ON THE CLASS OF SATURATION IN THE THEORY OF APPROXIMATION III[1)]

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1. Introduction. Let $f(x)$ be integrable $(-\pi,\pi)$ and be periodic with period 2π, and let

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x).$$

We denote the Riesz typical means of the above series by

$$X_n^k(x) = \sum_{\nu=0}^{n} A_\nu(x)(1-\nu^k/n^k),$$

then the following results are known [A. Zygmund [6] and G. Sunouchi-C. Watari [5]).

(1°) $|f(x) - X_n^k(x)| = o(n^{-k})$ uniformly, implies that $f(x)$ is a constant.

(2°) $|f(x) - X_n^k(x)| = O(n^{-k})$ uniformly, implies

$|f^{(k)}(x)| \leqq M$ (when k is an even integer)

$|\tilde{f}^{(k)}(x)| \leqq M$ (when k is an odd integer).

(3°) If $|f^{(k)}(x)| \leqq M$ (when k is an even integer)

$|\tilde{f}^{(k)}(x)| \leqq M$ (when k is an odd integer),

then

$$|f(x) - X_n^k(x)| = O(n^{-k}) \quad \text{uniformly}.$$

We denote the Riesz means of the α-th[2)] order of the Fourier series of $f(x)$ by

$$X_n^{k,(\alpha)}(x) = \sum_{\nu=0}^{n} A_\nu(x)(1-\nu^k/n^k)^\alpha,$$

then we have proved the same results. In fact, the propositions (1°) and (2°)

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2) We assume α is a positive integer. $X_n^{k,(\alpha)}(x)$ is different from ordinary Riesz means which have a continuous parameter n. But (C,α)-summability implies $X_n^{k,(\alpha)}$-summability. See M. Riesz [3].

are proved in the paper of Sunouchi-Watari [5] and (3°) is proved in the following way. When $\alpha = 2$,

$$|f(x) - X_n^{k,(2)}(x)| \leqq |f(x) - X_n^{(k)}(x)| + |X_n^k(x) - X_n^{k,(2)}(x)| = I_1 + I_2,$$

say. $\qquad I_1 = O(n^{-k})$, by (3°) and

$$\begin{aligned} |I_2| &= \left| \sum_{\nu=0}^{n} A_\nu(x)\left(1 - \frac{\nu^k}{n^k}\right) - \sum_{\nu=0}^{n} A_\nu(x)\left(1 - \frac{\nu^k}{n^k}\right)^2 \right| \\ &= \left| \sum_{\nu=0}^{n} A_\nu(x)\left(1 - \frac{\nu^k}{n^k}\right)\frac{\nu^k}{n^k} \right| \\ &= \frac{1}{n^k}\left| \sum_{\nu=0}^{n} \nu^k A_\nu(x)(1 - \nu^k/n^k) \right|. \end{aligned}$$

Since $\sum \nu^k A_\nu(x)$ is the Fourier series of $f^{(k)}(x)$ or $\widetilde{f}^{(k)}(x)$ according to an even k or an odd k, we get $I_2 = O(n^{-k})$. Repeating this argument, we get the desired properties.

But if we consider the local approximation, there is an essential difference between $X_n^{k,(\alpha)}(x)$ $(\alpha < k)$ and $X_n^{k,(\alpha)}(x)$ $(\alpha \geqq k)$.

If

$$|X_n^{k,(\alpha)}(x) - f(x)| = O(n^{-k}),$$

uniformly in an interval, then it is necessary to be

$$a_n = O(n^{-k+\alpha}),\ b_n = O(n^{-k+\alpha}).$$

This is a modification of the well-known limitation theorem of Rieszian means (Chandrasekharan-Minakshisundaram [1], p. 13).

Hence if we consider the local saturation problem,we have to take $X_n^{k,(\alpha)}(x)$-means for $\alpha \geqq k$. In this paper we shall confine ourselves to $X_n^{k,(k)}(x)$-means only. The case $\alpha > k$ is similar.

2. A lemma.

LEMMA 1. (1°) *If k is an even integer and $f^{(k)}(x)$ is continuous over $[-\pi, \pi]$, then*

$$\lim_{n\to\infty} n^k\{X_n^{k,(k)}(x) - f(x)\} = (-1)^{\frac{k}{2}-1} k f^{(k)}(x)$$

boundedly.

(2°) *If k is an odd integer and $\widetilde{f}^{(k)}(x)$ is continuous over $[-\pi, \pi]$, then*

$$\lim_{n\to\infty} n^k\{X_n^{k,(k)}(x) - f(x)\} = (-1)^{\frac{k-1}{2}} k \widetilde{f}^{(k)}(x)$$

boundedly.

The case $k=1$ has been proved previously by the author [4].

PROOF. In the first place, we consider $X_n^k(x)$, where k is even. From the formulas of Zygmund [6, pp. 698-700],

$$X_n^k(x)-f(x)=\frac{1}{\pi}\int^{\infty}\{f(x+u/n)+f(x-u/n)-2f(x)\}\lambda(u)du$$

where

$$\lambda(u)=\frac{\sin u}{u}-(-1)^{\frac{k}{2}}\left(\frac{\sin u}{u}\right)^k.$$

Let us set

$$\Lambda_0(u)=\lambda(u),\ \Lambda_p(u)=\int_u^{\infty}\Lambda_{p-1}(t)dt,$$

then

$$\Lambda_1(0)=\pi/2,\ \Lambda_3(0)=\Lambda_5(0)=\dots\dots=\Lambda_{k-1}(0)=0$$

and

$$\Lambda_{k+1}(0)=\int_0^{\infty}\Lambda_k(t)dt=(-1)^{\frac{k}{2}-1}\frac{\pi}{2}.$$

By the successive integration by parts,

$$\begin{aligned}&X_n^k(x)-f(x)\\&=\frac{1}{\pi n^k}\int_0^{\infty}\left\{f^{(k)}\left(x+\frac{u}{n}\right)+f^{(k)}\left(x-\frac{u}{n}\right)\right\}\Lambda_k(u)du\end{aligned}$$

and

$$\begin{aligned}&n^k\{X_n^k(x)-f(x)\}\\&=\frac{1}{\pi}\int^{\infty}\left\{f^{(k)}\left(x+\frac{u}{n}\right)+f^{(k)}\left(x-\frac{u}{n}\right)-2f^{(k)}(x)\right\}\Lambda_k(u)du\\&+(-1)^{\frac{k}{2}-1}f^{(k)}(x).\end{aligned}$$

Now we shall show that the first term of the right-hand side tends to zero.

Since $\Lambda_k(u)$ is absolutely integrable $(0,\infty)$, for a given ε we can take a δ such that

$$\int_{\delta}^{\infty}|\Lambda_k(u)|du<\varepsilon,$$

and split the integral into two parts,

$$\frac{1}{\pi}\int^{\infty}\left\{f^{(k)}\left(x+\frac{u}{n}\right)+f^{(k)}\left(x-\frac{u}{n}\right)-2f^{(k)}(x)\right\}\Lambda_k(u)du$$
$$=\int_0^{\delta}+\int_{\delta}^{\infty}=I_1+I_2,$$

say. We denote by M the maximum of $|f^{(k)}(x)|$, then

$$|I_2|\leqq 2\varepsilon M.$$

Next we take n so large that

$$|f^{(k)}\left(x+\frac{u}{n}\right)+f^{(k)}\left(x-\frac{u}{n}\right)-2f^{(k)}(x)|<\varepsilon,$$

then

$$|I_1|\leqq\int_0^{\delta}\varepsilon\,|\Lambda_k(u)|du\leqq\varepsilon\int_0^{\infty}|\Lambda_k(u)|du.$$

Hence we get

$$\lim_{n\to\infty} n^k[X_n^k(x)-f(x)]=(-1)^{\frac{k}{2}-1}f^{(k)}(x).$$

Concerning with the $X_n^{k,(2)}(x)$, we proceed

$$\begin{aligned}&n^k[X_n^{k,(2)}(x)-f(x)]\\&\quad=n^k[X_n^{k,(2)}(x)-X_n^k(x)]+n^k[X_n^k(x)-f(x)]\\&\quad=J_1+J_2\end{aligned}$$

say We have proved already

$$\lim_{n\to\infty} J_2=(-1)^{\frac{k}{2}-1}f^{(k)}(x).$$

Since

$$\begin{aligned}J_1&=n^k\left\{\sum_{\nu=0}^{n}\left(1-\frac{\nu^k}{n^k}\right)^2A_\nu(x)-\sum_{\nu=0}^{n}\left(1-\frac{\nu^k}{n^k}\right)A_\nu(x)\right\}\\&=-\left\{\sum_{\nu=0}^{n}\left(1-\frac{\nu^k}{n^k}\right)\nu^kA(x)\right\}\end{aligned}$$

and $f^{(k)}(x)$ is continuous,

$$\lim_{n\to\infty} J_1=(-1)^{\frac{k}{2}-1}f^{(k)}(x).$$

Hence

$$n^k\{X_n^{k,(2)}(x)-f(x)\}=2(-1)^{\frac{k}{2}-1}f^{(k)}(x).$$

Repeating this argument, we get

$$n^k\{X_n^{k,(k)}(x) - f(x)\} = (-1)^{\frac{k}{2}-1}kf(x).$$

In the case k is odd, interchanging the role $f(x)$ and $\tilde{f}(x)$, we have (Zygmund [6], pp. 702-703),

$$\begin{aligned}
&\tilde{X}_n^k(x) - \tilde{f}(x) \\
&= \frac{1}{\pi}\int_0^\infty\left\{f\left(x+\frac{u}{n}\right)-f\left(x-\frac{u}{n}\right)\right\}\mu(u)\,du \\
&= \frac{1}{\pi n^k}\int_0^\infty\left\{f^{(k)}\left(x+\frac{u}{n}\right)+f^{(k)}\left(x-\frac{u}{n}\right)-2f^{(k)}(u)\right\}M_k(u)du \\
&+(-1)^{\frac{k-1}{2}}f^{(k)}(x)
\end{aligned}$$

where

$$\mu(u) = -\frac{\cos u}{u} + (-1)^{\frac{k-1}{2}}\left(\frac{\sin u}{u}\right)^k$$

and

$$M_0(u) = \mu(u),\ M_p(u) = \int_u^\infty M_{p-1}(t)dt.$$

Hence, arguing to the similar with the first case, we get

$$\lim_{n\to\infty} n^k\{X_n^{k,(k)}(x) - f(x)\} = (-1)^{\frac{k-1}{2}}kf^{(k)}(x).$$

That is

$$\lim_{n\to\infty} n^k\{X_n^{k,(k)}(x) - f(x)\} = (-1)^{\frac{k-1}{2}}k\tilde{f}^{(k)}(x).$$

3. Local saturation of Rieszian means.

THEOREM 1. (1°) *If*

$X_n^{k,(k)}(x) - f(x) = o(n^{-k})$ *uniformly in* $[a, b]$, *then* $f(x)$ *or* $\tilde{f}(x)$ *is at most a* $(k-1)$*-th polynomial in* $[a,b]$ *according to an even* k *or an odd* k.

(2°) *If* $X_n^{k,(k)}(x) - f(x) = O(n^{-k})$ *uniformly in* $[a, b]$, *then* $f^{(k)}(x)$ *or* $\tilde{f}^{(k)}(x)$ *is bounded in* $[a,b]$ *according to an even* k *or an odd* k.

PROOF. We denote by $C_0^{(k)}$ the class of functions $g(x)$ such that $g(x)=0$ outside of $[a, b]$ and $g^{(k)}(x)$ is continuous in $[0, 2\pi]$ when k is even.

From the hypothesis of (1°), we have

$$\lim_{n\to\infty} n^k \{X_n^{k,(k)}(x,f) - f(x)\} = 0,$$

uniformly in $[a, b]$, and

$$\lim_{n\to\infty} \int_0^{2\pi} n^k \{X_n^{k,(k)}(x,f) - f(x)\} g(x) dx = 0$$

for all $g(x) \in C^{(k)}$.

Since $X_n^{k,(k)}(x,f)$ has a symmetric kernel representation,

$$\int_0^{2\pi} n^k \{X_n^{k,(k)}(x,f) - f(x)\} g(x) dx$$
$$= \int_0^{2\pi} n^k \{X_n^{k,(k)}(x, g) - g(x)\} f(x), dx.$$

Since we have from Lemma 1

$$\lim_{n\to\infty} n^k \{X_n^{k,(k)}(x, g) - g(x)\} = (-1)^{\frac{k}{2}-1} k g^{(k)}(x)$$

we get

$$\int_0^{2\pi} f(x) g^{(n)}(x) dx = 0.$$

Hence by the well-known lemma (Courant-Hilbert [2], p. 201), $f(x)$ is a polynomial of $(k-1)$-th degree.

In the case k is odd, we have

$$\int_0^{2\pi} f(x) \widetilde{g^{(k)}}(x) dx = 0$$

by the same argument, and this is equivalent with, by the Parseval relation,

$$\int_0^{2\pi} \widetilde{F}(x) g^{(k+1)}(x) dx = 0$$

where $F(x)$ is an indefinite integral of $f(x)$. Hence we get $f(x)$ is at most a polynomial of $(k-1)$-th degree.

(2°) If

$$n^k \{X_n^{k,(k)}(x,f) - f(x)\} = O(1)$$

uniformly in $[a, b]$, by the weak compactness of the space $L_\infty[a, b]$, we can take a subsequence n_ν and a function $h(x) \in L_\infty(a, b)$ such that

$$\lim_{\nu\to\infty} \int_0^{2\pi} n_\nu \{X_{n_\nu}^{k,(k)}(x,f) - f(x)\} g(x) dx$$

$$= \int_0^{2\pi} h(x)g(x)dx.$$

But the right-hand side is equal to

$$\int_0^{2\pi} f(x)g^{(k)}(x)dx$$

and the left-hand side is equal to

$$\int_0^{2\pi} H_k(x)g^{(k)}(x)dx$$

where $H_k(x)$ is a k-th integral of $h(x)$.
Hence

$$H_k(x) - f(x)$$

is at most a polynomial of $(k-1)$-th degree and $f^{(k)}(x)$ is bounded in $[a, b]$. The case where k is odd, is proved in the same way.

THEOREM 2. (1°) *If $f(x) \in L(0, 2\pi)$ and $f^{(k)}(x)$ or $\tilde{f}^{(k)}(x)$ is vanished in $[a, b]$ according to an even k or an odd k, then*

$$X_n^{k,(k)}(x) - f(x) = o(n^{-1})$$

uniformly in $[a+\delta, b-\delta]$ for any $\delta > 0$.

(2°) *If $f^{(k)}(x)$ or $\tilde{f}^{(k)}(x)$ is bounded in $[a,b]$ according to an even k or an odd k, then*

$$X_n^{k,(k)}(x) - f(x) = O(n^{-k})$$

uniformly in $[a+\delta, b-\delta]$ for any $\delta > 0$.

PROOF. (1°) Suppose that k is even, $f(x) \in L(0, 2\pi)$ and $f(x)$ is a polynomial of $(k-1)$-th degree in $[a, b]$ and set

$$f(x) \sim \sum_{\nu=0}^{\infty} A_\nu(x).$$

Now we consider a trigonometric series

$$S_1: \quad \sum_{\nu=0}^{\infty} \nu^k A_\nu(x)$$

and another function $g(x)$ which is a constant in $[0, 2\pi]$. We denote by $F_2(x)$ and $G_2(x)$ the second integrals of $f(x)$ and $g(x)$ respectively. Then, since $F_2(x) - G_2(x)$ is at most a polynomial of $(k+1)$-th degree and the coefficient S_1 is $o(n^k)$, we can conclude that S_1 is uniformly summable (C, k) to zero in

$[a+\delta,\ b-\delta]$ (See Zygmund [7] p. 367). Hence S_1 is uniformly (R, n^k, k)-summable to zero in $[a+\delta,\ b-\delta]$. That is

$$\lim_{n\to\infty}\sum_{\nu=1}^{n}\left(1-\frac{\nu^k}{n^k}\right)^k \nu^k A_\nu(x)=0,$$

uniformly in $[a', b']$.

We set $\qquad \nu^k A_\nu(x)=B_\nu(x), \left(1-\frac{\nu^k}{n^k}\right)^k = T^k_{n,\nu}$

and

$$P_n(x)=\sum_{\nu=0}^{n}\left(1-\frac{\nu^k}{n^k}\right)^k A_\nu(x).$$

Then

$$\begin{aligned}
&P_n(x)-P_{n-1}(x)\\
&=\sum_{\nu=1}^{n}\left(1-\frac{\nu^k}{n^k}\right)^k\frac{B_\nu(x)}{\nu^k}-\sum_{\nu=1}^{n-1}\left\{1-\frac{\nu^k}{(n-1)^k}\right\}^k\frac{B_\nu(x)}{\nu^k}\\
&=\sum_{\nu=1}^{n}\left[\left(1-\frac{\nu^k}{n^k}\right)^k-\left\{1-\frac{\nu^k}{(n-1)^k}\right\}^k\right]\frac{B_\nu(x)}{\nu^k}\\
&=\frac{n^k-(n-1)^k}{n^k(n-1)^k}\sum_{\nu=1}^{n-1}\{T^{k-1}_{n,\nu}+T^{k-2}_{n,\nu}T_{n-1,\nu}+\dots+T_{n,\nu}T^{k-2}_{n-1,\nu}+T^{k-1}_{n-1,\nu}\}B_\nu\\
&=\frac{n^k-(n-1)^k}{n^k(n-1)^k}T_n(B),
\end{aligned}$$

say. Summing up this from N to M, and set

$$\sum_{n=1}^{m}\{n^k-(n-1)^k\}T_n(B)=S_m(B)$$

then

$$\begin{aligned}
&P_M(x)-P_N(x)\\
&=\sum_{n=N+1}^{M}\frac{n^k-(n-1)^k}{n^k(n-1)^k}T_n(B)\\
&=\sum_{n=N+1}^{M}S_n(B)\left\{\frac{1}{n^k(n-1)^k}-\frac{1}{(n+1)^k n^k}\right\}+\frac{S_M(B)}{M^k(M-1)^k}-\frac{S_N(B)}{N^k(N-1)^k}.
\end{aligned}$$

Since

$$\lim_{n\to\infty}\sum_{\nu=1}^{n}\left(1-\frac{\nu^k}{n^k}\right)^k\nu^k A_\nu(x)=0,$$

we have

$$S_n(B) = o(n^k)$$

and

$$|P_M(x) - P_N(x)| = \sum_{n=N+1}^{M-1} \frac{o(n^k)\{(n+1)^k-(n-1)^k\}}{n^k(n-1)^k(n+1)^k} + \frac{o(M^k)}{M^{2k}} + \frac{o(N^k)}{N^{2k}}$$

$$= \sum_{n=N+1}^{M-1} o\left\{\frac{1}{(n-1)^k} - \frac{1}{(n+1)^k}\right\} + o\left(\frac{1}{M^k}\right) + o\left(\frac{1}{N^k}\right)$$

Letting $M \to \infty$, we get $P_M(x) \to f(x)$ and

$$f(x) - P_N(x) = o(N^{-k})$$

uniformly in $[a + \delta, b - \delta]$. Thus we prove the proposition (1°). Another cases are proved in the same way.

From this, we can get the following theorem concerning with local saturaiton.

THEOREM 3. *The local saturation class and order of Rieszian means, is* $\{f(x)$ *is a polynomial of* $(k-1)$*-th degree,* $f^{(k)}(x)$ *is bounded,* $n^{-k}\}$, *when* k *is even and* $\{\tilde{f}(x)$ *is a polynomial of* $(k-1)$*-th degree,* $\tilde{f}^{(k)}(x)$ *is bounded,* $n^{-k}\}$, *when* k *is odd.*

REMARK. Results analogous to Theorem 1, 2, 3 hold for approximation in mean.

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