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INVARIANT SUBMANIFOLDS IN A SASAKIAN MANIFOLD

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1. Introduction. The theory of invariant submanifolds in a contact Riemannian manifold was initiated by M. Okumura [5].

In this note we wish to classify invariant submanifolds of codimension 2 which are η-Einstein manifolds in a Sasakian manifold with constant ϕ-sectional curvature.

To state the main theorem we prepare the followings. An odd dimensional Euclidean space E^{2n+1} (resp. an odd dimensional sphere S^{2n+1}) has the standard Sasakian structure with constant ϕ-sectional curvature $H=-3$ (resp. $H>-3$) [9]. By CD^n, L and (L, CD^n) we denote the open unit ball in a complex n-dimensional Euclidean space C^n, a real line and the product bundle $L\times CD^n$. The (L, CD^n) also has a Sasakian structre with constant ϕ-sectional curvature $H<-3$ [9]. Let Q^{n-1} be an $(n-1)$-dimensional complex quadric in a complex projective space $P^n(C)$ of complex dimension n. By (S, Q^{n-1}) we denote a circle bundle over Q^{n-1}. Then (cf. [1, p. 61]) since Q^{n-1} is a Kaehlerian manifold of restricted type, (S, Q^{n-1}) defines a Sasakian structure [6]. Since Q^{n-1} $(n\geqq 3)$ is Einsteinian, (S, Q^{n-1}) is η-Einsteinian [8]. Henceforth let $\widetilde{M}$ be one of the E^{2n+1}, S^{2n+1} and (L, CD^n) and $\widetilde{B}$ be C^n (if $\widetilde{M}=E^{2n+1}$), $P^n(C)$ (if $\widetilde{M}=S^{2n+1}$) and CD^n (if $\widetilde{M}=(L, CD^n)$). $\widetilde{M}$ is a principal G^1-bundle over $\widetilde{B}$ and G^1 is a circle or a line. $\pi:\widetilde{M}\longrightarrow\widetilde{B}$ denotes the projection. We may prove the following theorem.

THEOREM. i) *S^{2n-1} and (S, Q^{n-1}) are the only connected complete invariant submanifolds in S^{2n+1} which are η-Einsteinian.*

ii) *(L, CD^{n-1}) (resp. E^{2n-1}) is the only connected complete invariant submanifold in (L, CD^n) (resp. E^{2n+1}) which is η-Einsteinian.*

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2. Local results. (ϕ, ξ, η, g) denote the tensors of the Sasakian structure of $\widetilde{M}$. Let M be a connected contact manifold of dimension $2n-1$ which is a submanifold of $\widetilde{M}$, the (almost) contact structure of M being given by (ϕ_0, ξ_0, η_0). *An invariant immersion* $i: M \longrightarrow \widetilde{M}$ (of (almost) contact manifolds) is an immersion such that $i \cdot \phi_0 = \phi \cdot i$ and $i\xi_0 = \xi$. $i(M)$ is called an invariant submanifold of $\widetilde{M}$. All metric properties on M will refer to the metric g_0 induced on M by the immersion i. The structure tensors g_0, ϕ_0, ξ_0 and η_0 may be respectively identified with the restrictions of the structure tensors g, ϕ, ξ and η to $i(M)$. It is known that $i(M)$ is a Sasakian manifold and a minimal submanifold in $\widetilde{M}$ (for instance [5]). Henceforth we assume that M is connected and complete.

On $\widetilde{B}$ there is a Kaehlerian structure (J, Ω, h) which has following properties [6]:

$$\pi \cdot \phi = J \cdot \pi, \tag{1}$$

$$d\eta = \pi^* \Omega, \tag{2}$$

$$g = \pi^* h + \eta \otimes \eta. \tag{3}$$

As M is complete and clearly a regular contact manifold, there is a fibering $\pi_0: M \longrightarrow B = M/\xi$. Henceforth X^*, Y^* and Z^* over M will be horizontal lifts of X, Y and Z over B respectively with respect to the connection η. If we define a (1, 1)-tensor field $'J$ on B by

$$'J_y X = \pi_0 \phi X^*_v, \tag{4}$$

where $X \in T_y(B)$, $\pi_0(v) = y$ and define a metric tensor field h_0 on B by

$$h_0(X, Y) = g_0(X^*, Y^*), \tag{5}$$

then $('J, h_0)$ is a Kaehlerian structure on B and B is complete.

We define a mapping of B into $\widetilde{B}$ as follows:

$$f(x) = \pi \cdot i(u), \quad \pi_0(u) = x, \qquad x \in B. \tag{6}$$

Then f is clearly well-defined and an immersion of B into $\widetilde{B}$. Making use of (1), (4) and (6), it is easily verified that f is an almost complex mapping. Since B and $\widetilde{B}$ are complex manifolds and f is a differentiable mapping of B into $\widetilde{B}$ such that

$$f_* \cdot {}'J = J \cdot f_*,$$

by the well-known fact, f is a complex analytic mapping. Therefore we obtain

PROPOSITION 1. *Let M^{2n-1} be an invariant submanifold of $\widetilde{M}^{2n+1}$ which is complete and $\widetilde{B}$ the base space of the principal G^1-bundle $\widetilde{M}^{2n+1}$. Then $\pi M = B$ is a complex hypersurface of $\widetilde{B}$ which is complete.*

Let W be a neighborhood of a point x of B on which we can choose a unit vector field A normal to B. By [7, (2)] for a vector field X on B tangent to B we have

$$'\widetilde{\nabla}_X A = -'\alpha(X) + 's(X)JA\,, \tag{7}$$

where $'\alpha(X)$ is tangent to B and $'\widetilde{\nabla}_X$ denotes the covariant differentiation for the Riemannian metric h on B. α is the second fundamental form of B.

When A^* denotes the horizontal lift of A with respect to η, by virtue of (3), A^* is a unit vector field normal to M (for the metric to g). For a vector field U tangent to M we have

$$\widetilde{\nabla}_U A^* = -\alpha(U) + s(U)\phi A^*\,, \tag{8}$$

where $\alpha(U)$ is tangent to M and $\widetilde{\nabla}_U$ denotes the covariant differentiation for the Riemannian metric on $\widetilde{M}$. α is the second fundamental form of M.

PROPOSITION 2. *Under the same assumption as Proposition 1, the relations*

$$\pi\cdot\alpha = \alpha\cdot\pi\,, \tag{9}$$

$$s + \eta = \pi^*\cdot{}'s \tag{10}$$

hold good.

PROOF. By Lemma 1 and lemma 2 of [6, §35] we have

$$[A^*, \xi] = 0\,, \tag{11}$$

$$\eta([X^*, Y^*]) = -2g(X^*, \phi Y^*)\,, \tag{12}$$

$$\widetilde{\nabla}_{X^*} Y^* = ('\widetilde{\nabla}_X Y)^* + \frac{1}{2}\eta([X^*, Y^*])\,\xi\,. \tag{13}$$

From (12) we have

$$\eta([X^*, A^*]) = 0\,. \tag{14}$$

From (7), (8), (13) and (14) we have

$$\alpha(X^*) = {}'\alpha(X)^*, \quad s(X^*) = {}'s(X)\cdot\pi. \tag{15}$$

Making use of (11) and the fact that the torsion tensor is zero, we have

$$\alpha(\xi) = 0, \quad s(\xi) = -1. \tag{16}$$

By (15) and (16), for any vector field U tangent to M it follows that

$$\pi\cdot\alpha(U) = {}'\alpha\cdot\pi(U), \quad (s+\eta)(U) = \pi^*{}'s(U). \tag{17}$$

COROLLARY 1. *Under the same assumption as Proposition* 1, *M is totally geodesic if and only if B is totally geodesic.*

PROOF. By (9) and (16), Corollary 1 is proved.

The following Theorem is due to Tanno [9].

THEOREM A. *In the fibering $\pi: M \longrightarrow B$ of a regular K-contact Riemannian manifold M, M is an η-Einstein manifold if and only if B is an Einstein almost Kaehlerian manifold.*

The following Theorem is due to Chern-Nomizu-Smyth [2], [4].

THEOREM B. *Let B be a complex hypersurface in a space $\tilde{B}$ of constant holomorphic sectional curvature K. If B is complete and Einsteinian, then either B is totally geodesic in $\tilde{B}$ or B is holomorphically isometric to the complex quadric Q^{n-1} in $P^n(C)$, the latter case arising only when* $K > 0$.

PROPOSITION 3. *Let $i: M \longrightarrow \tilde{M}$ be an invariantly immersed submanifold of codimension 2 such that the induced metric is complete and η-Einsteinian. Then, if $\tilde{M} = S^{2n+1}$, M is totally geodesic or πM is holomorphically isometric to Q^{n-1}. If $\tilde{M} = E^{2n+1}$ or (L, CD^n), M is totally geodesic.*

PROOF. Proposition 3 follows immediately from Proposition 1, Corollary 1, Theorem A and Theorem B.

3. The proof of Theorem. When M is totally geodesic in $\tilde{M}$, $\tilde{M}$ having a property of a free mobility [9], M is unique up to an automorphism of $\tilde{M}$. With a view to obtain Theorem we need to prove only the following Proposition.

PROPOSITION 4. *Let M be a non-totally geodesic connected invariant submanifold of codimension 2 in S^{2n+1} such that the induced metric is complete and η-Einsteinian. Then there is an automorphism $\widetilde{\theta}$ of S^{2n+1} such that $\widetilde{\theta}M=(S, Q^{n-1})$.*

PROOF. By Proposition 3, $B=\pi M$ is holomorphically isometric to Q^{n-1}. By Theorem 1 of [4] there is a holomorphic isometry of $P^n(C)$ such that $\theta B=Q^{n-1}$. Let $x \in B$ and $\theta x=y$ and let $\tau(t)$, $0 \leqq t \leqq 1$, be a curve joining x and y. Then we have a continuous family of J-basis $(\tau(t), e_i(t), Je_i(t))$, $i=1, \cdots, n$, on $\tau(t)$ such that $e_i(1)=\theta_*(e_i(0))$. Therefore θ is contained in the connected component of the automorphism group of $P^n(C)$. Hence there are finite numbers of infinitesimal automorphisms $X_1, X_2, \cdots, X_s$ of $P^n(C)$ such that $\theta=\exp t_s X_s \cdots \exp t_1 X_1$. By Lemma 5.1 of [9] there are infinitesimal automorphisms Y_k $(k=1, \cdots, s)$ of S^{2n+1} such that

$$\pi Y_k = X_k\,, \tag{18}$$

$$\pi(\exp t_k Y_k)(u) = \exp t_k X_k(\pi u)\,, \qquad u \in M\,. \tag{19}$$

Putting $\widetilde{\theta}=\exp t_s Y_s \cdots \exp t_1 Y_1$, we have $\pi\widetilde{\theta}M=Q^{n-1}$. Since $\widetilde{\theta}M$ and (S, Q^{n-1}) have the same fibre, we have $\widetilde{\theta}M=(S, Q^{n-1})$. This completes the proof of Proposition 4.

The Theorem follows immediately from Proposition 4.

REMARK. This proof is due to S. Tanno. Our original one is slightly different and more complicated.

For an invariant submanifold of S^{2n+1}, we have an improved result.

COROLLARY 2. *S^{2n-1} and (S, Q^{n-1}) are the only connected complete invariant submanifolds in S^{2n+1} which have constant scalar curvature.*

PROOF. πM is a complex hypersurface in $P^n(C)$ and also constant scalar curvature [8]. By [3] πM is an Einstein space. It follows that M is an η-Einstein space in S^{2n+1}.

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