ON DISTORTIONS IN CERTAIN QUASICONFORMAL MAPPINGS

KAZUO IKOMA AND KÊICHI SHIBATA

(Received October 21, 1960)

Let $w = f(z)$ be a quasiconformal mapping of $|z| < 1$ into the w-plane in the sense of Pfluger-Ahlfors, whose maximal dilatation is not greater than a finite constant $K(\geqq 1)$, then it will be simply referred to a K-QC mapping in $|z| < 1$.

First, we formulate, in §1, a theorem producing Schwarz-Pfluger's theorem [5], next determine in §2 the range of a real number α such that there is no positive finite $\lim_{z\to 0} |f(z)|/|z|^{\alpha}$ for any K-QC mapping $w = f(z)$ in $|z| < 1$ satisfying $f(0) = 0$, and finally in §3, we establish, as applications, some distortion theorems supplementing completely our preceding results [3].

1. A.Pfluger [5] reported that for any K-QC mapping $w = f(z)$ of $|z| < 1$ onto $|w| < 1$ with the limit $\lim_{z\to 0} |f(z) - f(0)|/|z|^{1/K} = c$, $c \leqq 1 - |f(0)|^2 \leqq 1$ holds and $c = 1$ arises when $w = f(z) = e^{i\phi} z|z|^{(1/K)-1}$.

Now, we prove the following theorem producing the above Pfluger's result, and state its corollary.

THEOREM 1. *Let $w = f(z)$ be a K-QC mapping of $|z| < 1$ onto $|w| < 1$ such that $f(0) = 0$. If $\alpha \leqq 1/K$, then there holds*

$$\liminf_{z\to 0} |f(z)|/|z|^{\alpha} \leqq 1,$$

where the equality holds only if $f(z) = e^{i\phi}|z|^{1/K} e^{i\arg z}$ with a real constant ϕ.

PROOF. Denote by $L(r)$ and $A(r)$ respectively the length and the area of the images of $|z| = r$ and $|z| < r$ under $w = f(z)$. Then we have for almost all $r \in (0, 1)$,

$$L(r) = \int_0^{2\pi} \left| \frac{\partial f(re^{i\theta})}{r\partial\theta} \right| r\, d\theta,$$

and for arbitrary r, r' such that $0 < r < r' < 1$,

$$A(r') - A(r) \geqq \int_0^{2\pi} \int_r^{r'} J[f(re^{i\theta})] r d\theta dr,$$

where $J[f]$ means the Jacobian of f.

By using Schwarz's inequality and the well known formula $|\partial f(re^{i\theta})/r\partial\theta|$

$\leqq KJ[f(re^{i\theta})]$ valid for almost all $z = re^{i\theta}$ in $|z| < 1$, we can obtain

$$\frac{dA(r)}{dr} \geqq \frac{[L(r)]^2}{2\pi r K}.$$

Applying the isoperimetric inequality $[L(r)]^2 \geqq 4\pi A(r)$, it follows that

$$\frac{dA(r)}{dr} \geqq \frac{2A(r)}{rK}.$$

From this, we see easily for almost all $r \in (0, 1)$,

$$\frac{d}{dr}\{A(r)/r^{2/K}\} \geqq 0.$$

Since $A(r)$ is an increasing function of r, it is shown immediately by Vallée Poussin's theorem that $A(r)/r^{2/K}$ is a non-decreasing function of r, therefore we have

$$A(r)/\pi r^{2/K} \leqq 1.$$

Put $\min\limits_{|z|=r<1} |f(z)| = m(r)$, then it is evident from $f(0) = 0$ that $\pi\{m(r)\}^2 \leqq A(r)$, hence we obtain

$$\begin{aligned}\liminf_{z\to 0} |f(z)|/|z|^{\alpha} &\leqq \liminf_{z\to 0} |f(z)|/|z|^{1/K} \\ &= \liminf_{r\to 0} m(r)/r^{1/K} \leqq \liminf_{r\to 0} \{A(r)/\pi r^{2/K}\}^{1/2} \leqq 1.\end{aligned}$$

Next, if $\liminf\limits_{z\to 0} |f(z)|/|z|^{\alpha} = 1$, then obviously $A(r) = \pi r^{2/K}$ holds. This implies that the image contour L_r of $|z| = r$ by $w = f(z)$ is a circle with radius $r^{1/K}$ lying in $|w| < 1$. After some computations using the cross ratio, it can be asserted that the modulus of the annular domain bounded by L_r and $|w| = 1$ is not larger than $\log(1/r^{1/K})$ and its modulus equals to the maximum $\log(1/r^{1/K})$ if and only if the center of L_r coincides with $w = 0$. On the other hand, by a well known property of a K-QC mapping, the modulus of the image of $r < |z| < 1$ under any K-QC mapping is not less than $\log(1/r^{1/K})$. Hence, the center of L_r for $0 < r < 1$ is always $w = 0$, and so $w = f(z)$ reduces to a K-QC mapping of $0 < r < |z| < 1$ onto $r^{1/K} < |w| < 1$. Therefore, we can see, by a theorem of A.Mori [4], that $f(z) = e^{i\phi}|z|^{1/K} e^{i\arg z}$.
The converse is trivial, and so our proof is completed.

As an immediate consequence of Theorem 1, we have the following

COROLLARY 1. *Let $w = f(z)$ be a K-QC mapping of $|z| < 1$ onto $|w| < 1$ such that $f(0) = 0$. If $\alpha \geqq K$, then there holds*

$$\limsup_{z\to 0} |f(z)|/|z|^{\alpha} \geqq 1,$$

where the equality holds only if $f(z) = e^{i\phi}\,|z|^{K} e^{i\arg z}$ *with a real constant* ϕ.

2. We denote by $\mathfrak{S}_\alpha$ the family of K-QC mappings in $|z|<1$ satisfying $f(0)=0$ and $\lim_{z\to 0} |f(z)|/|z|^{\alpha} = 1$, where α is real. Before we consider the distortion of the mapping belonging to $\mathfrak{S}_\alpha$, we precede with the following theorem indicating the range of such α as $\mathfrak{S}_\alpha$ is empty.

THEOREM 2. *If* $w=f(z)$ *is a* K-QC *mapping in* $|z|<1$ *such that* $f(0)=0$ *and the positive finite* $\lim_{z\to 0} |f(z)|/|z|^{\alpha}$ (α *is real*) *exists, then there holds* $1/K \leqq \alpha \leqq K$.

PROOF. Let $\zeta = h(w)$ be a mapping which maps the image of $|z|<1$ under $w=f(z)$ conformally onto $|\zeta|<1$ and transforms the origin onto itself, then, by our assumption, the positive finite limit

$$\begin{aligned}\lim_{z\to 0} |h\{f(z)\}|/|z|^{\alpha} &= \lim_{w\to 0} |h(w)|/|w| \cdot \lim_{z\to 0} |f(z)|/|z|^{\alpha} \\ &= h'(0)\cdot \lim_{z\to 0} |f(z)|/|z|^{\alpha}\end{aligned}$$

exists, which shall be denoted by $1/\gamma$.

Moreover, we put $W=\gamma h\{f(z)\} = F(z)$, then it is obvious that $W=F(z)$ is a K-QC mapping of $|z|<1$ onto $|W|<\gamma$, $F(0)=0$ and $\lim_{z\to 0} |F(z)|/|z|^{\alpha} = 1$. From this, corresponding to an arbitrary positive number ε, there is a positive number δ such that

$$(1-\varepsilon)|z|^{\alpha} < |F(z)| < (1+\varepsilon)|z|^{\alpha}$$

for $0<|z|<\delta$. Denote by A the circular annulus bounded by $|z|=r$ with $0<r<\delta$ and $|z|=1$, and by mod $F(A)$ the modulus of the image $F(A)$ of A under $W=F(z)$, then it is easily found that

$$\log\frac{\gamma}{(1+\varepsilon)r^{\alpha}} < \operatorname{mod} F(A) < \log\frac{\gamma}{(1-\varepsilon)r^{\alpha}}.$$

On the other hand, by a well known result of a K-QC mapping, there holds in general

$$\frac{1}{K}\log\frac{1}{r} \leqq \operatorname{mod} F(A) \leqq K\log\frac{1}{r}.$$

Thus, we obtain for such ε and r as above that

$$\log\frac{\gamma}{(1+\varepsilon)r^{\alpha}} < K\log\frac{1}{r}$$

and further

$$\frac{1}{K}\log\frac{1}{r} < \log\frac{\gamma}{(1-\varepsilon)r^{\alpha}},$$

from which follow

$$\frac{\log\dfrac{\gamma}{1+\varepsilon}}{\log\dfrac{1}{r}} + \alpha < K$$

and

$$\frac{1}{K} < \frac{\log\dfrac{\gamma}{1-\varepsilon}}{\log\dfrac{1}{r}} + \alpha.$$

Here, by making $r \to 0$, it is concluded that $\alpha \leqq K$ and $1/K \leqq \alpha$ i.e. $1/K \leqq \alpha \leqq K$. q. e. d.

Theorem 2 implies that the family $\mathfrak{S}_\alpha$ is empty for $\alpha < 1/K$ or $\alpha > K$. Furthermore, it will be shown in §3 that $\mathfrak{S}_\alpha$ is not empty for $1/K \leqq \alpha \leqq K$.

3. Applying our theorems in §1 and §2, we have the following theorems concerning the existence of the positive lower bound of $\min\limits_{0<|z|=r<1} |f(z)|$ and the upper bound of $\max\limits_{0<|z|=r<1} |f(z)|$ for $f(z) \in \mathfrak{S}_\alpha$.

THEOREM 3. *The positive lower bound of* $\min\limits_{|z|=r<1} |f(z)|$ for $f(z) \in \mathfrak{S}_\alpha$ *exists if and only if* $\alpha = 1/K$.

THEOREM 4. *The finite upper bound of* $\max\limits_{|z|=r<1} |f(z)|$ *for* $f(z) \in \mathfrak{S}_\alpha$ *exists if and only if* $\alpha = K$.

The latter implies immediately the following

COROLLARY 2. *The family* $\mathfrak{S}_\alpha$ *is normal if and only if* $\alpha = K$.

By Theorem 2, $\mathfrak{S}_\alpha$ is empty for $\alpha < 1/K$ or $\alpha > K$, and so it is sufficient to prove in the case where $1/K \leqq \alpha \leqq K$. As proof for the necessity in Theorems 3 and 4, we shall present some examples of quasiconformal mappings in the sense of Grötzsch whose dilatations are not larger than K.*)

PROOF OF THEOREM 3. First, Pfluger's estimate [6] :

*) As is well known, these mappings are equivalent to continuously differentiable *K-QC* mappings. (see e. g. Hersch [1])

$$\min_{|z|=r<1} |f(z)| \geqq \frac{1}{4} \{4r/(1+r)^2\}^{1/K}$$

proves the sufficiency.

Next, in the case $1/K < \alpha \leqq 1$, consider the following mapping $w = f_n(z)$:

(1) $$w = |z|^{\alpha}\{1-(1-r_n)|z|^{(\alpha K-1)r_n/K(1-r_n)}\}e^{i\arg z},$$

where $|z|<1$, $0<r_n<1$, and $r_n \to 0$ as $n\to\infty$.

After some elementary calculations, it can be seen that every dilatation of (1) on $|z| = r < 1$ is equal to

$$\{1-(1-r_n)r^{(\alpha K-1)r_n/K(1-r_n)}\}/\alpha\{1-(1-r_n/\alpha K)r^{(\alpha K-1)r_n/K(1-r_n)}\}$$

which is a number lying between 1 and K. Moreover, the mapping (1) transforms the origin onto itself and $\lim_{z\to 0} |w|/|z|^{\alpha} = 1$. Thus (1) is a K-QC mapping of $|z|<1$ onto $|w|<r_n$, and hence $w=f_n(z)$ belongs to $\mathfrak{S}_\alpha$.

On the other hand, it is evident that $\lim_{n\to\infty} f_n(r) = 0$.

In the case $1<\alpha\leqq K$, make the composite mapping $w = f_n(z)$ of the following

(2) $$r_0 s/(1-s)^2 = z/(1-z)^2$$

(3) $$t = |s|^{\alpha} e^{i\arg s},$$

(4) $$w = r_0^{\alpha} t/(1-t)^2,$$

where $|z|<1, r_0 = 4rn/(n+r)^2$, $n = 1, 2, \ldots\ldots\ldots$, and r is fixed arbitrarily in the open interval $(0, 1)$.

Then, it is easily ascertained quite similarly to the argument in [3] that $w = f_n(z)$ belongs to $\mathfrak{S}_\alpha$, while it can be obtained by some elementary computations that

$$\begin{aligned}\lim_{n\to\infty} f_n(-r) &= -\lim_{n\to\infty} \{4nr/(n+r)^2\}^{\alpha}(n+r)^2/4n(1+r)^2 \\ &= -\{r^{\alpha}/(1+r)^2\}\cdot\lim_{n\to\infty}\{4n/(n+r)^2\}^{\alpha-1} = 0. \qquad \text{q.e.d.}\end{aligned}$$

PROOF OF THEOREM 4. First, we consider the case $\alpha = K$. By a well known result of Stoïlow, $w = f(z)$ can be represented in the form $f(z) = g\{\varphi(z)\}$, where $\zeta = \varphi(z)$ is a K-QC mapping of $|z|<1$ onto $|\zeta|<1$ and $w = g(\zeta)$ is a regular schlicht function in $|\zeta|<1$. In particular, we shall choose $\varphi(z)$ such that $\varphi(0) = 0$.

Denote by ρ the largest distance from $\zeta = 0$ to the image contour Λ_r of $|z| = r$ under $\zeta = \varphi(z)$, then obviously

$$\max_{|z|=r} |f(z)| = \max_{\zeta \text{ on } \Lambda_r} |g(\zeta)| \leqq \max_{|\zeta|=\rho} |g(\zeta)|.$$

According to a generalization of Schwarz-Grötzsch's theorem by Hersch-Pfluger [2] or A.Mori [4], there holds for $0 < |z| < 1$,

$$|\varphi(z)| \leqq k\{q^{1/K}(|z|)\},$$

where $k\{q\} = \theta_2^2(0)/\theta_3^2(0)$ and θ_2, θ_3 are elliptic theta functions. Hence we have

$$\max_{|\zeta|=\rho} |g(\zeta)| \leqq \max_{|\zeta|=k\{q^{1/K}(r)\}} |g(\zeta)|.$$

Further, Koebe's distortion theorem implies that

$$\max_{|\zeta|=k\{q^{1/K}(r)\}} |g(\zeta)| \leqq |g'(0)| k\{q^{1/K}(r)\}/[1 - k\{q^{1/K}(r)\}]^2.$$

From our normalization, it follows that

$$\begin{aligned} 1 &= \lim_{z\to 0} |f(z)|/|z|^\alpha = \lim_{\zeta\to 0} |g(\zeta)|/|\zeta| \cdot \lim_{z\to 0} |\varphi(z)|/|z|^\alpha \\ &= |g'(0)| \cdot \lim_{z\to 0} |\varphi(z)|/|z|^\alpha. \end{aligned}$$

Here, since $\lim\limits_{z\to 0} |\varphi(z)|/|z|^\alpha \geqq 1$ from Corollary 1 in §1, there holds $|g'(0)| \leqq 1$. Thus, we obtain

$$\max_{|\zeta|=k\{q^{1/K}(r)\}} |g(\zeta)| \leqq k\{q^{1/K}(r)\}/[1 - k\{q^{1/K}(r)\}]^2,$$

so that

$$\max_{|z|=r<1} |f(z)| \leqq k\{q^{1/K}(r)\}/[1 - k\{q^{1/K}(r)\}]^2.$$

Next, in the case $1 \leqq \alpha < K$, take the following mapping $w = f_n(z)$:

$$w = |z|^\alpha \{1 + (r_n - 1)|z|^{\alpha/(r_n-1)}\} e^{i\arg z}, \tag{5}$$

where $|z| < 1$, $r_n > K/(K-\alpha)$ and $r_n \to \infty$ as $n \to \infty$.

Then, it can be found without great difficulty that every dilatation of (5) on $|z| = r < 1$ equals to

$$\alpha(1 + r_n r^{\alpha/(r_n-1)})/(1 + r_n r^{\alpha/(r_n-1)} - r^{\alpha/(r_n-1)})$$

which is a number lying between 1 and K. Hence, (5) is a K-QC mapping of $|z| < 1$ onto $|\zeta| < r_n$ with $\zeta(0) = 0$ and $\lim\limits_{z\to 0} |\zeta|/|z|^\alpha = 1$, and so $w = f_n(z)$ belongs to $\mathfrak{S}_\alpha$.

On the other hand, it is obvious that $\lim\limits_{n\to\infty} f_n(r) = \infty$.

Finally, in the case $1/K \leqq \alpha < 1$, consider the composite mapping $w = f_n(z)$ of those with the same forms as (3), (4) and (5) mentioned above, then it is shown as in [3] that $w = f_n(z)$ belongs to $\mathfrak{S}_\alpha$, while it can be obtained by formally the same computation as before that

$$\lim_{n\to\infty} f_n(-r) = -\lim_{n\to\infty} \{4nr/(n+r)^2\}^{\alpha}\cdot(n+r)^2/4n(1+r)^2$$
$$= -\{r^{\alpha}/(1+r)^2\}\cdot\lim_{n\to\infty}\{(n+r)^2/4n\}^{1-\alpha} = -\infty.$$

Thus our proof is completed.

REFERENCES

[1] J. HERSCH, Contribution à la théorie des fonctions pseudo-analytiques, Comm. Math. Helv., 30 (1956), 1-19.
[2] J. HERSCH and A. PFLUGER, Généralisation du lemme de Schwarz et du principe de la mesure harmonique pour les fonctions pseudo-analytiques, C. R. Acad. Sci. Paris, 234 (1952), 43-45.
[3] K. IKOMA, Note on the distortions in certain quasiconformal mappings, Jap. Journ. Math., 29 (1959), 9-12.
[4] A. MORI, On quasi-conformality and pseudo-analyticity, Sûgaku, 7 (1955), 75-89 (in Japanese), or, Trans. Amer. Math. Soc., 84 (1957), 56-77.
[5] A. PFLUGER, Quelques théorèmes sur une classe de fonctions pseudo-analytiques, C. R. Acad. Sci. Paris, 231 (1950), 1022-1023.
[6] A. PFLUGER, Quasikonforme Abbildungen und logarithmische Kapazität, Ann. Inst. Fourier, 2 (1950-1951), 69-80.

YAMAGATA UNIVERSITY AND UNIVERSITY OF ÔSAKA PREFECTURE.