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EXISTENCE OF ALMOST PERIODIC SOLUTIONS BY LIAPUNOV FUNCTIONS

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1. Introduction. The existence of almost periodic solutions of almost periodic systems has been studied by many authors. Generally, the existence of a bounded solution does not imply the existence of almost periodic solutions [4]. To obtain almost periodic solutions, we need additional conditions, for example, separation conditions and stability conditions. Another approach is to assume the existence of a Liapunov function with some properties ([2], [5]). Relationships between separation conditions and stability conditions have been discussed by the author [3].

In this paper, by assuming the existence of some Liapunov function, we shall obtain an existence theorem for an almost periodic solution, which improves Fink and Seifert's result [2] and proves Yoshizawa's result [5] as a corollary.

We denote by R^n the Euclidean n-space and set $R = R^1$ and $R^+ = [0, \infty)$. Let $|x|$ be the Euclidean norm of $x \in R^n$.

2. Theorem and some remarks. Consider the almost periodic system

$$x' = f(t, x) \qquad (' = d/dt)\ , \tag{2.1}$$

where $x, f \in R^n$ and $f(t, x)$ is defined on $R \times D$, D open set of R^n, and is almost periodic in t uniformly for $x \in D$. The following theorem is an improvement of Fink and Seifert's result [2].

THEOREM. *Suppose that the system* (2.1) *has a solution* $\phi(t)$ *such that* $\phi(t) \in K$ *on* R^+, *where* K *is a compact subset of* D, *and assume that there exists a continuous scalar function* $V(t, x)$ *defined on* $R^+ \times D$, *which satisfies the following conditions*:

(i) $V(t, \phi(t))$ *is bounded on* R^+,

(ii) $|V(t, x) - V(t, y)| \leqq L|x - y|$ *for* $x, y \in S$, $t \in R^+$, *where* S *is any compact subset of* D *and* L *may depend on* S,

(iii) $\dot{V}(t, x) \geqq a(|x - \phi(t)|)$, *where* $a(r)$ *is continuous and positive definite and*

$$\dot{V}(t, x) = \overline{\lim_{h \to +0}} \frac{1}{h} \{V(t + h, x + hf(t, x)) - V(t, x)\}\ .$$

Then the system (2.1) *has a unique almost periodic solution in D whose module is contained in the module of $f(t, x)$.*

The proof shall be given in the next section. In order to obtain a unique almost periodic solution in K, Fink and Seifert have assumed the following conditions; in addition to the conditions in our theorem, $V(t, x)$ is defined on $R \times D$ and is continuous in t uniformly for $(t, x) \in R \times S$ for each compact subset S of D, and $V(t, \phi(t)) = 0$. Our theorem shows that we can drop these conditions and furthermore we can verify the uniqueness in D of the almost periodic solution. As will be seen from the example below, the uniqueness of the almost periodic solutions in any compact subset of D does not necessarily imply the uniqueness in D.

Consider

$$\begin{cases} x' = \left(1 - \dfrac{x - \phi(t)}{3 - \phi(t)}\right)\phi'(t) \\ y' = (x - \phi(t))^2(x - 3)^2 + y^2 \,, \end{cases}$$

where $\phi(t) = \sin t + \sin \sqrt{2}\, t$. Let $D = (-2, \infty) \times (-\infty, \infty)$. Then there are exactly two almost periodic solutions in D, that is, $\{x = \phi(t), y = 0\}$ and $\{x = 3, y = 0\}$. However, any compact subset of D contains at most one almost periodic solution $\{x = 3, y = 0\}$, because $\inf_{t \in R} \phi(t) = -2$.

In our theorem, we have to know what $\phi(t)$ is. However, there is often a case where we know only the existence of a compact solution of (2.1). For such a case, the following corollary is useful and it also improves Yoshizawa's result [5], except the result on stability.

COROLLARY. *Suppose that there exists a continuous scalar function $V(t, x, y)$ defined on $R^+ \times D \times D$ which satisfies the following conditions:*

(i) *$V(t, x, x)$ is bounded for $t \in R^+$, $x \in S$, where S is any compact subset of D,*

(ii) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \leqq L\{|x_1 - x_2| + |y_1 - y_2|\}$ *for $t \in R^+$, $x_1, x_2, y_1, y_2 \in S$, where L may depend on S,*

(iii) *$\dot{V}(t, x, y) \geqq a(|x - y|)$, where $a(r)$ is continuous and positive definite and*

$$\dot{V}(t, x, y) = \overline{\lim_{h \to +0}} \frac{1}{h}\{V(t + h, x + hf(t, x), y + hf(t, y)) - V(t, x, y)\} \,.$$

Moreover, assume that the system (2.1) *has a solution which remains in a compact subset of D for $t \geqq 0$.*

Then the system (2.1) *has a unique almost periodic solution in D whose module is contained in the module of $f(t, x)$.*

Let $\phi(t)$ be a given compact solution and consider $V(t, x, \phi(t))$ as the Liapunov function in Theorem. Then this corollary follows immediately from our theorem.

3. Proof of Theorem. The following lemma is well known (cf. [1]).

LEMMA. *Let S be a compact subset of D. For each g in the hull of f, assume that the system*

$$x' = g(t, x) \tag{3.1}$$

has one and only one solution which remains in S for all $t \in R$.

Then the system (2.1) *has an almost periodic solution whose module is contained in the module of $f(t, x)$.*

Under our assumption, we shall show that for each g in the hull of f, the system (3.1) has one and only one solution in K for all $t \in R$. Since $f(t, x)$ is almost periodic in t, there is a sequence $\{t_k\}$ such that $t_k \longrightarrow \infty$ as $k \longrightarrow \infty$ and

$$f(t + t_k, x) \longrightarrow g(t, x) \tag{3.2}$$

uniformly on $R \times K$ as $k \longrightarrow \infty$. Since $\{\phi(t + t_k)\}_{k=1}^{\infty}$ is uniformly bounded and equicontinuous on any compact interval in R, we can assume that

$$\phi(t + t_k) \longrightarrow \psi(t) \tag{3.3}$$

uniformly on any compact interval in R as $k \longrightarrow \infty$.

Then $\psi(t) \in K$ for all $t \in R$ and $\psi(t)$ is a solution of (3.1). We shall show that if system (3.1) has a solution $x(t)$ such that $x(t) \in K$ for all $t \in R$, then $x(t) = \psi(t)$ for all $t \in R$.

Let V_k be defined by

$$V_k(t) = V(t + t_k, x(t)) \quad \text{for} \quad t \geqq -t_k ,$$

and set

$$D^+ V_k(t) = \overline{\lim_{h \to +0}} \frac{1}{h}\{ V(t + t_k + h, x(t + h)) - V(t, x(t))\} .$$

Then, by condition (ii), we have

$$D^+ V_k(t) \geqq \dot{V}(t + t_k, x(t)) - A_k(t) ,$$

where $A_k(t) = L \mid g(t, x(t)) - f(t + t_k, x(t)) \mid$ and $L = L(K')$ is the constant in condition (ii) for K', K' compact neighbourhood of K. Clearly we have

$$\lim_{k \to \infty} A_k - (t) = 0 \quad \text{uniformly on} \quad R. \tag{3.4}$$

By condition (iii), we have

$$D^+ V_k(t) \geqq a(|x(t) - \phi(t + t_k)|) - A_k(t) .$$

On any interval $[b, c]$, if k is sufficiently large so that $b + t_k \geqq 0$, we obtain

$$V_k(c) - V_k(b) \geqq \int_b^c a(|x(s) - \phi(s + t_k)|)ds - \int_b^c A_k(s)ds . \tag{3.5}$$

By conditions (i) and (ii), there exists a $B > 0$ such that

$$|V_k(c) - V_k(b)| = |V(c + t_k, x(c)) - V(b + t_k, x(b))| \leqq B \quad \text{for all} \quad k .$$

Therefore we have

$$\int_b^c a(|x(s) - \phi(s + t_k)|)ds - \int_b^c A_k(s)ds \leqq B .$$

Letting $k \to \infty$, it follows from (3.3) and (3.4) that

$$\int_b^c a(|x(s) - \psi(s)|)ds \leqq B .$$

Since b and c are arbitrary, we have

$$\int_{-\infty}^{\infty} a(|x(s) - \psi(s)|)ds \leqq B ,$$

and hence, there exist sequences $\{\tau_m\}$ and $\{\sigma_m\}$ such that $\tau_m \to -\infty$, $\sigma_m \to +\infty$, as $m \to \infty$ and that $a(|x(\tau_m) - \psi(\tau_m)|) \to 0$, $a(|x(\sigma_m) - \psi(\sigma_m)|) \to 0$ as $m \to \infty$. This shows

$$|x(\tau_m) - \psi(\tau_m)| \to 0 , \quad |x(\sigma_m) - \psi(\sigma_m)| \to 0 \quad \text{as} \quad m \to \infty \tag{3.6}$$

since $a(r)$ is continuous, positive definite and $|x(\tau_m) - \psi(\tau_m)|$, $|x(\sigma_m) - \psi(\sigma_m)|$ are bounded.

In (3.5), let $b = \tau_m$ and $c = \sigma_m$. Then, if k is sufficiently large so that $\tau_m + t_k \geqq 0$, we have

$$V_k(\sigma_m) - V_k(\tau_m) \geqq \int_{\tau_m}^{\sigma_m} a(|x(s) - \phi(s + t_k)|)ds - \int_{\tau_m}^{\sigma_m} A_k(s)ds$$

and

$$\begin{aligned}
&\int_{\tau_m}^{\sigma_m} a(|x(s) - \phi(s + t_k)|)ds - \int_{\tau_m}^{\sigma_m} A_k(s)ds - V(\sigma_m + t_k, \phi(\sigma_m + t_k)) \\
&\qquad + V(\tau_m + t_k, \phi(\tau_m + t_k)) \\
&\leqq V_k(\sigma_m) - V_k(\tau_m) - V(\sigma_m + t_k, \phi(\sigma_m + t_k)) + V(\tau_m + t_k, \phi(\tau_m + t_k)) \\
&\leqq L\{|x(\sigma_m) - \phi(\sigma_m + t_k)| + |x(\tau_m) - \phi(\tau_m + t_k)|\} \\
&\leqq L\{|x(\sigma_m) - \psi(\sigma_m)| + |\psi(\sigma_m) - \phi(\sigma_m + t_k)| + |x(\tau_m) - \psi(\tau_m)| \\
&\qquad + |\psi(\tau_m) - \phi(\tau_m + t_k)|\} .
\end{aligned}$$

Hence, letting $k \to \infty$, we can see that for a fixed m,

$$
\begin{aligned}
&\int_{\tau_m}^{\sigma_m} a(|x(s) - \psi(s)|)ds - \overline{\lim_{k\to\infty}} \{V(\sigma_m + t_k, \phi(\sigma_m + t_k)) \\
&\qquad - V(\tau_m + t_k, \phi(\tau_m + t_k))\} \\
&\leqq L\{|x(\sigma_m) - \psi(\sigma_m)| + |x(\tau_m) - \psi(\tau_m)|\} .
\end{aligned} \tag{3.7}
$$

However, since $V(t, \phi(t))$ is bounded and $D^+V(t, \phi(t)) \geqq 0$, $V(t, \phi(t)) \to v_0$ as $t \to \infty$ for some constant v_0, and hence, (3.7) implies

$$\int_{\tau_m}^{\sigma_m} a(|x(s) - \psi(s)|)ds \leqq L\{|x(\sigma_m) - \psi(\sigma_m)| + |x(\tau_m) - \psi(\tau_m)|\} .$$

Letting $m \to \infty$, it follows from (3.6) that

$$\int_{-\infty}^{\infty} a(|x(s) - \psi(s)|)ds = 0 ,$$

which implies $a(|x(s) - \psi(s)|) = 0$, that is, $x(s) = \psi(s)$ for all $s \in R$.

Now we shall show the uniqueness of the almost periodic solution in D. Let $\{t_k\}$ be a sequence such that $t_k \to \infty$, $f(t + t_k, x) \to f(t, x)$ uniformly on $R \times S$, S any compact set in D, and $\phi(t + t_k) \to \psi(t)$ uniformly on any compact interval in R as $k \to \infty$. Then $\psi(t) \in K$ for all $t \in R$ and, as was seen above, $\psi(t)$ is the unique solution in K of system (2.1). Thus $\psi(t)$ is an almost periodic solution of system (2.1). Therefore it is sufficient to show that $\psi(t) = p(t)$ for any almost periodic solution $p(t)$ of (2.1) in D.

Suppose that there exists an almost periodic solution $p(t)$ of (2.1) such that $p(t) \in D$ for all $t \in R$ and $|p(t_0) - \psi(t_0)| = \varepsilon$ at some $t_0 \in R$ for some $\varepsilon > 0$. Since $p(t_0) \in D$, there exists an open set O with the compact closure $\bar{O} \subset D$ such that $p(t_0) \in O \subset \bar{O} \subset D$. Since $p(t)$ is almost periodic, there exists a sequence $\{\sigma_m\}$ such that $\sigma_m \to \infty$ as $m \to \infty$ and $p(\sigma_m) \in \bar{O}$ for all m.

Let $V_k(t) = V(t + t_k, p(t))$. Then, by the same argument as used in obtaining (3.5), we have

$$V_k(\sigma_m) - V_k(t_0) \geqq \int_{t_0}^{\sigma_m} a(|p(t) - \phi(t + t_k)|)dt - \int_{t_0}^{\sigma_m} A_k(m, t)dt , \tag{3.8}$$

where $A_k(m, t) = L_m |f(t + t_k, p(t)) - f(t, p(t))|$ and L_m may depend on a compact set K_m in D which is a neighbourhood of the compact set $\{p(t); t_0 \leqq t \leqq \sigma_m\}$. Clearly, for a fixed m,

$$\lim_{k\to\infty} A_k(m, t) = 0 \quad \text{uniformly for} \quad t \in [t_0, \sigma_m] .$$

Since $p(\sigma_m) \in \bar{O}$ and we have conditions (i), (ii), there exists a $B > 0$ such that

$$|V_k(\sigma_m) - V_k(t_0)| \leqq B \quad \text{for all} \quad m .$$

Letting $k \to \infty$ in (3.8), we have

$$\int_{t_0}^{\sigma_m} a(|p(t) - \psi(t)|)dt \leqq B ,$$

which implies

$$\int_{t_0}^{\infty} a(|p(t) - \psi(t)|)dt \leqq B . \tag{3.9}$$

Since $p(t) - \psi(t)$ is almost periodic, there exists a sequence $\{\tau_m\}$ such that

$$|p(t_0) - \psi(t_0) - p(\tau_m) + \psi(\tau_m)| < \varepsilon/3 \quad \text{for all} \quad m \tag{3.10}$$

and

$$\tau_m \to \infty \quad \text{as} \quad m \to \infty , \quad \tau_m + 2 < \tau_{m+1} . \tag{3.11}$$

The uniform continuity of $p(t) - \psi(t)$ implies the existence of a δ, $0 < \delta < 1$, such that

$$|p(t) - \psi(t) - p(\tau_m) + \psi(\tau_m)| < \varepsilon/3 \quad \text{for} \quad \tau_m - \delta < t < \tau_m + \delta . \tag{3.12}$$

From (3.10), (3.12) and $|p(t_0) - \psi(t_0)| = \varepsilon$, it follows that

$$\varepsilon/3 < |p(t) - \psi(t)| < 5\varepsilon/3 \quad \text{for} \quad \tau_m - \delta < t < \tau_m + \delta \quad \text{and all} \quad m .$$

Let

$$a_0 = \min\{a(r); \varepsilon/3 \leqq r \leqq 5\varepsilon/3\} \ (>0) .$$

Then we have

$$B \geqq \sum_{m=1}^{\infty} \int_{\tau_m-\delta}^{\tau_m+\delta} a(|p(t) - \psi(t)|)dt \geqq \sum_{m=1}^{\infty} 2\delta a_0 = \infty$$

since the intervals $(\tau_m - \delta, \tau_m + \delta)$ are disjoint by (3.11). This is a contradiction. Thus $p(t) = \psi(t)$. This completes the proof.

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