A FIBERING OF RIEMANNIAN MANIFOLDS ADMITTING 1-PARAMETER GROUPS OF MOTIONS

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(Received June 23, 1965)

1. Introduction. Let M be an n-dimensional connected complete differentiable Riemannian manifold[1] admitting an effective 1-parameter group H of motions. The group H induces a Killing vector field on M. Each of its maximal integral curves corresponds to a *trajectory* under H. On the other hand, the group H can be regarded as a 1-parameter subgroup of the Lie group G of all motions of M and the closure of H, in G, forms a subgroup which is a connected abelian Lie group. This Lie group we denote by $\bar{H}$. Our object is to prove the following theorems:

THEOREM 1. *Suppose that a trajectory is dense in* M[2]. *Then,*

1) *every trajectory is dense in* M,
2) M *is homeomorphic to a torus,*
3) *the Riemannnian metric of* M *is Euclidean, and*
4) *the Killing vector field reduces to a parallel field.*

THEOREM 2. *The closure of a trajectory consists of one point, or is homeomorphic to a straight line or a torus of dimension* r $(1 \leqq r \leqq n)$. *In the latter two cases, it has the structure of a regularly imbedded*[3] *differentiable submanifold on which a Euclidean metric is naturally induced from* M.

Thus, a point or a differentiable submanifold, which is the closure of a trajectory, is called the *closure manifold* of the trajectory and we have

THEOREM 3. *In order that* M *becomes a fiber bundle whose fibers are the closure manifolds of trajectories, it is necessary and sufficient that* $\bar{H}$ *acts on* M *without fixed point. This fiber bundle can be regarded as a differentiable principal fiber bundle with group* $\bar{H}$.[4]

1) $n > 1$. The word "differentiable" means "C^∞-differentiable".
2) A trajectory is sometimes regarded as a subset of M, as is here the case.
3) This means that the topology of the submanifold coincides with the relative one.
4) The base manifold may consist of a single point.

2. Poofs of theorems. First we explain two notations. Let $H(x)$ denote the trajectory passing through a point $x \in M$. Let $d(x_1, x_2)$ denote the length of a minimizing geodesic arc from x_1 to x_2, where $x_1, x_2 \in M$.

LEMMA 1. *For $x_0 \in M$, let x_1, y_0 be two points of $H(x_0)$ and J_0 be an element of H such that $J_0(x_0) = x_1$. Then,*

$$d(x_0, y_0) = d(x_1, J_0(y_0)), \qquad d(x_0, x_1) = d(y_0, J_0(y_0)).$$

PROOF. The first relation is obvious. To prove the second, take $J \in H$ such that $J(x_0) = y_0$. Since $J_0 \cdot J = J \cdot J_0$, we have

$$J(x_1) = J_0 \cdot J \cdot J_0^{-1}(x_1) = J_0(y_0).$$

Hence the second relation follows immediately.

LEMMA 2. *Let T be a trajectory under H and y_0 be a point of the closure of T. Then the closures of T and $H(y_0)$ coincide.*

PROOF. It is sufficient to prove the case $y_0 \notin T$. Denote the closures of T and $H(y_0)$ by $\overline{T}$ and $\overline{H(y_0)}$ respectively and let $\{x_\lambda\}$[5] $\subset T$ be a sequence converging to y_0. First take any $y \in H(y_0)$. And choose $J \in H$ such that $J(y_0) = y$. A sequence $\{x'_\lambda\} \subset T$, where $x'_\lambda \equiv J(x_\lambda)$, converges to y. So $y \in \overline{T}$. Hence, $\overline{H(y_0)} \subset \overline{T}$. Next take any $x \in T$. And choose $J_\lambda \in H$ such that $J_\lambda(x_\lambda) = x$. As $d(y_0, x_\lambda) = d(y_\lambda, x)$ where $y_\lambda \equiv J_\lambda(y_0) \in H(y_0)$, $\{y_\lambda\}$ converges to x. So, $x \in \overline{H(y_0)}$. Hence, $\overline{T} \subset \overline{H(y_0)}$. This completes our proof.

LEMMA 3. *Let A be a connected abelian Lie group. Then any invariant differentiable Riemannian metric on A is Euclidean.*

PROOF. An invariant vector field on A is a field of vectors with constant components in a canonical coordinate neighborhood, and becomes a Killing vector field under an invariant differentiable Riemannian metric. By using this fact the lemma is easily proved.

PROOF OF THEOREM 1. The assertion 1) is evident by Lemma 2. Take an n-frame F_{x_0} at a point $x_0 \in M$ whose first vector is tangent to $H(x_0)$ and such that the remaining vectors are orthogonal to $H(x_0)$. By transferring the frame by the elements of H, we obtain on $H(x_0)$ a field of n-frames. This

5) $\lambda = 1, 2, \cdots$ (to ∞).

field can be extended to a continuous field of n-frames over M. This is easily verified by using Lemma 1, since $H(x_0)$ is dense. In this continuous field, we denote the n-frame at $x \in M$ by F_x. As is immediately seen, for any $x \in M$ there exists a motion of M carrying F_{x_0} to F_x. This motion belongs to $\bar{H}$ and conversely an element of $\bar{H}$ is such one. Hence dim $\bar{H} > 1$ and so $\bar{H}$ must be a toral group. Moreover the map

$$f : \bar{H} \longrightarrow M \quad \text{defined by} \quad f(J) = J(x_0),$$

where $J \in \bar{H}$, is one-to-one onto and continuous. The group $\bar{H}$ being compact, the map f becomes a homeomorphism. Therefore the assertion 2) holds good. Indeed, the map f is a diffeomorphism. So we can induce on $\bar{H}$ the Riemannian metric of M by the map f. As the induced metric becomes invariant on $\bar{H}$, it must be Euclidean by Lemma 3. Thus the assertion 3) is proved. On the other hand, since the Killing vector field reduces to a vector field of constant length by the assumption, all the trajectories are geodesics. This fact proves our assertion 4).

PROOF OF THEOREM 2. The case $\dim \bar{H}=1$. Then $\bar{H}$ coincides with H. Hence the closure of any trajectory T is the trajectory itself. So, it consists of one point, or is homeomorphic to a straight line or a circle. Further, the latter part of the theorem is verified by using the facts that a straight line satisfies the second axiom of countability and that a circle is compact. So, our theorem holds good.

The case $\dim \bar{H} > 1$. Then $\bar{H}$ is a toral group. Let T be a trajectory and take a point $x_0 \in T$. The closure $\bar{T}$ of T consists of the points $h(x_0)$ for all $h \in \bar{H}$. Hereafter we treat $\bar{T}$ as a topological subspace of M. Let R denote the subgroup of $\bar{H}$ which consists of the elements leaving x_0 fixed. Being a closed subgroup, the factor group $\bar{H}/R$ consists of the identity only or forms a toral group. Then the map

$$\phi : \bar{H}/R \longrightarrow \bar{T} \quad \text{defined by} \quad \phi([h]) = h(x_0),$$

where $[h]$ denotes the element of $\bar{H}/R$ containing $h \in \bar{H}$, is one-to-one onto and continuous. So, the map ϕ becomes a homeomorphism. Therefore $\bar{T}$ consists of one point or is homeomorphic to a torus of dimension r ($1 \leqq r \leqq n$). In the latter case, give $\bar{T}$ the same differentiability with $\bar{H}/R$ by the map ϕ. Since $h(x_0)$ is differentiable with respect to h, the inclusion map of $\bar{T}$ into M becomes differentiable and everywhere non-singular. $\bar{T}$ being compact, $\bar{T}$ has the structure of a regularly imbedded differentiable submanifold of M. If $\dim \overline{T}=1$, it is obvious that the induced metric on $\overline{T}$ is Euclidean. If $\dim \overline{T} > 1$, then $\overline{T}$ consists of the union of some non-closed trajectories by Lemma 2. And under the induced metric, $\overline{T}$ reduces to the same manifold as M in Theorem 1. Accordingly the induced metric on $\overline{T}$ must be Euclidean. This completes the proof.

PROOF OF THEOREM 3. 1) Suppose that $\overline{H}$ acts on M without fixed point. Then, there exists no trajectory which consists of a single point.

The case $\dim \overline{H}=1$. $\overline{H}$ coincides with H. The closure manifold of a trajectory consists of the trajectory only and is regularly imbedded in M by Theorem 2. If M has both of closed trajectory and non-closed one, there exists a nonzero element of H which leaves a point of M fixed. This contradicts with our assumption. Accordingly, it is sufficient to consider the two cases where all the trajectories are non-closed and are closed. In the respective case, it is easily verified that M can be regarded as a differentiable principal fiber bundle with group $\overline{H}$ whose fibers are the closure manifolds.

The case $\dim \overline{H} > 1$. Then $\overline{H}$ is a toral group. Take $x_0 \in M$. Let $H(x_0)$ denote the closure manifold of $H(x_0)$. The map

$$\varphi: \overline{H} \longrightarrow \overline{H(x_0)} \quad \text{defined by} \quad \varphi(h) = h(x_0),$$

where $h \in \overline{H}$, is a diffeomorphism onto from the assumption. So, $1 < \dim \overline{H} \leqq n$. It is easily shown that M can be regarded as a differentiable principal fiber bundle with group $\overline{H}$ whose fibers are the closure manifolds.

2) Suppose that some nonzero element of $\overline{H}$ leaves a point $x_0 \in M$ fixed. To conclude our proof, it is sufficient to consider the case where there exists no trajectory consisting of a single point, as H acts effectively on M. Let R denote the subgroup of $\overline{H}$ which consists of the elements leaving x_0 fixed. This is a closed subgroup and so a Lie group. Let $\overline{H(x_0)}$ denote the closure manifold of $H(x_0)$. The group R acts on M leaving every point of $\overline{H(x_0)}$ fixed, as is seen from proof of Theorem 1. Accordingly, $1 \leqq \dim \overline{H(x_0)} < n$.

The case $\dim R=0$. Then R is a finite group. Take a unit vector v at x_0 orthogonal to $\overline{H(x_0)}$ such that, if we transfer v by every element of R, we obtain m[6] unit vectors distinct from each other. Let y_0 be the terminal point of a geodesic arc g issuing from x_0, with initial tangent vector v, and let $\overline{H(y_0)}$ be the closure manifold of $H(y_0)$. If the arc-length of g is sufficiently small, we obtain an m-covering map

$$p: \overline{H(y_0)} \longrightarrow \overline{H(x_0)} \quad \text{defined by} \quad p(y) = x,$$

6) m suffices to be an integer > 1.

where $x \in \overline{H(x_0)}$ and $y \in \overline{H(y_0)}$, are the initial and the terminal points respectively of every geodesic arc obtained by transferring g by every element of $\overline{H}$.

The case $\dim R > 0$. By the same way as above, we can see that near $\overline{H(x_0)}$ there exists a closure manifold which has higher dimension than $\overline{H(x_0)}$.

These two cases show that, in a fibering of M, the closure manifolds can not become its fibers.

The above 1), 2) prove our theorem.

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