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# SEPARATION CONDITIONS AND STABILITY PROPERTIES IN ALMOST PERIODIC SYSTEMS

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**1. Introduction.** The existence of almost periodic solutions in almost periodic systems has been studied by many authors. Generally, the existence of a bounded solution does not imply the existence of almost periodic solutions [6], and hence we need some additional conditions to obtain almost periodic solutions. In linear systems, one of conditions is Favard's separation condition [2], and for general systems, there are Amerio's separation condition [1] and stability conditions ([5], [7]).

Seifert [7] has shown that the existence of a bounded solution with some global stability implies Amerio's separation condition. In this paper, we shall discuss relationships between separation conditions and local stability conditions. In Section 2, we shall consider a linear system and show that the uniform stability implies Favard's separation condition. In Section 3, Amerio's separation condition will be characterized in terms of conditional stability, and we shall also show that the existence of a bounded solution with uniformly asymptotic stability implies some kind of separation condition.

We denote by $R^n$ the real Euclidean $n$-space and set $R^1 = R$ and $R^+ = [0, \infty)$. For $x \in R^n$, let $|x|$ be the Euclidean norm of $x$. If $A$ and $B$ are topological spaces, $C(A; B)$ denotes the set of continuous functions on $A$ into $B$.

**2. Favard's separation condition and uniform stability.** Consider the linear systems

$$x' = A(t)x \qquad (' = d/dt) \tag{2.1}$$

and

$$x' = A(t)x + f(t)\,, \tag{2.2}$$

where the $n \times n$ matrix $A(t)$ and the $n$-vector $f(t)$ are continuous and almost periodic in $t$. To show the existence of an almost periodic solution, Favard [2] has assumed the condition below. We shall discuss a relationship between Favard's condition and the uniform stability of solutions.

DEFINITION 2.1. The system (2.1) is said to satisfy Favard's separation condition if for each $B \in H(A)$, where $H(A)$ is the hull of $A(t)$, the system

$$x' = B(t)x \tag{2.3}$$

has no nontrivial bounded solution $x(t)$ defined on $R$ which satisfies

$$\inf_{t \in R} |x(t)| = 0 .$$

THEOREM 2.1. *If the zero solution of the system* (2.1) *is uniformly stable, then the system* (2.1) *satisfies Favard's separation condition.*

PROOF. For each $B \in H(A)$, consider the system (2.3). Let $x_1(t), \cdots, x_m(t)$, $m \leqq n$, be a basis of the space of bounded solutions on $R$ of (2.3). Since $B(t)$ is almost periodic, there is a sequence $\{t_k\}$ such that

$$t_k \to -\infty \quad \text{as} \quad k \to \infty \tag{2.4}$$

and $B(t + t_k) \to B(t)$ uniformly on $R$ as $k \to \infty$. Since $\{x_j(t + t_k)\}_{k=1}^{\infty}$ $(1 \leqq j \leqq m)$ is uniformly bounded and equicontinuous, it follows from Ascoli-Arzela's theorem that there exists a subsequence of $\{t_k\}$, which will be denoted by $\{t_k\}$ again, and functions $y_j(t)$ such that

$$x_j(t + t_k) \to y_j(t)$$

uniformly on any compact interval in $R$ for all $j$, $1 \leqq j \leqq m$, as $k \to \infty$. Since $x_j(t + t_k)$ is a solution of the system

$$x' = B(t + t_k)x ,$$

clearly $y_j(t)$ is a bounded solution of (2.3).

We shall show that $y_1(t), \cdots, y_m(t)$ are linearly independent. Suppose that $\sum_{j=1}^{m} c_j y_j(0) = 0$ for some constants $c_1, \cdots, c_m$ and set $z(t) = \sum_{j=1}^{m} c_j x_j(t)$. Then $z(t)$ is a solution of (2.3) and clearly

$$z(t_k) \to 0 \quad \text{as} \quad k \to \infty . \tag{2.5}$$

Since the zero solution of (2.3) is also uniformly stable, (2.4) and (2.5) imply that

$$z(t) = 0 .$$

Therefore

$$c_1 = \cdots = c_m = 0$$

by the linear independence of $x_1(t), \cdots, x_m(t)$. This shows that $y_1(t), \cdots, y_m(t)$ are linearly independent.

Now let $x(t)$ be any nontrivial bounded solution on $R$ of (2.3). Then there correspond constants $\lambda_1, \cdots, \lambda_m$ such that

$$x(t) = \sum_{j=1}^{m} \lambda_j y_j(t) .$$

Since $x(0) \neq 0$, we have

$$\left| \sum_{j=1}^{m} \lambda_j x_j(t_{k_0}) \right| \geqq \varepsilon$$

for sufficiently large $k_0$ and for some $\varepsilon > 0$. Since the zero solution of (2.3) is uniformly stable, there exists a positive constant $\delta$ such that

$$\left| \sum_{j=1}^{m} \lambda_j x_j(t) \right| \geqq \delta \text{ for } t \leqq t_{k_0} .$$

Therefore, for each $t \in R$

$$|x(t)| = \lim_{k \to \infty} \left| \sum_{j=1}^{m} \lambda_j x_j(t + t_k) \right| \geqq \delta$$

by (2.4). This shows that the system (2.1) satisfies Favard's separation condition.

The following result follows immediately from Theorem 2.1 and Favard's result [2, p.64].

COROLLARY 2.1. *If the zero solution of* (2.1) *is uniformly stable and the system* (2.2) *has a bounded solution on* $R^+$, *then the system* (2.2) *has an almost periodic solution whose module is contained in the module of* $(A, f)$.

Clearly the converse of Theorem 2.1 is not necessarily true. One of counter examples is the case where the zero solution of the system (2.1) is not uniformly stable and the system has an exponential dichotomy which is a special case of Favard's separation condition. Now we shall prove that the converse of Theorem 2.1 holds under some supplementary conditions.

THEOREM 2.2. *Assume that the zero solution of the system* (2.1) *is positively and negatively stable. If the system satisfies Favard's separation condition, then the zero solution is uniformly stable.*

PROOF. Let $X(t)$ be a fundamental matrix of (2.1). By the first assumption, $X(t)$ is bounded on $R$. Now we shall show that

$$\inf \{ |X(t)x_0|; t \in R, x_0 \in R^n, |x_0| = 1 \} \neq 0 . \tag{2.6}$$

Suppose not. Then there exists sequences $\{t_k\} \subset R$, $\{x_k\} \subset R^n$ $(|x_k| = 1)$ such that

$$\lim_{k \to \infty} |X(t_k)x_k| = 0 .$$

Clearly $\{x_k\}$ can be assumed to converge to an $x_\infty \in R^n$ ($|x_\infty| = 1$), and we have

$$|X(t_k)x_k - X(t_k)x_\infty| \leqq |X(t_k)| \times |x_k - x_\infty| ,$$

where $|X|$ is the operator norm of the matrix $X$. Since $X(t)$ is bounded on $R$, we have

$$\lim_{k\to\infty} |X(t_k)x_\infty| = 0 .$$

This contradicts Favard's separation condition.

Since we have (2.6), there exists a positive constant $c$ such that

$$|X(t)x| \geqq c|x| \quad \text{for} \quad t \in R , \quad x \in R^n .$$

This implies

$$|X(t)x| \leqq c'|X(s)x| \quad \text{for} \quad t, s \in R , \quad x \in R^n ,$$

where $c' = \sup_{t\in R} |X(t)|/c$, which shows the uniform stability of the zero solution of (2.1).

REMARK 2.1. As will be seen from the example below (for the details, see [3, p.300]), we cannot drop Favard's separation condition in Theorem 2.2 without any other supplementary condition.

Consider a scalar almost periodic equation

$$x' = -\alpha(t)x ,$$

where $\alpha(t) = \sum_{k=1}^{\infty} c_k \sin \lambda_k t$ ($\lambda_k > 0$, $c_k > 0$, $\sum_{k=1}^{\infty} c_k < \infty$, $\sum_{k=1}^{\infty} c_k/\lambda_k = \infty$) and $\lambda_k$ are linearly independent. Since $\exp\left(-\int_0^t \alpha(s)ds\right)$ can be easily verified to be bounded on $R$, the zero solution is positively and negatively stable. But it is not uniformly stable. Suppose that it is uniformly stable. Noting that $\lambda_k$ are linearly independent, we can see $\alpha(t) \in H(-\alpha)$, and hence, the equation

$$x' = \alpha(t)x$$

has a bounded nontrivial solution on $R^+$. On the otherhand, we can show that $\exp\left(\int_0^t \alpha(s)ds\right)$ is unbounded on $R^+$. Thus there arises a contradiction.

REMARK 2.2. As was stated before, Favard's separation condition is not equivalent to the uniform stability. However, the zero solution of (2.1) is uniformly stable with respect to bounded solutions on $R$ if and only if Favard's separation condition is satisfied, where the zero solution of (2.1) is said to be uniformly stable with respect to bounded solutions on $R$ if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$|x(t)| < \varepsilon \quad \text{for} \quad t \geqq t_0\,,$$

whenever $x(t)$ is a bounded solution on $R$ of (2.1) and $|x(t_0)| < \delta(\varepsilon)$ for some $t_0 \in R$. This can be proved by the same argument as in the proofs of Theorem 2.1 and 2.2.

**3. Amerio's separation condition and stability properties.** Consider the systems

$$x' = f(t, x) \tag{3.1}$$

and

$$x' = g(t, x)\,, \tag{3.2}$$

where $f(t, x) \in C(R \times R^n; R^n)$ and $f(t, x)$ is almost periodic in $t$ uniformly for $x \in R^n$ and $g \in H(f)$.

Throughout this section, let $K$ be a compact subset of $R^n$. For each $g \in H(f)$, we denote by $A(g, K)$ the set of solutions $x(t)$ of the system (3.2) such that $x(t) \in K$ for all $t \geqq t_0$, and for each $x \in A(g, K)$, $t(x)$ denotes the infimum of $t_0$, where $t(x)$ may be $-\infty$. Let $B(g, K) = \{x \in A(g, K); t(x) = -\infty\}$. Clearly, if the system (3.1) has a solution in $A(f, K)$, then $B(g, K)$ is not empty for each $g \in H(f)$. In the following definitions, we shall consider the case where $B(g, K)$ is not empty for each $g \in H(f)$.

DEFINITION 3.1. The system (3.1) is said to satisfy Amerio's separation condition in $K$ if there exists a positive constant $\lambda = \lambda(g)$ for each $g \in H(f)$ such that any distinct solutions $x$, $y$ in $B(g, K)$ satisfy

$$\inf_{t \in R} |x(t) - y(t)| \geqq \lambda\,.$$

REMARK 3.1. Under Amerio's separation condition, the constant $\lambda$ can be chosen independently of $g \in H(f)$. Hence we assume that $\lambda$ does not depend on $g \in H(f)$.

DEFINITION 3.2. $x \in B(f, K)$ is said to be conditionally uniformly stable in $K$, if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|x(t) - y(t)| \leqq \varepsilon$ for $t \geqq t_0$, whenever $y \in A(f, K)$ and $|x(t_0) - y(t_0)| < \delta(\varepsilon)$ for some $t_0 \geqq t(y)$. $x \in B(f, K)$ is said to be conditionally uniformly asymptotically stable in $K$, if $x$ is conditionally uniformly stable in $K$ and if there exists a $\delta_0 > 0$ and for any $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that $|x(t) - y(t)| < \varepsilon$ for $t \geqq t_0 + T(\varepsilon)$, whenever $y \in A(f, K)$ and $|x(t_0) - y(t_0)| < \delta_0$ for some $t_0 \geqq t(y)$.

DEFINITION 3.3. The system (3.1) is said to be conditionally uniformly asymptotically stable in $K$ if every $x \in B(f, K)$ is conditionally uniformly asymptotically stable in $K$.

Now we shall show that Amerio's separation condition will be characterized in terms of conditionally uniformly asymptotic stability of the system.

THEOREM 3.1. *The system* (3.1) *satisfies Amerio's separation condition in* $K$ *if and only if for each* $g \in H(f)$, *the system* (3.2) *is conditionally uniformly asymptotically stable in* $K$ *with a common triple* $(\delta_0, \delta(\cdot), T(\cdot))$.

PROOF. Assume that for each $g \in H(f)$ the system (3.2) is conditionally uniformly asymptotically stable in $K$ with a common triple $(\delta_0, \delta(\cdot), T(\cdot))$. First of all, we shall see that any distinct solutions $x$, $y$ in $B(g, K)$, $g \in H(f)$, satisfy

$$\varliminf_{t \to -\infty} |x(t) - y(t)| \geqq \delta_0 . \tag{3.3}$$

Suppose not. Then for some $g \in H(f)$, there exists two distinct solutions $x$, $y$ in $B(g, K)$ which satisfy

$$\varliminf_{t \to -\infty} |x(t) - y(t)| < \delta_0 . \tag{3.4}$$

Since $x(t) \neq y(t)$, we have $|x(t_0) - y(t_0)| = \varepsilon$ at some $t_0$ and for some $\varepsilon > 0$. Then there is a $t_1$ such that $t_1 < t_0 - T(\varepsilon/2)$ and

$$|x(t_1) - y(t_1)| < \delta_0$$

since we have (3.4). The conditionally uniformly asymptotic stability of $x(t)$ implies

$$|x(t_0) - y(t_0)| < \varepsilon/2 ,$$

which contradicts $|x(t_0) - y(t_0)| = \varepsilon$. Thus we have (3.3).

Since $K$ is a compact set, there are a finite number of coverings which consist of $m_0$ balls with diameter $\delta_0/4$. We shall show that the number of solutions in $B(g, K)$ is at most $m_0$. Suppose not. Then there are $m_0 + 1$ solutions in $K$, $x_j(t)$, $j = 1, 2, \cdots, m_0 + 1$, and a $t_2$ such that

$$|x_i(t_2) - x_j(t_2)| \geqq \delta_0/2 \quad \text{for} \quad i \neq j , \tag{3.5}$$

because we have (3.3). However some two of these solutions, say $x_i(t)$, $x_j(t)$ $(i \neq j)$, are in one ball at time $t_2$, and hence

$$|x_i(t_2) - x_j(t_2)| < \delta_0/4 ,$$

which contradicts (3.5). Therefore the number of solutions in $B(g, K)$ is $m \leqq m_0$. Thus

$$B(g, K) = \{x_1(t), x_2(t), \cdots, x_m(t)\} \tag{3.6}$$

and

$$\varliminf_{t\to-\infty} |x_i(t) - x_j(t)| \geqq \delta_0 , \qquad i \neq j . \tag{3.7}$$

Consider a sequence $\{\tau_k\}$ such that $\tau_k \to -\infty$ as $k \to \infty$ and $g(t + \tau_k, x) \to g(t, x)$ uniformly on $R \times K$ as $k \to \infty$. Since $\{x_j(t + \tau_k)\}_{k=1}^{\infty}$ $(1 \leqq j \leqq m)$ is uniformly bounded and equicontinuous, there exists a subsequence of $\{\tau_k\}$, which will be denoted by $\{\tau_k\}$ again, and functions $y_j(t)$ such that

$$x_j(t + \tau_k) \to y_j(t)$$

uniformly on any compact interval in $R$ for $j$, $1 \leqq j \leqq m$, as $k \to \infty$. Clearly $y_j(t)$ is a solution in $B(g, K)$. Since we have

$$|y_i(t) - y_j(t)| = \lim_{k\to\infty} |x_i(t + \tau_k) - x_j(t + \tau_k)|$$

for $t \in R$, it follows from (3.7) that

$$|y_i(t) - y_j(t)| \geqq \delta_0 \text{ for all } t \in R \text{ and } i \neq j . \tag{3.8}$$

Since the number of solutions in $B(g, K)$ is $m$, $B(g, K)$ consists of $y_1(t), \cdots, y_m(t)$ and we have (3.8), which shows that the system (3.1) satisfies Amerio's separation condition in $K$.

Now we assume that the system (3.1) satisfies Amerio's separation condition in $K$. First of all, we shall see that for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for any $g \in H(f)$ and any $x \in B(g, K)$, $|x(t) - y(t)| \leqq \varepsilon$ for all $t \geqq t_0$, whenever $y \in A(g, K)$, $|x(t_0) - y(t_0)| \leqq \delta(\varepsilon)$ for some $t_0 \geqq t(y)$. Suppose not. Then there exists an $\varepsilon > 0$ and sequences $g_k \in H(f)$, $x_k \in B(g_k, K)$, $y_k \in A(g_k, K)$, $t_k$ and $\tau_k$, $\tau_k > t_k$, such that

$$|x_k(t_k) - y_k(t_k)| < 1/k , \qquad t_k \geqq t(y_k) , \tag{3.9}$$

$$|x_k(\tau_k) - y_k(\tau_k)| = \varepsilon , \tag{3.10}$$

where we can assume that $\varepsilon \leqq \lambda/2$ for the constant $\lambda$ in Def. 3.1.

If we set $u_k(t) = x_k(t + \tau_k)$ and $v_k(t) = y_k(t + \tau_k)$, then $u_k(t)$ and $v_k(t)$ are solutions of

$$x' = g_k(t + \tau_k, x) \tag{3.11}$$

such that $u_k(0) = x_k(\tau_k)$ and $v_k(0) = y_k(\tau_k)$. Clearly

$$u_k(t) \in K \text{ for all } t \in R$$

and

$$v_k(t) \in K \text{ for } t \geqq t_k - \tau_k \quad (t_k - \tau_k < 0) .$$

Since $g_k(t + \tau_k, x) \in H(f)$ and $H(f)$ is compact by the uniform norm on $R \times K$, $\{g_k(t + \tau_k, x)\}$ has a subsequence, which we shall denote by $\{g_k(t + \tau_k, x)\}$ again, such that

$$g_k(t+\tau_k, x) \to h(t, x) \text{ uniformly on } R \times K \text{ as } k \to \infty ,$$

where $h \in H(f)$. We can also assume that $t_k - \tau_k$ tends to a $\tau\,(< 0)$ as $k \to \infty$, where $\tau$ may be $-\infty$.

Since $\{v_k(t)\}$ is uniformly bounded and equicontinuous on any compact interval in $(\tau, \infty)$, $v_k(t)$ can be assumed to tend to a function $\eta(t)$ defined on $(\tau, \infty)$ uniformly on any compact interval of $(\tau, \infty)$ as $k \to \infty$. Since $v_k(t)$ is a solution of (3.11), $\eta(t) \in A(h, K)$ and $\eta(t) \in K$ for $t > \tau$. By the same argument, there exists a function $\xi(t)$ such that $u_k(t)$ tends to $\xi(t)$ uniformly on any compact interval in $R$ as $k \to \infty$ and $\xi(t) \in B(h, K)$. If $\tau > -\infty$, $\eta(t) \in K$ for $t \geqq \tau$ and $\lim_{k\to\infty} v_k(t_k - \tau_k) = \eta(\tau)$. Therefore

$$\begin{aligned} |\xi(\tau) - \eta(\tau)| &= \lim_{k\to\infty} |u_k(t_k - \tau_k) - v_k(t_k - \tau_k)| \\ &= \lim_{k\to\infty} |x_k(t_k) - y_k(t_k)| = 0 . \end{aligned}$$

Thus we have a solution $\eta^* \in B(h, K)$, where

$$\eta^*(t) = \begin{cases} \eta(t) & \text{for } t \geqq \tau \\ \xi(t) & \text{for } t < \tau . \end{cases}$$

If $\tau = -\infty$, we set $\eta^*(t) = \eta(t) \in B(h, K)$. Thus we have two solutions $\eta^*(t)$, $\xi(t)$ in $B(h, K)$. But

$$\begin{aligned} |\eta^*(0) - \xi(0)| = |\eta(0) - \xi(0)| &= \lim_{k\to\infty} |v_k(0) - u_k(0)| \\ &= \lim_{k\to\infty} |x_k(\tau_k) - y_k(\tau_k)| = \varepsilon > 0 , \end{aligned}$$

which shows that $\eta^*(t)$ and $\xi(t)$ are distinct solutions in $B(h, K)$. Therefore $|\eta^*(t) - \xi(t)| \geqq \lambda$ for all $t \in R$. However $|\eta^*(0) - \xi(0)| = \varepsilon < \lambda/2$. Thus there arises a contradiction.

Now let $\delta_0$ be a positive constant such that $\delta_0 < \delta(\lambda/2)$. For this $\delta_0$, we shall show that for any $\varepsilon > 0$, there is a $T(\varepsilon) > 0$ such that every solution $x \in B(g, K)$ satisfies

$$|x(t) - y(t)| < \varepsilon \quad \text{for all } t \geqq t_0 + T(\varepsilon) ,$$

whenever $y \in A(g, K)$ and $|x(t_0) - y(t_0)| < \delta_0$ for some $t_0 \geqq t(y)$.

Suppose not. Then there exists an $\varepsilon > 0$ and sequences $g_k \in H(f)$, $x_k \in B(g_k, K)$, $y_k \in A(g_k, K)$, $t_k \geqq t(y_k)$, and $\tau_k$, $\tau_k > t_k + k$, such that

$$|x_k(t_k) - y_k(t_k)| < \delta_0 \,(< \delta(\lambda/2)) \tag{3.12}$$

and

$$|x_k(\tau_k) - y_k(\tau_k)| \geqq \varepsilon . \tag{3.13}$$

Since (3.12) implies $|x_k(t) - y_k(t)| < \lambda/2$ for all $t \geqq t_k$, we have

$$\varepsilon \leqq |x_k(\tau_k) - y_k(\tau_k)| \leqq \lambda/2 \tag{3.14}$$

by (3.13). If we set $u_k(t) = x_k(t + \tau_k)$ and $v_k(t) = y_k(t + \tau_k)$, then $u_k(t)$ and $v_k(t)$ are solutions of

$$x' = g_k(t + \tau_k, x) ,$$

$u_k(t) \in K$ for all $t \in R$ and $v_k(t) \in K$ for $t \geqq -k$. Thus we can assume that there exists an $h \in H(f)$, $\xi \in B(h, K)$ and $\eta \in B(h, K)$ such that

$$g_k(t + \tau_k, x) \longrightarrow h(t, x)$$

uniformly on $R \times K$ as $k \longrightarrow \infty$ and

$$u_k(t) \longrightarrow \xi(t) , \qquad v_k(t) \longrightarrow \eta(t)$$

uniformly on any compact interval in $R$ as $k \longrightarrow \infty$.

On the other hand, we have

$$|\xi(0) - \eta(0)| = \lim_{k\to\infty} |u_k(0) - v_k(0)| = \lim_{k\to\infty} |x_k(\tau_k) - y_k(\tau_k)| ,$$

which implies that, by (3.14)

$$\varepsilon \leqq |\xi(0) - \eta(0)| \leqq \lambda/2 . \tag{3.15}$$

Since $\xi \in B(h, K)$ and $\eta \in B(h, K)$, (3.15) contradicts the separation condition. This shows that for any $g \in H(f)$, the system (3.2) is conditionally uniformly asymptotically stable in $K$ with a common triple $(\delta_0, \delta(\cdot), T(\cdot))$. The proof is completed.

We shall show a relationship between usual stability and separation condition. When $\varphi(t)$ is a solution in $A(f, K)$, $H(f, \varphi)$ denotes the set of the pair $(g, x(t))$ such that for some sequence $\{t_k\}$, $t_k \longrightarrow \infty$ as $k \longrightarrow \infty$, $f(t + t_k, x) \longrightarrow g(t, x)$ as $k \longrightarrow \infty$ uniformly on $R \times S$ for each compact subset $S$ of $R^n$ and $\varphi(t + t_k) \longrightarrow x(t)$ as $k \longrightarrow \infty$ uniformly on any compact interval in $R$. Clearly $H(f, \varphi)$ is not empty.

DEFINITION 3.4. $H(f, \varphi)$ is said to satisfy a separation condition if there exists a positive constant $\lambda = \lambda(g)$ for each $g \in H(f)$ such that any two distinct elements $(g, x)$, $(g, y)$ in $H(f, \varphi)$ satisfy

$$\inf_{t \in R} |x(t) - y(t)| \geqq \lambda .$$

REMARK 3.2. The constant $\lambda$ can be chosen independently of $g \in H(f)$ by the same argument as in Amerio's proof.

Clearly the separation condition on $H(f, \varphi)$ is a generalization of Amerio's condition. In the proof of Amerio's theorem for the existence of almost periodic solutions, Amerio has used essentially the separation condition on $H(f, \varphi)$, and hence, Amerio's existence theorem can be proved, replacing his separation condition by the condition on $H(f, \varphi)$. This existence theorem follows also from Fink's result [4, Theorem 3]. For the existence of almost periodic solutions, Fink has considered more general separation condition than Amerio's condition, and our separation condition on $H(f, \varphi)$ is a special case of it.

THEOREM 3.2. *For the system* (3.1), *assume that there exists a solution* $\varphi(t)$ *in* $A(f, K)$ *and that for any* $(g, x) \in H(f, \varphi)$, $x$ *is uniformly asymptotically stable with a common triple* $(\delta_0, \delta(\cdot), T(\cdot))$. *Then* $H(f, \varphi)$ *satisfies the separation condition.*

The proof is similar to the proof of sufficiency in Theorem 3.1. Replace $B(g, K)$ by $G(g)$, where

$$G(g) = \{x; (g, x) \in H(f, \varphi)\} \quad \text{for each } g \in H(f) .$$

REMARK 3.3. Theorem 3.2 shows that the stability on $\varphi$ is a sufficient condition for the existence of almost periodic solutions. This is already known by Kato [5].

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