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K-CONTACT RIEMANNIAN MANIFOLDS ISOMETRICALLY IMMERSED IN A SPACE OF CONSTANT CURVATURE

TOSHIO TAKAHASHI AND SHÛKICHI TANNO

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Introduction. A K-contact Riemannian manifold (M, ξ, g) is a Riemannian manifold (M, g) admitting a unit Killing vector field ξ satisfying

$$R(X, \xi)\xi = g(X, \xi)\xi - X \tag{1.1}$$

where R donotes the Riemannian curvature tensor of (M, g). A K-contact Riemannian manifold is Sasakian, if we have

$$R(X, \xi)Z = g(X, Z)\xi - g(\xi, Z)X . \tag{1.2}$$

In the preceding papers [3] and [4], each of the present authors studied isometric immersions of Sasakian manifolds (M^m, ξ, g) in a space $({}^*M^{m+1}, G)$ of constant curvature. Now we show that the results are generalized to K-contact Riemannian manifolds.

THEOREM A. *If a K-contact Riemannian manifold (M^m, ξ, g) is isometrically immersed in a space $({}^*M^{m+1}, G)$ of constant curvature, then (M^m, ξ, g) is Sasakian.*

This theorem gives a sufficient condition for a K-contact Riemannian manifold to be Sasakian.

By Theorem A above and the first theorem in [4], we have

THEOREM B. *Let (M^m, ξ, g) be a K-contact Riemannian manifold which is isometrically immersed in a space $({}^*M^{m+1}, G)$ of constant curvature* 1. *Then*

(i) *the type number $k \leqq 2$, and*

(ii) *(M^m, ξ, g) is of constant curvature* 1 *if and only if the scalar curvature* $S = m(m-1)$.

By a theorem of B.O'Neill and E.Stiel [1] that a complete Riemannian manifold (M^m, g) of constant curvature $C > 0$ which is isometrically immersed in a complete Riemannian manifold $(^*M^{m+k}, G)$, $2k<m$, of constant curvature C is totally geodesic, we have

COROLLARY. *If a complete K-contact Riemannian manifold* (M^m, ξ, g) *with* $S=m(m-1)$ *is isometrically immersed in a unit sphere* S^{m+1}, *then* (M^m, g) *is a unit sphere* S^m.

With respect to Theorem 1 of [3], we have

THEOREM C. *Let* (M^m, ξ, g) *be a K-contact Riemannian manifold which is isometrically immersed in a space* $(^*M^{m+1}, G)$ *of constant curvature* $C\neq 1$. *Then* $C<1$ *and* (M^m, g) *is of constant curvature* 1.

REMARK. Theorems A, B and C are also true for a properly and isometrically immersed K-contact pseudo-Riemannian manifold M^m in a pseudo-Riemannian manifold $^*M^{m+1}$ of constant curvature (cf. [2], [3], [4]). We only have to remove $C<1$ in the conclusion of Theorem C.

2. A lemma. Let (M^m, ξ, g) be an m-dimensional K-contact Riemannian manifold. Denote by R_1 the Ricci curvature tensor. Then, we have

$$R_1(X, \xi) = (m-1)g(X, \xi) \tag{2.1}$$

for each vector field X on M^m (cf. [4], etc.). We assume that (M^m, ξ, g) is isometrically immersed in a space $(^*M^{m+1}, G)$ of constant curvature C and of dimension $m+1$. The Gauss and Codazzi equations are

$$-R(X, Y)Z = C[g(Y, Z)X - g(X, Z)Y] + g(AY, Z)AX - g(AX, Z)AY, \tag{2.2}$$

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{2.3}$$

where A denotes the operator defined by the second fundamental form with respect to some (local) field of unit normals. By (2.2), we see that the Ricci curvature tensor R_1 is given by

$$R_1(X, Y) = (m-1)Cg(X, Y) + \theta g(AX, Y) - g(AAX, Y), \tag{2.4}$$

where $\theta = \text{trace } A$. By (2.1) and (2.4), we get

$$\theta g(AX, \xi) - g(AAX, \xi) + (m-1)(C-1)\, g(X, \xi) = 0. \tag{2.5}$$

In (2.2), we put $Y=Z=\xi$. Then, using (1.1), we get

$$(2.6) \qquad (C-1)[X-g(X,\xi)\xi]+g(A\xi,\xi)AX-g(AX,\xi)A\xi=0\,.$$

LEMMA 2.1. *If ξ is not an eigenvector of A at a point p of M^m, then $C=1$ and $\xi=ae_1+be_2$ at p, where $Ae_1=\lambda e_1\,(\lambda\neq 0)$ and $Ae_2=0$.*

PROOF. Let $e_i\,(i=1,\cdots,m)$ be a unit eigenvectors of A at p such that $Ae_i=\lambda_i e_i$. Put $\xi=\alpha^i e_i$, where α^i are constant. Since ξ is not an eigenvector of A by the assumption, at least two of α^i are non-zero. Assume that $\alpha^1,\cdots,\alpha^r$ are non-zero. If we put $X=e_j$ in (2.5), we get

$$-\lambda_j^2+\theta\lambda_j+(m-1)(C-1)=0, \qquad j=1,\cdots,r\,.$$

Hence, $\lambda_1,\cdots,\lambda_r$ take at most two values λ and μ. So we may assume $\lambda_1=\lambda_2=\cdots=\lambda_s=\lambda\ (s<r)$, $\lambda_{s+1}=\cdots=\lambda_r=\mu$ and $\lambda\neq\mu$. λ and μ satisfy

$$(2.7) \qquad \lambda\mu=(m-1)(1-C)\,.$$

By a change of eigenvectors, we can assume

$$\xi=ae_1+be_r, \qquad Ae_1=\lambda e_1,\ Ae_r=\mu e_r,$$

where $a^2+b^2=1$, $a\neq 0$ and $b\neq 0$. In (2.6), we put $X=e_1$ and consider the inner product with e_1. Then we have

$$(C-1)(1-a^2)+g(Ag\xi,\xi)\lambda-a^2\lambda^2=0\,.$$

Since $A\xi=a\lambda e_1+b\mu e_r$, we get

$$(C-1)(1-a^2)+b^2\lambda\mu=0\,.$$

By (2.7) and $1-a^2=b^2$, we have $(m-2)(1-C)=0$. Hence, $C=1$ and $\lambda\mu=0$ follow.

3. The case $C=1$.

PROPOSITION 3.1. *If a K-contact Riemannian manifold (M^m,ξ,g) is isometrically immersed in a space $(^*M^{m+1},G)$ of constant curvature 1, then (M^m,ξ,g) is Sasakian.*

PROOF. If ξ is not an eigenvector of A at p, then $\xi = ae_1 + be_r$ and $A\xi = a\lambda e_1$ $(a, b, \lambda \neq 0)$ by Lemma 2.1. Applying this to (2.6), we get $a^2\lambda AX = g(X, a\lambda e_1)a\lambda e_1$, i.e., $AX = \lambda\, g(X, e_1)e_1$. This shows that A is of rank 1, and hence, (M^m, ξ, g) is of constant curvature 1 at p.

Next, if ξ is an eigenvector of A at p $(A\xi = \nu\xi)$, then (2.6) implies that $\nu AX = \nu^2 g(X, \xi)\xi$. If $\nu \neq 0$, A is of rank 1 and (M^m, ξ, g) is of constant curvature 1 at p. If $\nu = 0$, putting $C = 1$ and $Y = \xi$ in (2.2), we have (1.2). Thus (M^m, ξ, g) is Sasakian.

4. **The case $C \neq 1$.** By Lemma 2.1, ξ is an eigenvector of A and so we put $A\xi = \nu\xi$. Let e be any (local) vector field which is orthogonal to ξ. Then (2.6) implies that

$$(C-1)e + \nu Ae = 0\,.$$

Hence, we have $\nu \neq 0$ and e is an eigenvector (field) of A such that

$$Ae = -\nu^{-1}(C-1)e\,. \tag{4.1}$$

In (2.3) we put $X = e$ and $Y = \xi$. Then, by $A\xi = \nu\xi$ and (4.1), we get

$$\begin{aligned}(\nabla_e \nu)\xi + (\nabla_\xi(\nu^{-1}(C-1)))e + (\nu + \nu^{-1}(C-1))\nabla_e\xi \\ - \nu^{-1}(C-1)[e, \xi] - A[e, \xi] = 0\,.\end{aligned} \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} g([e, \xi], \xi) &= g(\nabla_e\xi - \nabla_\xi e, \xi) \\ &= -g(\phi e, \xi) + g(e, \nabla_\xi\xi) = 0\,, \end{aligned}$$

where we have used $\nabla_X\xi = -\phi X$ and $\phi\xi = 0$. Hence, $A[e, \xi] = -\nu^{-1}(C-1)[e, \xi]$. Therefore, in (4.2), the first three terms remain. Since ξ, e, and $\nabla_e\xi = -\phi e$ are linearly independent, we get $\nu^2 = 1 - C$ and we see that (M^m, ξ, g) is totally umbilic. Thus,

THEOREM 4.1. *Let (M^m, ξ, g) be a K-contact Riemannian manifold which is isometrically immersed in a space $({}^*M^{m+1}, G)$ of constant curvature $C \neq 1$. Then $C < 1$ and (M^m, ξ, g) is of constant curvature* 1.

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KUMAMOTO UNIVERSITY
KUMAMOTO, JAPAN
AND
TÔHOKU UNVIERSITY
SENDAI, JAPAN