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ON 5-DIMENSIONAL SASAKI-EINSTEIN SPACE WITH SECTIONAL CURVATURE $\geqq 1/3$

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1. Introduction. S. I. Goldberg has proved the following theorems [4];

THEOREM. *If a compact, simply connected regular Sasakian space*[1] *has positive sectional curvature and constant scalar curvature, then it is isometric to a Euclidean sphere of the same dimension.*

THEOREM. *If a compact regular Sasakian space has positive sectional curvature, then its second Betti number vanishes.*

For the proofs of these theorems, the assumption of regularity of the contact structure is inevitable. Without the assumption of regularity of the first theorem we have proved the following [6]

THEOREM. *If a complete* $2m+1(\geqq 5)$*-dimensional Sasakian space has sectional curvature* $> 1/2m$, *then the second Betti number vanishes.*

On the other hand, M. Berger proved the following [3]

THEOREM. *If a complete, Kähler-Einstein space has positive sectional curvature, then it is isometric to a complex projective space with a metric of constant holomorphic sectional curvature.*

In a former paper [2], he has proved the following theorem as a special case of this theorem.

If a 4-dimensional compact Kähler-Einstein space has non-negative sectional curvature, then it is a locally symmetric space.

To exclude regularity condition of the second theorem of Goldberg, we apply the Berger's method to 5-dimensional Sasaki-Einstein space and obtain

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1) In this note, manifolds are assumed to be connected and C^∞-differentiable.

THEOREM. *If a* 5-*dimensional compact simply connected Sasaki-Einstein space has sectional curvature* $\geqq 1/3$, *then it is a Euclidean sphere of* 5-*dimension.*

2. Preliminaries. Let M be an n-dimensional Riemannian space. We denote by M_p its tangent space at p, and by $g_{\lambda\mu}$[2] the Riemannian structure of M. If we denote by $R_{\lambda\mu\nu}{}^{\omega}$ the Riemannian curvature tensor, the sectional curvature at p with respect to a 2-plane spanned by the orthonormal vectors $X, Y \in M_p$ is defined by

$$\rho(X, Y) = -R_{\lambda\mu\nu\omega} X^\lambda Y^\mu X^\nu Y^\omega .$$

The Ricci tensor $R_{\lambda\mu}$ is defined by $R_{\lambda\mu} = \sum R_{\omega\lambda\mu}{}^{\omega}$. If the relation $R_{\lambda\mu} = k\, g_{\lambda\mu}$ holds for a scalar k, then the Riemannian structure is called an Einstein metric. This scalar k is necessarily constant provided that the dimension>2.

A Riemannian space with a unit Killing vector field $Z=(\eta^\lambda)$ such that

$$\nabla_\lambda \nabla_\mu \eta_\nu = \eta_\mu g_{\lambda\nu} - \eta_\nu g_{\lambda\mu}$$

is called a Sasakian space.

In the following we only consider an n-dimensional Sasakian space M. It is known that M is orientable, and n is necessarily odd: $n=2m+1$. We define tensor fields $\varphi_{\lambda\mu}$, φ_λ^μ by

$$\varphi_{\lambda\mu} = \nabla_\lambda \eta_\mu , \qquad \varphi_\lambda{}^\mu = \varphi_{\lambda\nu}\, g^{\nu\mu} ,$$

then the following formulas are valid:

$$\varphi_\lambda{}^\nu \varphi_\nu{}^\mu = -\delta_\lambda^\mu + \eta_\lambda \eta^\mu ,$$

$$\varphi_\lambda{}^\nu \eta_\nu = 0 , \qquad \varphi_{\lambda\mu} = -\varphi_{\mu\lambda} .$$

For any vector $X = (X^\lambda)$, we mean φX the vector $(\varphi_\mu{}^\lambda X^\mu)$. As for the curvature tensor, we have [5]

$$R_{\lambda\mu\nu\omega}\, \eta^\omega = \eta_\lambda\, g_{\mu\nu} - \eta_\mu\, g_{\lambda\nu} , \tag{2.1}$$

$$\varphi_\lambda{}^\varepsilon R_{\varepsilon\mu\rho\sigma} = \varphi_\mu{}^\varepsilon R_{\varepsilon\lambda\rho\sigma} + \varphi_{\rho\lambda}\, g_{\sigma\mu} - \varphi_{\rho\mu}\, g_{\sigma\lambda} + \varphi_{\sigma\mu}\, g_{\rho\lambda} - \varphi_{\sigma\lambda}\, g_{\rho\mu} , \tag{2.2}$$

$$\varphi_\mu{}^\beta \varphi_\lambda{}^\alpha R_{\alpha\beta\nu\omega} = R_{\lambda\mu\nu\omega} + \varphi_{\nu\lambda}\varphi_{\mu\omega} - \varphi_{\omega\lambda}\varphi_{\mu\nu} + g_{\nu\lambda}\, g_{\mu\omega} - g_{\omega\lambda}\, g_{\mu\nu} . \tag{2.3}$$

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2) Indices $\lambda, \mu, \cdots$ run from 1 to n.

From (2. 1), we have

$$\rho(X, Z) = 1 \tag{2.4}$$

for any vector X which is linearly independent to Z on M.

For any point p of M, we can take an orthonormal basis $X_1, X_{1^*}, \cdots, X_m, X_{m^*}, X_n = Z$, $(X_{i^*} = \varphi X_i)$, and with respect to this basis, the component of the tensors $g_{\lambda\mu}$, $\varphi_{\lambda\mu}$ and η_λ are given by

$$g_{\lambda\mu} = \delta_{\lambda\mu},$$

$$\varphi_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda=i,\ \mu=i^*, \\ -1 & \text{if } \lambda=i^*,\ \mu=i, \\ 0 & \text{otherwise,} \end{cases}$$

$$\eta_\lambda = (0, \cdots, 0, 1).$$

We call such an orthonormal basis an adapted basis.

By virtue of (2. 2) and (2. 3), we have the following formulas[3] with respect to the adapted basis.

$$R_{\lambda\mu^*\lambda^*\mu} = -R_{\lambda\mu^*\lambda\mu^*} + (\delta_{\lambda\mu} - 1), \tag{2.5}$$

$$R_{\lambda\mu\lambda^*\mu^*} = R_{\lambda\mu\lambda\mu} + (1 - \delta_{\lambda\mu}), \tag{2.6}$$

$$R_{\lambda\mu^*\lambda^*\mu^*} = R_{\lambda\mu^*\lambda\mu}. \tag{2.7}$$

If a Sasakian space is an Einstein space at the same time,

$$R_{\lambda\mu} = k g_{\lambda\mu}, \tag{2.8}$$

then the constant k is equal to $n-1$.

3. Lemmas.

LEMMA 1. *In a 5-dimensional Sasaki-Einstein space, if we take an orthonormal basis* $(X_1, X_2, X_3, X_4, X_5 = Z)$ *of* M_p *for any* $p \in M$, *then we have*

$$\rho(X_1, X_2) = \rho(X_3, X_4).$$

PROOF. From (2. 8) and (2. 4) we have

3) S. Tachibana and Y. Ogawa [6].

$$\rho(X_1, X_2) + \rho(X_1, X_3) + \rho(X_1, X_4) = 3,$$
$$\rho(X_2, X_1) + \rho(X_2, X_3) + \rho(X_2, X_4) = 3,$$
$$\rho(X_3, X_1) + \rho(X_3, X_2) + \rho(X_3, X_4) = 3,$$
$$\rho(X_4, X_1) + \rho(X_4, X_2) + \rho(X_4, X_3) = 3.$$

Hence it can be easily deduced that $2\rho(X_1, X_2) - 2\rho(X_3, X_4) = 0$.

LEMMA 2. *Let M be a 5-dimensional Sasaki-Einstein space. Then we can take for any $p \in M$ an adapted basis $(X_1, X_2 = \varphi X_1, X_3, X_4 = \varphi X_3, X_5 = Z)$ of M_p such that*

(i) $R_{1212} = R_{3434}$ $(=a)$, $R_{1313} = R_{2424}$ $(=b)$, $R_{1414} = R_{2323}$ $(=c)$,

$R_{1324} = b+1$, $R_{2341} = c+1$, $R_{1234} = b+c+2$, $R_{5i5i} = -1$ $(i=1, \cdots, 4)$,

and all the other $R_{\lambda\mu\nu\omega} = 0$.

(ii) $\rho(X_1, X_2) \geqq 2\{\rho(X_1, X_3) + \rho(X_1 X_4)\} - 3$.

PROOF. Let W_p be an orthogonal hyperplane to Z in M_p. We select a unit vector X_1 such that

$$\rho(X_1, \varphi X_1) = \operatorname*{Max}_{X \in W_p} \rho(X, \varphi X).$$

Let V_p be an ortho-complementary subspace to $\{X_1, \varphi X_1\}$ in W_p. Then V_p is spanned by some unit vectors Y and φY. Therefore, for the symmetric quadratic form on V_p defined by

$$h(X, Y) = -R_{\lambda\mu\nu\omega} X_1^\lambda X_1^\nu X^\mu Y^\omega, \quad X, Y \in V_p,$$

there exists a unit vector X_3 in V_p such that

$$h(X_3, \varphi X_3) = 0.$$

Now this orthonormal basis $\{X_1, X_2 = \varphi X_1, X_3, X_4 = \varphi X_3, X_5 = Z\}$ of M_p is a desired one.

In fact, taking account of Lemma 1, we have

$$R_{1212} = R_{3434}, \quad R_{1313} = R_{2424}, \quad R_{1414} = R_{2323}.$$

From (2. 6) and (2. 5), it holds

$$R_{1324} = R_{131^*3^*} = R_{1313} + 1\,,$$

$$R_{2341} = -R_{13^*1^*3} = R_{1414} + 1\,.$$

By virtue of (2. 1), we have easily

$$R_{i5j5} = -\delta_{ij}\,, \quad (i, j = 1, \cdots, 4)\,.$$

From the selection of the vector X_3, we have $R_{1314} = 0$. From (2. 8), we have $R_{2324} = 0$. Hence applying (2. 7) to R_{1314} and R_{2324}, we see $R_{2414} = 0$, $R_{1332} = 0$. Next we consider the sectional curvature

$$\begin{aligned}&\rho(\alpha X_1 + \beta X_3, \varphi(\alpha X_1 + \beta X_3))\\ &\qquad = (\alpha^2 + \beta^2)^{-2}\{A\alpha^4 + B\beta^4 + 2C\alpha^2\beta^2 + 2D\alpha^3\beta + 2E\alpha\beta^3\}\end{aligned}$$

where α, β are any real numbers and $A = \rho(X_1, X_2)$, $B = \rho(X_3, X_4)$, $C = \rho(X_1, X_3) + 3\rho(X_1, X_4) - 3$, $D = -2R_{1214}$, $E = -2R_{3414}$. From the choice of X_1 and Lemma 1, we have

$$\alpha^2\beta^2(C - \rho(X_1, X_2)) + \alpha^3\beta D + \alpha\beta^3 E \leqq 0$$

for any real α, β. If we substitute $-\beta$ for β in it, and adding these two inequalities, we can get $C - \rho(X_1, X_2) \leqq 0$. If $\alpha\beta > 0$, then we have

$$(C - \rho(X_1, X_2))\,\alpha\beta \leqq \alpha^2 D + \beta^2 E \leqq -(C - \rho(X_1, X_2))\,\alpha\beta\,.$$

From this we have easily $D = E = 0$. Therefore we have $R_{1214} = R_{3414} = 0$ and $\rho(X_1, X_2) \geqq \rho(X_1, X_3) + 3\rho(X_1, X_4) - 3$. By the same process for $\rho(\alpha X_1 + \beta X_4, \varphi(\alpha X_1 + \beta X_4))$, we have

$$R_{1213} = R_{3414} = R_{4243} = R_{2414} = 0\,,$$

$$\rho(X_1, X_2) \geqq 3\rho(X_1, X_3) + \rho(X_1, X_4) - 3\,.$$

Hence we have $\rho(X_1, X_2) \geqq 2\{\rho(X_1, X_3) + \rho(X_1, X_4)\} - 3$. This proves the lemma.

4. Proof of the theorem. It is known that in a compact orientable Einstein space M, if the scalar

$$K(p) = \sum \{-R_{\lambda\nu\mu\nu} R_{\lambda\omega\rho\sigma} R_{\mu\omega\rho\sigma} + \frac{1}{2} R_{\lambda\mu\nu\omega} R_{\nu\omega\rho\sigma} R_{\rho\sigma\lambda\mu} + 2R_{\lambda\nu\mu\omega} R_{\lambda\rho\mu\sigma} R_{\nu\rho\omega\sigma}\}$$

satisfies $K(p) \geqq 0$ for all $p \in M$, then M must be a locally symmetric space (Lichnerowicz [8]). In our 5-dimensional Sasaki-Einstein space, taking the basis of Lemma 2, we can calculate $K(p)$ explicitly as follows:

$$-\sum R_{\lambda\nu\mu\nu} R_{\lambda\omega\rho\sigma} R_{\mu\omega\rho\sigma} = 32\{a^2 + 3(b^2+c^2) + 6(b+c) + 2bc+8\},$$

$$\frac{1}{2}\sum R_{\lambda\mu\nu\omega} R_{\nu\omega\rho\sigma} R_{\rho\sigma\lambda\mu}$$
$$= 8\{a^3 + 3a(b+c+2)^2 + 4(b^3+c^3) + 6(b^2+c^2) + 3(b+c) - 2\},$$

$$2\sum R_{\lambda\nu\mu\omega} R_{\lambda\rho\mu\sigma} R_{\nu\rho\omega\sigma}$$
$$= 24\{2a(2bc+b+c+1) + (a+b+c) - 2(b+c+2)(2bc+b+c)\}.$$

Now we have

$$a + b + c = -3.$$

Defining $x=b+c$ and $y=bc$, we can calculate directly

$$-K(p)/32 = 5x^2 + 15x + (9x+22)y + 6.$$

By virture of (ii) of Lemma 2, we get

$$b + c \geqq -2.$$

Moreover, if the Sasakian space in consideration has sectional curvature $\geqq \delta$, then it satisfies that $b+c \leqq -2\delta$. Therefore the range on which (x, y) exists is

$$D = \{-2 \leqq x \leqq -2\delta,\quad \delta^2 \leqq y \leqq x^2/4\}.$$

If we put $f(x, y)$ the right hand side of the above equation, then for $(x, y) \in D$, we have

$$f(x, y) \leqq \frac{3}{4}(3x+2)(x+2)^2 \leqq -\frac{9}{2}\left(\delta - \frac{1}{3}\right)(x+2)^2.$$

Hence if $\delta \geqq 1/3$, then we see that $f(x, y) \leqq 0$ and $K(p) \geqq 0$ for all $(x, y) \in D$. This means that M is a locally symmetric space. On the other hand, it is known that a locally symmetric Sasakian space is a space of constant curvature (M. Okumura [7]). Hence M is a space of constant curvature, we get our theorem.

REMARK. If $\delta > 1/3$, then we can conclude immediately the theorem from the fact that $K(p) = 0$ if and only if $a=b=c=-1$ and $R_{\lambda\mu\nu\omega}$ has the properties showed in Lemma 2.

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