vector field w_i in an almost Tachibana space, taking account of (3. 10) and of (5. 7), we have

$$- (K^*_{ji} - K_{ji})w^j = (\nabla_j F^{ba})(\nabla_i F_{ba})w^j = 0,$$

that is, $K^*_{ji} w^j = K_{ji}w^j$. And consequently, from Theorem 5. 7, we have

COROLLARY 3. *A necessary condition for a vector field w_i in an almost Tachibana space to be covariant almost analytic is that*

$$(\nabla_j F_i^{\ a})(\nabla^j w^i)F_a^{\ h} w_h = 0 \tag{5.26}$$

and

$$\nabla^a \nabla_a w_i - K_{ji}w^j = 0 \tag{5.27}$$

are satisfied and a sufficient condition for w_i in a compact almost Tachibana space to be covariant analytic is

$$\nabla^a \nabla_a w_i - (2K^*_{ji} - K_{ji})w^j + F_i^{\ c} \nabla^b (F_{cba} w^a) - 2(\nabla_c F_{ba})(\nabla^c w^b)F_i^{\ a} = 0. \tag{5.28}$$

From (5. 27) we have

COROLLARY 4. *A covariant almost analytic vector in a compact almost*

Tachibana space is harmonic.

By a similar computation, we can prove the following integral formula which is valid in a compact almost Hermitian space:

$$\begin{aligned}(5.29)\quad \int_M \Big[&\{\nabla^a\nabla_a w_i - (2K^*_{ji} - K_{ji})w^j + F_i^{\ c}\nabla^b(F_{cba}w^a) \\ &+ (\nabla^c w^b)G_{cba}F_i^{\ a} + F_i^{\ a}(w^b\nabla_b F_a + F_b\nabla^b w_a)\}w^i \\ &- \frac{1}{2}F_i^{\ c}\{w_b(\nabla_c F^b) + F^b\nabla_b w_c + F_{cba}(\nabla^b w^a)\}w^i + \frac{1}{2}T^{ji}T_{ji} \\ &+ \frac{1}{2}\Big\{\frac{1}{2}(\nabla^b\tilde{w}^a - \nabla^a\tilde{w}^b)(\nabla_b\tilde{w}_a - \nabla_a\tilde{w}_b) + (\nabla_a\tilde{w}^a)^2\Big\}\Big]d\sigma = 0,\end{aligned}$$

from which we have

THEOREM 5.8. *A necessary and sufficient condition for a vector w_i in a compact almost Kähler space to be covariant almost analytic is*

$$\nabla^a\nabla_a w_i - (2K^*_{ji} - K_{ji})w^j + (\nabla^c w^b)G_{cba}F_i^{\ a} = 0.$$

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