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SUBNORMAL OPERATOR WITH A CYCLIC VECTOR

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In this paper, we aim to characterize the non-trivial closed invariant subspace of a subnormal operator and to study the existence of such subspaces.

An operator A acting on a Hilbert space H is said to be subnormal if, on some space K containing H, there exists a normal operator B such that $Bx=Ax$ for every x in H; then B is called a normal extension of A.

The normal extension B, acting on K, of a subnormal operator A, acting on H, a subspace of K, is the minimal normal extension of A if the smallest subspace of K that contains H and reduces B is K itself; Halmos has shown that any two minimal normal extensions are unitarily equivalent ([3]).

If A is subnormal on H, we call that a vector x is cyclic with respect to A if the smallest subspace containing x and invariant under A is H; in this case we say that H is cyclic with respect to A.

For our purpose, it is natural to assume that the subnormal operator A on H has a cyclic vector x; because, if $\bigvee\{A^n x;\ n \geqq 0\}$ (which denotes the smallest closed subspace containing $A^n x;\ n \geqq 0$) is properly included in H, then it is clearly a non-trivial closed invariant subspace of A.

Bram proved in [1] that when A is normal and acts on H, the fact that H is cyclic with respect to A in the sense just defined is equivalent to the fact that H is cyclic in the usual sense, i.e., that there exists x in H such that H is the smallest closed subspace that contains x and reduces A.

It is known that if B is a normal operator on K with a cyclic vector, then there exists a unitary mapping U of K onto a suitable function space $L^2(d\mu(\lambda); \sigma(B))$ such that UBU^{-1} has the form of "multiplication by λ" ([2]).

Applying this representation theorem of normal operators to the minimal normal extension B on K of a subnormal operator A on H with a cyclic vector, we can show that H admits a representation relative to A onto a subspace $H^2(d\mu(\lambda);\ \sigma(B))$ of $L^2(d\mu(\lambda);\ \sigma(B))$.

In the next section, we show this representation of a subnormal operator with a cyclic vector and using this, we give the sufficient conditions of the existence of non-trivial closed invariant subspaces of subnormal operators.

We state here a characterization of subnormal operators given by Halmos [3] and Bram [1] without the proof.

THEOREM 1. (Halmos[3]) *An operator A on a Hilbert space H is subnormal if and only if*

(1) $\sum_{m,n=0}^{r}(A^n x_m, A^m x_n) \geqq 0$ *for every finite set* $x_0, x_1, \cdots, x_r$ *in* H, *and*

(2) *there exists a positive constant c such that*

$$\sum_{m,n=0}^{r}(A^{n+1}x_m, A^{m+1}x_n) \leqq c\cdot \sum_{m,n=0}^{r}(A^n x_m, A^m x_n)$$

for every finite set $x_0, \cdots, x_r$ *in* H.

THEOREM 2. (Bram[1]) *Let A be an operator on H, and suppose that* $\sum_{m,n=0}^{r}(A^n x_m, A^m x_n) \geqq 0$ *for every finite set* $x_0, x_1, \cdots, x_r$ *in H. Then*

$$\sum_{m,n=0}^{r}(A^{n+1}x_m, A^{m+1}x_n) \leqq \|A\|^2\cdot \sum_{m,n=0}^{r}(A^n x_m, A^m x_n)$$

for every finite set $x_0, x_1, \cdots, x_r$ in H.

LEMMA 1. *Let H be cyclic with respect to a subnormal operator A on H, and let B, acting on K, be the minimal normal extension of A. Then K is cyclic with respect to B.*

PROOF. Let x be a cyclic vector for H with respect to A, i.e., $H=\vee\{A^n x;\ n\geqq 0\}$. Let $M=\vee\{B^{*m}B^n x;\ m, n\geqq 0\}$. Then, since $B^n x=A^n x$ for all $n\geqq 0$, we have $H\subset M$; moreover M reduces B. But B is the minimal normal extension of A so that $M=K$, K is cyclic with respect to B.

Let x be a cyclic vector for a subnormal operator A on H, let $\mu=(E(\lambda)x, x)$ where $E(\lambda)$ denotes the resolution of the identity for the minimal normal extension B, acting on K, of A and let D_1 be the linear manifold in K consisting of all vectors of the form $f(B)x$ where f is a bounded Borel function on the spectrum $\sigma(B)$ of B.

By Lemma 1, D_1 is dense in K and we see easily that the operator V_1 from D_1 to $L_2(d\mu;\ \sigma(B))$ defined by $V_1 f(B)x=f$ has a unique extension V from the closure $\widetilde{D_1}=K$ of D_1 to the L^2-closure of the set of all bounded Borel functions, i.e., to $L^2(d\mu;\ \sigma(B))$ and that V is an isometric isomorphism between K and $L^2(d\mu;\ \sigma(B))$.

LEMMA 2. *If A is a subnormal operator on H with a cyclic vector x in H and if $H^2(d\mu\,;\sigma(B))$ be the L^2-closure of the set P of all complex polynomials in λ, defined on the spectrum $\sigma(B)$ of the minimal normal extension B, acting on K, of A with respect to the Lebesgue-Stieltjes measure $d\mu=d(E(\lambda)x,x)$, then H and $H^2(d\mu\,;\sigma(B))$ are isomorphic by the mapping V defined as above.*

PROOF. Since $H=\bigvee\{A^n x; n\geqq 0\}$, for any y in H, there exists some sequence p_n in P such that $y=\lim_{n\to\infty} p_n(A)x$, and since $p_n(A)x=p_n(B)x$ *for all* $n\geqq 0$, *we have*

$$\int |p_n(\lambda)-p_m(\lambda)|^2 d(E(\lambda)x,x)\to 0 \text{ as } m,\ n\to\infty,$$

so that there exists a function p_y in $H^2(d\mu;\sigma(B))$ such that

$$\int |p_y(\lambda)-p_n(\lambda)|^2 d(E(\lambda)x,x)\to 0 \text{ as } n\to\infty.$$

Since the existence of p_y is independent of the choice of the sequence p_n in P, the operator V defined by $Vy=p_y$ is well-defined and clearly V is an isometry from H into $H^2(d\mu\,;\sigma(B))$.

Conversely, by the definition of $H^2(d\mu;\sigma(B))$, for any p in $H^2(d\mu;\sigma(B))$, there exists a sequence p_n in P such that

$$\int |p(\lambda)-p_n(\lambda)|^2 d\,\mu\to 0 \text{ as } n\to\infty.$$

Hence, $\{p_n(B)x\}$ is a Cauchy sequence in H, and hence there exists a vector y in H such that $\|p_n(A)x-y\|\to 0$ as $n\to\infty$. This means that the operator V is an isometry from H onto $H^2(d\mu;\sigma(B))$, and the proof is completed.

THEOREM 3. *If A is a subnormal operator on H with a cyclic vector x in H and if T is a bounded linear operator on H which commutes with A, then T is subnormal and there exists a Borel measurable function $p_T(\lambda)$ in*

$$H^\infty(d\mu;\sigma(B))=H^2(d\mu;\sigma(B))\cap L^\infty(d\mu;\sigma(B))$$

such that $Ty=p_T(B)y$ for all y in H, where B denotes the minimal normal extension of A and $d\mu=d(E(\lambda)x,x)$, $E(\lambda)$ denotes the spectral measure of B.

PROOF. Since $H=\vee\{A^n x; n\geqq 0\}$, $Tx=\lim_{n\to\infty} p_n(A)x$ for some sequence $\{p_n(A)\}$, $p_n\in P$ and since $p_n(A)x=p_n(B)x$, we have

$$\int |p_n(\lambda)-p_m(\lambda)|^2 d(E(\lambda)x,x)\to 0 \text{ as } m,n\to\infty,$$

so that there exists a Borel measurable function p_T in $H^2(d\mu;\sigma(B))$ such that

$$\int |p_T(\lambda)|^2 d(E(\lambda)x,x)<\infty \text{ and} \int |p_T(\lambda)-p_n(\lambda)|^2 d(E(\lambda)x,x)\to 0$$

as $n\to\infty$(see [5; page 348]) and hence

$x\in D(p_T(B))=\{y\in K; \int |p_T(\lambda)|^2 d(E(\lambda)y,y)<\infty\}$ and $p_n(A)x=p_n(B)x\to p_T(B)x$ from which we have $Tx=p_T(B)x$.

Since T commutes with A, for any p in P, we have $Tp(A)x=p(A)Tx$ $=p(B)p_T(B)x=p_T(B)p(B)x=p_T(B)p(A)x$. Hence, if $y\in H$, and $q_n(A)x\to y$ with q_n in P, then $p_T(B)q_n(A)x=Tq_n(A)x\to Ty$ because T is bounded, and since $p_T(B)$ is closed, it follows that $H\subset D(p_T(B))$ and $p_T(B)y=Ty$ for all y in H. Hence, also, since $TH\subset H$, we have $H\subset D(p_T(B)^n)=D(p_T(B)^{*n})$ for all non-negative integers n, and $p_T(B)^n y=T^n y$ for all y in H.

Let $N=\vee\{p_T(B)^* y; y\in H, n\geqq 0\}$, then clearly we have $H\subset N\subset K$. If $y_0,y_1,\cdots,y_r$ in H, then we have

$$\begin{aligned}\sum_{m,n=0}^{r}(T^m y_n, T^n y_m)&=\sum_{m,n=0}^{r}(p_T(B)^m y_n, p_T(B)^n y_m)\\ &=\left\|\sum_{n=0}^{r} p_T(B)^{*n} y_n\right\|^2\geqq 0.\end{aligned}$$

Hence, by Theorem 1 and 2, T is subnormal. By Theorem 2, it follows that for any finite set $y_0,y_1,\cdots,y_r$ in H, we have

$$\sum_{m,n=0}^{r}(T^{m+1}y_n, T^{n+1}y_m)\leqq\|T\|^2\cdot\sum_{m,n=0}^{r}(T^m y_n, T^n y_m),$$

i.e., $$\sum_{m,n=0}^{r}(p_T(B)^{m+1}y_n, p_T(B)^{n+1}y_m)\leqq\|T\|^2\cdot\sum_{m,n=0}^{r}(p_T(B)^m y_n, p_T(B)^n y_m),$$

$$\text{or } \|p_T(B)\cdot\sum_{n=0}^{r} p_T(B)^{*n}y_n\|^2\leqq\|T\|^2\cdot\|\sum_{n=0}^{r} p_T(B)^{*n}y_n\|^2,$$

which shows that $p_T(B)$ is bounded on a dense linear subset of N.

Since $p_T(B)$ is closed, $N \subset D(p_T(B))$ and $\|p_T(B)y\| \leqq \|T\| \cdot \|y\|$ for all y in N. We observe that N reduces $p_T(B)$ and also B. Hence, by the minimality of $B, N=K$ and $K=D(p_T(B))$. This implies that $p_T(B)$ is bounded on K and hence $p_T(\lambda) \in L^\infty(d\mu;\sigma(B))$ which completes the proof.

COROLLARY 1. *If A is a subnormal operator on H with a cyclic vector and if T is a bounded linear operator on H which commutes with A and A^*, then T is normal.*

PROOF. By Theorem 3, T and T^* are subnormal. Since every subnormal operator S on H is hyponormal (i.e., $\|Sx\| \geqq \|S^*x\|$ for all $x \in H$), T is normal.

As the consequence, we see easily that if A is a subnormal operator with a cyclic vector and if $R(A)'$ is the commutant of the von Neumann algebra $R(A)$ generated by a single operator A, then $R(A)'$ is abelian, in particular, if A is normal, then $R(A)$ is maximal abelian.

Let A be a subnormal operator on H with a cyclic vector x, B its minimal normal extension and let $\alpha(A)$ be the set of all bounded linear operators on H which commutes with A, then, by Theorem 3, for any T in $\alpha(A)$, there exists a Borel measurable function $p_T(\lambda)$ in $H^\infty(d\mu;\sigma(B))$ such that $Ty=p_T(B)y$ for all y in H. Let $L_{\alpha(A)}$ be the set of all functions $p_T(\lambda)$, $T \in \alpha(A)$, then we can show

LEMMA 3. $L_{\alpha(A)}=H^\infty(d\mu;\sigma(B))$.

PROOF. If $p \in H^\infty(d\mu;\sigma(B))$, then, by the definition of $H^\infty(d\mu;\sigma(B))$, there exists a sequence p_n in P such that

$$\int |p_n(\lambda)-p(\lambda)|^2 d(E(\lambda)x, x) \to 0 \text{ as } n \to \infty .$$

Since $p \in L^\infty(d\mu;\sigma(B))$, we can define the bounded linear operator $p(B)$ on K such that $\|p_n(B)x-p(B)x\| \to 0$ as $n \to \infty$ and $p(B)x \in H$. (because $p_n(B)x = p_n(A)x \in H$). Since $H = \vee\{A^n x;\ n \geqq 0\}$ and for any q in P, $p(B)q(A)x=p(B)q(B)x=q(B)p(B)x=q(A)p(B)x \in H$, by the boundedness of $p(B)$, we have $p(B)y \in H$ for all $y \in H$. Therefore the restriction $p(B)|H$ of $p(B)$ on its invariant subspace H clearly commutes with A, i.e., $p(B)|H \in \alpha(A)$, and, by the definition of $L_{\alpha(A)}$, $p \in L_{\alpha(A)}$. The converse inclusion is clear by Theorem 3.

Let $\widetilde{r_0}(\sigma(A))$ be the uniform closure of the set $r_0(\sigma(A))$ of all rational functions with no pole in the spectrum $\sigma(A)$ of an operator A. If $r(\lambda)\in\widetilde{r_0}(\sigma(A))$, then $r(A)\in\alpha(A)$ and hence, if A is a subnormal operator on H with a cyclic vector, then, by Theorem 3 and Lemma 3, $r(\lambda)\in H^\infty(d\mu;\sigma(B))$. However, it might not be true that $\widetilde{r_0}(\sigma(B))\subset H^\infty(d\mu;\sigma(B))$ where B is the minimal normal extension of A.

THEOREM 4. *If A is a subnormal operator on H with a cyclic vector and if B is its minimal normal extension on K, then $\widetilde{r_0}(\sigma(B))\subset H^\infty(d\mu;\sigma(B))$ if and only if $\sigma(A)=\sigma(B)$.*

PROOF. Since $\widetilde{r_0}(\sigma(A))\subset H^\infty(d\mu;\sigma(B))$, we have only to show that $\widetilde{r_0}(\sigma(B))\subset H^\infty(d\mu;\sigma(B))$ implies $\sigma(A)=\sigma(B)$.

It is known that $\sigma(B)\subset\sigma(A)$ by Halmos [4] and that $\sigma(A)\subset\sigma(B)\cup h(B)$ by Bram[1], where $h(B)$ denotes the union of all holes of $\sigma(B)$. Hence we have only to show that $\widetilde{r_0}(\sigma(B))\subset H^\infty(d\mu;\sigma(B))$ implies $h(B)\subset\rho(A)$, where $\rho(A)$ denotes the resolvent set of A.

If λ_0 is an arbitrary point in $h(B)$ and if $r(\lambda)=(\lambda-\lambda_0)^{-1}$, then $r(\lambda)\in\widetilde{r_0}(\sigma(B))$ and hence $r(\lambda)\in H^\infty(d\mu;\sigma(B))$. Therefore, by Lemma 3, there exists an operator T in $\alpha(A)$ such that $Ty=r(B)y$ for all $y\in H$. Hence, for any $y\in H$, we have

$$(A-\lambda_0 I)Ty=T(A-\lambda_0 I)y=r(B)(B-\lambda_0 I)y$$

$$=\int r(\lambda)(\lambda-\lambda_0)dE(\lambda)y=y.$$

This means that $A-\lambda_0 I$ has a bounded inverse T, i.e., $\lambda_0\in\rho(A)$. Therefore $h(B)\subset\rho(A)$.

EXAMPLE. To show that a subnormal operator A need not be normal even when A has a cyclic vector and $\sigma(A)=\sigma(B)$, where B is the minimal normal extension of A, let D be the closed unit disk, μ the normalized Lebesgue measure in D, $K=L^2(d\mu;D)$, $B=L_z$, i.e., $L_zf(z)=z\cdot f(z)$ for all $f(z)\in L^2(d\mu;D)$. Let H be the L^2-closure of the set P of all complex polynomials in z, defined on D, with respect to μ, and set $A=L_z|H$, then clearly A is a subnormal operator on H with a cyclic vector $u(z)=1$ and its minimal normal extension is B on K. Since $\sigma(B)=D$, $h(B)=\emptyset$. And hence, by the same reason as in the proof of Theorem 4, $\sigma(A)=\sigma(B)$.

If we set $z=r\exp(i\theta)$, then $d\mu(z)=\dfrac{1}{\pi}r\,dr\,d\theta$; and hence,

$$\int_D zz^n d\mu(z)=\frac{1}{\pi}\int_0^1\int_0^{2\pi} r^{n+1}e^{i(n+1)\theta}r\,dr\,d\theta$$

$$=\frac{1}{\pi}\int_0^1 r^{n+2}dr\cdot\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta=0$$

for all $n\geqq 0$. This implies that $\bar{z}\perp H$ and $\bar{z}\in K$. Therefore $H\subsetneqq K$ and A is non-normal by the minimality of B.

THEOREM 5. *If B is the minimal normal extension of a subnormal operator A on H and if $\tilde{r}_0(\sigma(B))\cap\overline{\tilde{r}_0(\sigma(B))}\neq\{c\cdot 1\}$ (the bar denotes the complex conjugate), then there exists a non-trivial closed invariant subspace of H for A.*

PROOF. For any fixed non-zero vector x in H, we may assume that $H=\bigvee\{A^n x; n\geqq 0\}$, because if $\bigvee\{A^n x; n\geqq 0\}\subsetneqq H$, then $\bigvee\{A^n x; n\geqq 0\}$ is clearly a non-trivial closed invariant subspace of A.

In the case where $\sigma(A)=\sigma(B)$, for any $r(\lambda)$ in $\tilde{r}_0(\sigma(B))\cap\overline{\tilde{r}_0(\sigma(B))}$, $r(\lambda)\neq c\cdot 1$, we have $r(A), r(A)^*\in\alpha(A)$ and $r(A)\neq c\cdot I$; and hence $r(A)$ is normal by Corollary 1. In this case, clearly, A has reducing subspaces.

In the other case, we have $\sigma(A)\cap h(B)\neq\emptyset$ by the same reason as in the proof of Theorem 4. Since $Ay_n-\gamma y_n=By_n-\gamma y_n$ for all y_n in H, $\sigma_{ap}(A)\subset\sigma_{ap}(B)$ and easily we have $\sigma_p(S)\cup\sigma_c(S)\subset\sigma_{ap}(S)$ for any bounded linear operator S, where $\sigma_p(S)$, $\sigma_c(S)$ and $\sigma_{ap}(S)$ denote the point spectrum, the continuous spectrum and the approximate point spectrum of S, respectively. From this, we have $\gamma\in\sigma_r(A)$ if $\gamma\in\sigma(A)\cap h(B)$, where $\sigma_r(A)$ denotes the residual spectrum of A. Hence $\bar{\gamma}\in\sigma_p(A^*)$. Let $M=\{y\in H; A^*y=\bar{\gamma}y\}$, then the subspace $H\ominus M$ is clearly a non-trivial closed invariant subspace of A.

It is known that if $\sigma(B)$ has two dimensional Lebesgue measure zero, then $\tilde{r}_0(\sigma(B))=C(\sigma(B))$, where $C(\sigma(B))$ denotes the set of all continuous functions on $\sigma(B)$ (see[7]). Hence, we have

COROLLARY 2. (Wermer [6]) *If B is the minimal normal extension of a subnormal operator A and if the spectrum $\sigma(B)$ has two dimensional Lebesgue measure zero, then there exists a non-trivial closed invariant subspace of A.*

REMARK 1. It is clear that $H^\infty(d\mu;\sigma(B))=L^\infty(d\mu;\sigma(B))$ if and only if A is normal and also that $H^\infty(d\mu;\sigma(B))\cap\overline{H^\infty(d\mu;\sigma(B))}=\{c\cdot 1\}$ if and only if the von Neumann algebra $R(A)$ generated by a single operator A is the full operator algebra on H.

As an application of the representation theorem of a subnormal operator A on H with a cyclic vector, we can give the description of the existence of a non-trivial closed invariant subspace of A in terms of $L^2(d\mu; \sigma(B))$, $H^2(d\mu; \sigma(B))$ and $H^\infty(d\mu; \sigma(B))$ as follows:

A subnormal operator A on H with a cyclic vector x in H has a non-trivial closed invariant subspace if and only if

(*) there exists a function q, not identically zero, in $H^2(d\mu; \sigma(B))$ such that

$$[L^2(d\mu; \sigma(B)) \ominus \overline{H^\infty(d\mu; \sigma(B))} \cdot q] \cap H^2(d\mu; \sigma(B)) \neq \{0\},$$

where the bar denotes the complex conjugate and B is the minimal normal extension on K of A.

As special cases of the condition (*), we have

(case 1) there exists a function q in $H^2(d\mu; \sigma(B))$ such that

$\overline{H^\infty(d\mu; \sigma(B))} \cdot q \cap [L^2(d\mu; \sigma(B)) \ominus H^2(d\mu; \sigma(B))] \neq \{0\}$ and

(case 2) there exists a function q in $H^2(d\mu; \sigma(B))$ such that

$\overline{H^\infty(d\mu; \sigma(B))} \cdot q \cap H^2(d\mu; \sigma(B)) \neq \{c \cdot q\}$.

In the case 1, the subnormal operator A has a non-trivial closed invariant subspace M such that M is the closure of the range of some operator T in $\alpha(A)$. In fact, by the assumption, there exists a non-constant function p in $H^\infty(d\mu; \sigma(B))$ such that $\bar{p} \cdot q \in [L^2(d\mu; \sigma(B)) \ominus H^2(d\mu; \sigma(B))]$; and hence, for a vector y in H corresponding to q by Lemma 2 and an operator $p(B)|H$ in $\alpha(A)$ corresponding to p by Lemma 3, $(p(B)|H)^*y=0$. Let $H \ominus M = \{y \in H; (p(B)|H)^*y=0\}$, then M is the desirous subspace.

In the case 2, the subnormal operator A on H has a non-trivial closed invariant subspace M such that $M = \{y \in H; \|Ty\| = \|T^*y\|\}$ for some T in $\alpha(A)$. Hence, we have

THEOREM 6. *If A is a subnormal operator on H and if there exists a non-zero vector y in H such that $\|Ty\| = \|T^*y\|$ for some T in $\alpha(A)$, $0 \neq T \neq c \cdot I$, then A has a non-trivial closed invariant subspace.*

PROOF. We may assume that A has a cyclic vector and that T is non-normal. Let $M = \{y \in H; \|Ty\| = \|T^*y\|\}$, then M is non-trivial by the assumption. Since, by Theorem 3, T is subnormal and hence T is hyponormal, i.e., $(S=)T^*T - TT^* \geqq 0$. Therefore M is the null space of the non-negative self-adjoint operator S and is a closed subspace of H. Let B be the minimal normal extension on K of A, then, by Theorem 3, there exists a function $p_T(\lambda)$ in $H^\infty(d\mu; \sigma(B))$ such that $Ty = p_T(B)y$ for all y in H. The invariantness of M, under A, follows from $p_T(B)^*Ay = p_T(B)^*By = Bp_T(B)^*y = Ap_T(B)^*y \in H$ for all y in M because, for any y in H, $p_T(B)^*y \in H$ if and only if $\|Ty\| = \|T^*y\|$.

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