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NEGATIVE QUASIHARMONIC FUNCTIONS*

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1. The radial quasiharmonic function

$$s(r) = -\sum_{i=0}^{\infty} b_i r^{2i+2} ,$$

defined by $\Delta s = 1$, plays a crucial role in the problem of the existence of bounded quasiharmonic functions on the Poincaré ball $B_\alpha = \{r < 1, ds = (1 - r^2)^\alpha |dx|\}$ (see [18]). In the present paper we shall show that s has the striking property

$$s < 0 \quad on \quad B_\alpha \quad for\ every \quad \alpha .$$

This will lead us to the introduction of the class QN of negative quasiharmonic functions.

We shall carry out our reasoning for dimension $M = 3$. This is the essential case, as for $M = 2$ the harmonicity and the Dirichlet integral are independent of α. We conjecture that the reasoning developed in this paper will allow a generalization to an arbitrary M.

2. We start by stating our main result:

THEOREM 1. *The radial quasiharmonic function* $s(r) = -\sum b_i r^{2i+2}$ *belongs to* QN.

The proof will be given in Nos. 3–12.

3. First we determine the coefficients b_i.

LEMMA 1. *The function*

$$s(r) = -\sum_{i=0}^{\infty} b_i r^{2i+2} \tag{1}$$

with $\Delta s = 1$ *on* B_α *has*

$$b_0 = \frac{1}{6} , \tag{2}$$

and the other coefficients are determined by the recursion formula

$$b_i = p_i b_{i-1} + q_i . \tag{3}$$

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Here

$$p_i = \frac{2i(2i+1+2\alpha)}{(2i+2)(2i+3)} \tag{4}$$

and

$$q_i = \left(\prod_{j=1}^{i} \frac{j-2\alpha-2}{j}\right) \Big/ (2i+2)(2i+3) \,. \tag{5}$$

PROOF. On B_α, the metric tensor is

$$g_{ij} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 r^2 & 0 \\ 0 & 0 & \lambda^2 r^2 \sin^2 \psi \end{pmatrix},$$

the determinant is $g = \lambda^6 r^4 \sin^2 \psi$, and the Laplacian reduces to

$$\begin{aligned} \Delta s(r) &= -\frac{1}{\sqrt{g}} \frac{\partial}{\partial r}(\sqrt{g}\, g^{rr} s'(r)) \\ &= -\lambda^{-2}\left[s''(r) + \left(\frac{2}{r} - \frac{2\alpha r}{1-r^2}\right)s'(r)\right]. \end{aligned}$$

The equation $\Delta s = 1$ takes the form

$$-r^2(1-r^2)s''(r) - r[2(1-r^2) - 2\alpha r^2]s'(r) - r^2(1-r^2)^{2\alpha+1} = 1 \,. \tag{6}$$

On substituting $s(r)$ from (1) we obtain

$$\begin{aligned} & r^2(1-r^2)\sum_{i=0}^{\infty}(2i+2)(2i+1)b_i r^{2i} + r[2-2(1+\alpha)r^2]\sum_{i=0}^{\infty}(2i+2)b_i r^{2i+1} \\ & \quad -r^2 - r^2 \sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{j-2\alpha-2}{j}\right)r^{2i} = 0\,, \end{aligned}$$

that is,

$$\begin{aligned} & \sum_{i=0}^{\infty}[(2i+2)(2i+1) + 2(2i+2)]b_i r^{2i+2} \\ & \quad - \left[\sum_{i=0}^{\infty}(2i+2)(2i+1) + 2(1+\alpha)(2i+2)\right]b_i r^{2i+4} \\ & \quad - r^2 - \sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{j-2\alpha-2}{j}\right)r^{2i+2} = 0\,. \end{aligned}$$

This is equivalent to the following final form of our equation:

$$\begin{aligned} & \sum_{i=0}^{\infty}(2i+2)(2i+3)b_i r^{2i+2} - \sum_{i=1}^{\infty}2i(2i+1+2\alpha)b_{i-1}r^{2i+2} \\ & \quad - r^2 - \sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{j-2\alpha-2}{j}\right)r^{2i+2} = 0\,. \end{aligned} \tag{7}$$

To determine the constants b_i, we first equate to zero the coefficient of r^2 and obtain $6b_0 - 1 = 0$, that is, (2). The coefficient of r^{2i+2} for $i > 0$ gives

$$(2i + 2)(2i + 3)b_i = 2i(2i + 1 + 2\alpha)b_{i-1} + \prod_{j=1}^{i} \frac{j - 2\alpha - 2}{j},$$

hence (3)–(5).

4. The following consequence of Lemma 1 is immediate:

LEMMA 2. *The coefficients b_i are*

$$b_i = b_0 \prod_{j=1}^{i} p_j + \sum_{j=1}^{i-1} q_j \prod_{k=j+1}^{i} p_k + q_i, \tag{8}$$

with $b_0 = 1/6$.

We shall also use the notation

$$b_i = \sum_{j=0}^{i} \beta_{ij} \tag{9}$$

with

$$\begin{cases} \beta_{i0} = b_0 \prod_{j=1}^{i} p_j, & \beta_{ii} = q_i, \\ \beta_{ij} = q_j \prod_{k=j+1}^{i} p_k & \text{for} \quad 1 \leqq j \leqq i - 1. \end{cases} \tag{10}$$

An inspection of (8) shows readily:

LEMMA 3. *For a fixed i_0 and all $i > i_0$,*

$$b_i = b_{i_0} \prod_{j=i_0+1}^{i} p_j + \sum_{j=i_0+1}^{i} \beta_{ij}.$$

5. The signs of p_i and q_i will be instrumental. For a given $\alpha \in \boldsymbol{R}$ we set

$$\begin{cases} i_p = \max \left\{i \mid i < -\alpha - \dfrac{1}{2}\right\} \\ i_q = \max \{i \mid i < 2\alpha + 2\}. \end{cases} \tag{11}$$

The following immediate observations are compiled here for easy reference:

LEMMA 4. *If $\alpha > -3/2$, then all $p_i > 0$. If $\alpha = -3/2$, then $p_1 = 0$ and $p_i > 0$ for $i > 1$. If $\alpha < -3/2$, then $p_i < 0$ for $i \leqq i_p$, and $p_i \geqq 0$ for $i > i_p$, with equality at most for $i = i_p + 1$.*

LEMMA 5. *If $\alpha < -1/2$, then all $q_i > 0$. If $\alpha = -1/2$, then all $q_i = 0$.*

If $\alpha > -1/2$ *and* $i \leqq i_q$, *then* $q_i > 0$ *for* i *even and* $q_i < 0$ *for* i *odd. If* $\alpha > -1/2$ *and* $i > i_q$, *then* $q_i \geqq 0$ *for* i_q *even, and* $q_i \leqq 0$ *for* i_q *odd.*

These rules motivate the division of our discussion in the sequel into the cases $\alpha < -3/2$; $-3/2 \leqq \alpha \leqq -1/2$; and $\alpha \geqq 1$. If $\alpha \in (-1, 1)$, there exist functions $u \in QB$ (Sario-Wang [16]), and

$$u - \sup_{B_\alpha} u \in QN\,,$$

that is, $B_\alpha \notin O_{QN}$. Thus it will suffice to discuss the above three cases.

We shall first show, in Nos. 6–10, that the $b_i > 0$ for all sufficiently large i, and then in Nos. 11–12 that the series $s = -\sum_0^\infty b_i r^{2i+2}$ converges, hence $s - c \in QN$ for some constant c.

6. *Case* $\alpha < -3/2$. By Lemma 3, we have for $i > i_p$,

$$b_i = b_{i_p} \prod_{j=i_p+1}^{i} p_j + \sum_{j=i_p+1}^{i} \beta_{ij} \tag{12}$$

where

$$b_{i_p} = \sum_{j=0}^{i_p} \beta_{i_p j}\,.$$

LEMMA 6. *For* $\alpha < -3/2$, $\quad b_{i_p} > 0$.

PROOF. Set

$$\delta_{i_p j} = \frac{\beta_{i_p j}}{\beta_{i_p, j-1}}\,, \qquad 2 \leqq j \leqq i_p\,,$$

with

$$\beta_{i_p j} = q_j \prod_{k=j+1}^{i_p} p_k\,.$$

We have

$$\delta_{i_p j} = \frac{q_j}{q_{j-1} p_j} < 0 \tag{13}$$

and

$$\begin{aligned} |\delta_{i_p j}| &= 1 + \frac{4j^2 - 2(\alpha+1)(j+1)}{-j(2j+1+2\alpha)} \\ &> 1 + \frac{4j^2 + j + 1}{-j(2j+1+2\alpha)}\,. \end{aligned}$$

Therefore

$$|\delta_{i_p j}| > 1 \quad \text{for} \quad 2 \leqq j \leqq i_p\,. \tag{14}$$

Suppose first i_p even. Then

(15) $$b_{i_p} = \beta_{i_p0} + \sum_{j=1}^{(1/2)i_p} (\beta_{i_p,2j-1} + \beta_{i_p,2j}) .$$

Since $\beta_{i_pi_p} = q_{i_p} > 0$, we see by (13) and (14) that each sum in parentheses is > 0. The same is true of $\beta_{i_p0} = b_0 \prod_{j=1}^{i_p} p_j$, as each $p_j < 0$, and we conclude that $b_{i_p} > 0$.

If i_p is odd, we first observe that

$$\delta_{i_p1} = \frac{\beta_{i_p1}}{\beta_{i_p0}} = \frac{q_1}{b_0 p_1} = 3 \cdot \frac{-1-2\alpha}{3+2\alpha} < 0$$

for $\alpha < -3/2$, and

$$|\delta_{i_p1}| = 3\left(1 + \frac{2}{-3-2\alpha}\right) > 3 .$$

Since $\beta_{i_p0} < 0$ and $\beta_{i_p1} > 0$,

$$\beta_{i_p0} + \beta_{i_p1} > 0$$

and by (14)

$$\beta_{i_p2j} + \beta_{i_p,2j+1} > 0$$

for $1 \leqq j \leqq (1/2)(i_p - 1)$. Therefore

$$b_{i_p} = \sum_{j=0}^{(1/2)(i_p-1)} (\beta_{i_p2j} + \beta_{i_p,2j+1}) > 0 .$$

7. We can now go further than Lemma 6:

LEMMA 7. *For* $\alpha < -3/2$,

(16) $$b_i > 0 , \qquad i \geqq i_p$$

and

(17) $$\sum_{i=0}^{\infty} b_i = \infty .$$

PROOF. Inequality (16) is a direct consequence of (12). To prove (17) set $s = s_1 + s_2$ with

$$s_1 = -\sum_{i=0}^{i_p-1} b_i r^{2i+2} , \qquad s_2 = -\sum_{i=i_p}^{\infty} b_i r^{2i+2} .$$

Here $s_1 \in QB$ and $|s_2| < \sum_{i_p}^{\infty} b_i$. If this sum converges, we have $s_2 \in QB$, hence $s \in QB$, a contradiction since $\alpha \notin (-1, 1)$. This proves the lemma.

Note that the condition on α in Lemma 7 cannot be suppressed, as e.g. $\alpha = 0$ gives $b_i = 0$ for $i \geqq 1$.

8. *Case* $-3/2 \leqq \alpha \leqq -1/2$. For $\alpha = -3/2$, $p_1 = 0$, $p_i > 0$ for $i > 1$. For $-3/2 < \alpha \leqq -1/2$, all $p_i > 0$. For $\alpha = -1/2$, all $q_i = 0$. For $-3/2 \leqq$

$\alpha < -1/2$, all $q_i > 0$. For $-3/2 \leqq \alpha \leqq -1/2$ we therefore have $\beta_{i0} \geqq 0$, $\beta_{ij} \geqq 0, j > 1$.

LEMMA 8. *If* $-3/2 \leqq \alpha \leqq -1/2$,

$$b_i > 0 \quad \textit{for all} \quad i\,. \tag{18}$$

9. *Case* $\alpha \geqq 1$. Now we cannot specify an i beyond which all $b_i > 0$. However:

LEMMA 9. *For* $\alpha \geqq 1$, *there exists an* $i_0 \geqq i_q$ *such that*

$$b_{i_0} > 0\,. \tag{19}$$

PROOF. All b_i cannot vanish, since $\Delta s = 1$. Suppose there exists an i_0 such that $b_i \leqq 0$ for $i > i_0$. If s is bounded, we have $B_\alpha \notin O_{QB}$, a contradiction since $\alpha \notin (-1, 1)$. Thus s is unbounded and

$$-s + \sup_{B_\alpha} \left| \sum_{i=0}^{i_0} b_i r^{2i+2} \right| \in QP\,,$$

again a contradiction. We conclude that there exist infinitely many $b_i > 0$. In particular, there is some $i_0 \geqq i_q$ such that $b_{i_0} > 0$.

10. We can sharpen Lemma 9:

LEMMA 10. *For* $\alpha \geqq 1$, *and* i_0 *of Lemma* 9,

$$b_i > 0 \quad \textit{for } i \geqq i_0 \tag{20}$$

and

$$\sum_{i=0}^{\infty} b_i = \infty\,. \tag{21}$$

PROOF. For $i > i_0$,

$$b_i = b_{i_0} \prod_{j=i_0+1}^{i} p_j + \sum_{j=i_0+1}^{i} \beta_{ij}\,. \tag{22}$$

Each $p_j > 0$, hence the first term on the right is > 0. If i_q is even, then $q_i \geqq 0$ for $i > i_q$, and $\beta_{ij} \geqq 0$ for $i > i_q$. Therefore $b_i > 0$ for $i > i_0$. If i_q is odd, then $q_i \leqq 0$ for $i > i_q$. Suppose $b_{i_1} \leqq 0$ for some $i_1 \geqq i_q$. Then

$$b_{i_1+1} = p_{i_1+1} b_{i_1} + q_{i_1+1} \leqq 0 \tag{23}$$

and by induction we infer that $b_i \leqq 0$ for $i \geqq i_1$, a contradiction. Consequently $b_i > 0$ for $i \geqq i_q$.

The proof of (21) is the same as in that of Lemma 7.

Note that Lemma 10 cannot be sharpened to $b_i > 0$ for all $i \geqq 0$, since e.g. $b_1 = -\alpha/15$.

We have established that, in all cases, $b_i > 0$ for all but a finite number of i. It remains to show that the series $\sum b_i r^{2i+2}$ converges.

11. *Convergence when* $\alpha \leqq -1$. We claim:

LEMMA 11. *For* $\alpha \leqq -1$,

$$\sum_{i=0}^{\infty} b_i r^{2i+2} < \infty \ . \tag{24}$$

PROOF. The ratio of subsequent terms being $b_{i+1}r^2/b_i$, it suffices to show that $b_{i+1}/b_i \to 1$. In view of (3) we have

$$\frac{b_{i+1}}{b_i} = p_{i+1} + \frac{q_{i+1}}{b_i} \ , \tag{25}$$

where $p_{i+1} \to 1$ by (4). We shall show that $q_{i+1}/b_i \to 0$, that is, for any positive integer n, fixed henceforth, there exists an i_n such that $b_i/q_{i+1} > n$ for $i \geqq i_n$. For $i > i_p$,

$$\frac{b_i}{q_{i+1}} = \frac{b_{i_p}}{q_{i+1}} \prod_{j=i_p+1}^{i} p_j + \sum_{j=i_p+1}^{i-1} \frac{q_j}{q_{i+1}} \prod_{k=j+1}^{i} p_k + \frac{q_i}{q_{i+1}} \ , \tag{26}$$

where $b_{i_p} > 0$. Note that the case $-3/2 \leqq \alpha \leqq -1$ is included, for then $b_{i_p} = b_0 = 1/6$. Since $p_j \geqq 0$ for $j > i_p$, with equality at most for $j = i_p + 1$, and since $q_j > 0$ for all j, we obtain for $\alpha \leqq -1$ and $i \geqq i'_n = i_p + n + 1$,

$$\frac{b_i}{q_{i+1}} \geqq f(i) = \sum_{j=i-n}^{i-1} \frac{q_j}{q_{i+1}} \prod_{k=j+1}^{i} p_k + \frac{q_i}{q_{i+1}} \ . \tag{27}$$

It suffices to show that the function $f(i)$ introduced herewith dominates n for all sufficiently large i.

Since $f(i)$ and hence $f'(i)$ are rational in i, there exists an i''_n such that $f'(i)$ is of constant sign and $f(i)$ is monotone for $i \geqq i''_n$. In (27),

$$\frac{q_i}{q_{i+1}} = \frac{i+1}{i-1-2\alpha} \cdot \frac{(2i+4)(2i+5)}{(2i+2)(2i+3)} \to 1$$

as $i \to \infty$, and so does each q_j/q_{i+1} for $i - n \leqq j \leqq i-1$. Since also each $p_k \to 1$, we have $f(i) \to n + 1$, the convergence being monotone for $i \geqq i''_n$. We conclude that there exists an $i_n \geqq \max(i'_n, i''_n)$ such that

$$f(i) > n \quad \text{for} \quad i \geqq i_n \ .$$

This completes the proof of Lemma 11.

12. *Convergence when* $\alpha \geqq 1$. We proceed to show:

LEMMA 12. *For* $\alpha \geqq 1$,

$$\sum_{i=0}^{\infty} b_i r^{2i+2} < \infty \ . \tag{28}$$

PROOF. If i_q is even, then $q_i \geqq 0$ for $i > i_q$. Since each $p_i > 0$, the proof of Lemma 11 continues to be valid in the present case, with i_p replaced by i_0 of Lemma 9.

If i_q is odd, then $q_i \leqq 0$ for $i > i_q$. Again each $p_i > 0$, and since by Lemma 10, $b_i > 0$ for $i \geqq i_0$ we have by (25)

$$0 < \frac{b_{i+1}}{b_i} \leqq p_{i+1} \to 1 .$$

The proof of Theorem 1 is herewith complete.

13. Let O_G be the class of parabolic Riemannian manifolds, and O_{QX} the class of Riemannian manifolds which carry no functions in a given class QX, with $X = N, P, B$, or D, the class of negative, positive, bounded, or Dirichlet finite functions, respectively. In [16] we showed that

$$\begin{aligned} B_\alpha \notin O_G &\Leftrightarrow \alpha < 1 , \\ B_\alpha \notin O_{QP} &\Leftrightarrow \alpha \in (-1, 1) , \\ B_\alpha \notin O_{QB} &\Leftrightarrow \alpha \in (-1, 1) , \\ B_\alpha \notin O_{QD} &\Leftrightarrow \alpha \in \left(-\frac{3}{5}, 1\right) . \end{aligned}$$

From Theorem 1 we have the following consequences, which also can be established directly:

THEOREM 2. *There exist both parabolic and hyperbolic* 3-*manifolds which carry QN-functions but no QP-functions.*

Explicitly, if we denote by $\tilde{O}$ the complement of an O-class, then

$$B_\alpha \in \tilde{O}_G \cap \tilde{O}_{QN} \cap O_{QP} \Leftrightarrow \alpha \leqq -1 , \tag{29}$$

$$B_\alpha \in O_G \cap \tilde{O}_{QN} \cap O_{QP} \Leftrightarrow \alpha \geqq 1 . \tag{30}$$

THEOREM 3. *There exist Riemannian* 3-*manifolds which carry QN- and QP-functions but no QD-functions.*

Explicitly,

$$B_\alpha \in \tilde{O}_{QN} \cap \tilde{O}_{QP} \cap O_{QD} \Leftrightarrow -1 < \alpha \leqq -\frac{3}{5} . \tag{31}$$

We conjecture that Theorems 1–3 hold for manifolds of any dimension.

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