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A THEOREM ON LIMITS OF KLEINIAN GROUPS

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1. Let G be a group of conformal automorphisms of the extended complex plane $\hat{C} = C \cup \{\infty\}$. Every element of G is a Möbius transformation of the form

$$T: z \mapsto \frac{az + b}{cz + d},$$

where a, b, c and d are complex numbers with $ad - bc = 1$. This transformation T is often identified with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $PSL(2, C)$ and, in this case, $a + d$ is called the trace of T and is denoted by trace T.

If there does not exist a sequence of G which converges to the identity under the topology of $PSL(2, C)$, then G is called discrete.

A point $w \in \hat{C}$ is called a limit point of G provided that there exist a point $z \in \hat{C}$ and a sequence $\{T_i\}_{i=1}^{\infty}$ of elements of G such that $T_j \neq T_k (j \neq k)$ and such that $T_i(z) \to w$ as $i \to \infty$. If a point $w \in \hat{C}$ is not a limit point of G, it is called an ordinary point of G. Denote by $\Lambda(G)$ the set of all limit points of G and by $\Omega(G)$ the set of all ordinary points of G. If $\Omega(G)$ is not empty, then G is called a discontinuous group. If the limit set of a discontinuous group G contains more than two points, then G is called kleinian. A discontinuous group not being kleinian is said to be elementary. It is known that a kleinian group contains infinitely many loxodromic elements and the set of attracting fixed points of loxodromic elements in G is dense in $\Lambda(G)$.

An isomorphism ϕ of a kleinian group G_1 onto a kleinian group G_2 is said to be type preserving if $\phi(T)$ is parabolic if and only if T is parabolic.

Let T be a Möbius transformation of the form

$$T: z \mapsto \frac{az + b}{cz + d}, \qquad c \neq 0 .$$

Then we call two circles $I(T)$: $|z + d/c| = 1/|c|$ and $I(T^{-1})$: $|z - a/c| = 1/|c|$ the isometric circles of T and of T^{-1}, respectively. It is known that T maps the exterior of $I(T)$ onto the interior of $I(T^{-1})$. Since the radii of $I(T)$ and $I(T^{-1})$ are both equal to $1/|c|$ and since the distance of the center of $I(T)$ from that of $I(T^{-1})$ equals $|(a + d)/c|$, a necessary and sufficient condition in order that the two isometric circles $I(T)$ and $I(T^{-1})$ bound a doubly connected domain containing the point ∞ is $|\text{trace } T| = |a+d| > 2$.

The following theorem is due to Chuckrow [1].

CHUCKROW'S THEOREM. *Let* $G = \{S_1, S_2, \cdots\}$ *and* $G(n) = \{S_1(n), S_2(n), \cdots\}$ $(n = 1, 2, \cdots)$ *be kleinian groups. Assume that for every* m *there exists a Möbius transformation* Σ_m *such that* $\lim_{n\to\infty} S_m(n) = \Sigma_m$ *and denote by* Γ *the group* $\{\Sigma_1, \Sigma_2, \cdots\}$. *Assume further that all mappings* $\phi_n\colon S_m \mapsto S_m(n)$ *of* G *onto* $G(n)$ *are type preserving isomorphisms. Then the mapping* $\phi\colon S_m \mapsto \Sigma_m$ *is an isomorphism of* G *onto* Γ *and* Γ *contains no elliptic element of infinite order.*

The purpose of this paper is to supplement the above theorem in the following form.

THEOREM. *Under the same assumption of Chuckrow's theorem, the group* Γ *is discrete.*

REMARK 1. Our theorem is not valid if discontinuous groups G and $G(n)$ are elementary. The fact is easily verified from the following examples.

EXAMPLE 1. Let $G(n) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{2} + \sqrt{-1}/n \\ 0 & 1 \end{pmatrix} \right\rangle$, where $\langle T, U, \cdots \rangle$ denotes the group generated by the Möbius transformations $T, U, \cdots$. Then clearly $\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \right\rangle$ is not discrete.

EXAMPLE 2. Let $G(n) = \left\langle \begin{pmatrix} e(\theta) + \sqrt{-1}/n & 1 \\ (\sqrt{-1}/n)e(-\theta) & e(-\theta) \end{pmatrix} \right\rangle$, where θ is an irrational number and $e(\theta) = \exp(2\pi\sqrt{-1}\theta)$. Then clearly

$$\Gamma = \left\langle \begin{pmatrix} e(\theta) & 1 \\ 0 & e(-\theta) \end{pmatrix} \right\rangle,$$

which is not discrete.

REMARK 2. It is easily seen that our theorem implies the following.

MARDEN'S THEOREM. (Marden [3]). *A boundary group of the Schottky space is discrete.*

2. In this section we shall state lemmas which are concerned with discontinuous groups. The following lemma is due to Chuckrow and was used to prove Chuckrow's theorem stated above.

LEMMA 1. (Chuckrow [1]). *If* $\{\langle T_n, U_n \rangle\}_{n=1}^{\infty}$ *is a sequence of marked Schottky groups and if* U_n *converges to* U, *a Möbius transformation, then* T_n *does not converge to the identity.*

Next, we prove an elementary lemma.

LEMMA 2. *Let G be a kleinian group and let $\{T_i\}_{i=1}^{\infty}$ be a sequence of loxodromic elements in G. Then there exists a subsequence $\{T_{i_j}\}_{j=1}^{\infty}$ of $\{T_i\}_{i=1}^{\infty}$ such that all the fixed points of $\{T_{i_j}\}_{j=1}^{\infty}$ are in the complement of a domain $D \subset \hat{C}$ which contains at least a limit point of G.*

PROOF. Let D_1, D_2 and D_3 be domains in $\hat{C}$ satisfying

(i) $\bigcup_{p=1}^{3} \bar{D}_p \supset \Lambda(G)$,

(ii) $D_p \cap \Lambda(G) \neq \emptyset$, $p = 1, 2, 3$,

and

(iii) $D_p \cap D_q = \emptyset$, $p \neq q$, $p, q = 1, 2, 3$.

Here $\bar{D}_p$ is the closure of D_p.

Let (ξ_i, ξ_i') be the pair of fixed points of T_i, where ξ_i and ξ_i' are attracting and repelling fixed points of T_i, respectively.

If there is a set $\bar{D}_p$, say $\bar{D}_1$, containing infinitely many pairs $\{(\xi_{i_j}, \xi_{i_j}')\}_{j=1}^{\infty}$ of fixed points of elements $\{T_{i_j}\}_{j=1}^{\infty}$ belonging to the given sequence $\{T_i\}_{i=1}^{\infty}$, then clearly D_2 can be considered as a desired domain D.

In the other case, the property (i) implies that there is a set $\bar{D}_p$, say $\bar{D}_1$, which contains ξ_i for an infinite number of i and that there is a set $\bar{D}_q$ $(p \neq q)$, say $\bar{D}_2$, containing repelling fixed points ξ_{i_j}' of T_{i_j} for an infinite number of T_{i_j} whose attracting fixed points are contained in $\bar{D}_1$. By (ii) and (iii), we see that the domain D_3 is a desired domain D.

By the same argument as in the above proof, we can immediately show the following.

LEMMA 3. *Suppose that G is a group of Möbius transformations and has an infinite number of elements $\{T_i\}_{i=1}^{\infty}$ and at least three loxodromic elements and that fixed points of those loxodromic elements are different from each other. Then there exist a loxodromic element $L \in G$ and a subsequence $\{T_{i_j}\}_{j=1}^{\infty}$ of $\{T_i\}_{i=1}^{\infty}$ such that L does not fix any fixed point of T_{i_j} $(j = 1, 2, \cdots)$.*

PROOF. Let L_p $(p = 1, 2, 3)$ be loxodromic elements in G whose fixed points are different from each other and let D_p be a domain containing the fixed points of L_p and satisfying $\bigcup_{p=1}^{3} \bar{D}_p = \hat{C}$ and $D_p \cap D_q = \emptyset$ $(p \neq q)$. Then clearly the argument in the proof of Lemma 2 establishes our lemma.

The following lemma is well known. For the proof we refer to [2].

LEMMA 4. *Let G be a discontinuous group and let the point ∞ be an ordinary point of G. Then there are only a finite number of $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G such that $|c|$ is less than any preassigned real number c_0.*

The following lemma with Lemma 7 occupies the main part of the proof of our theorem.

LEMMA 5. *Let G be a kleinian group such that the point ∞ is an ordinary point and no element in G fixes the point ∞. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of the loxodromic elements in G. Then there exists a sequence of Schottky subgroups $\{\langle L, A_k\rangle\}_{k=1}^{\infty}$ of G such that any A_k is some T_i or is of the form $T_i T_{i'}^{-1}$, where $i' > i$ and i tends to ∞ as k tends to ∞.*

PROOF. We shall prove our lemma by classifying the situation into two cases: (i) the case where $|\text{trace } T_i| > 3$ for infinitely many T_i and (ii) the case otherwise.

In the case (i) we may assume that $|\text{trace } T_i| > 3$ for all i, so $I(T_i)$ and $I(T_i^{-1})$ are disjoint for every i. By Lemma 2 we can find a subsequence $\{T_{i_j}\}_{j=1}^{\infty}$ of $\{T_i\}_{i=1}^{\infty}$ such that all the fixed points of $\{T_{i_j}\}_{j=1}^{\infty}$ are contained in the complement of a domain D, which contains a point $w \in \Lambda(G)$.

Since the isometric circle of a loxodromic element contains the repelling fixed points of that element, Lemma 4 implies the existence of a subsequence $\{T'_k\}_{k=1}^{\infty}$ of $\{T_{i_j}\}_{j=1}^{\infty}$ and a subdomain D^* of D which contains the point $w \in \Lambda(G)$ such that $\hat{C} - D^*$ contains $I(T'_k)$ and $I(T_k'^{-1})$ $(k = 1, 2, \cdots)$ together with their interior. From $w \in \Lambda(G) \cap D^*$, we see that there exists a loxodromic element $U \in G$ such that its attracting fixed point ξ lies inside D^*. Let $V \in G$ be another loxodromic element, none of whose fixed points η and η' is ξ. For a sufficiently large integer M, fixed points $U^M(\eta)$ and $U^M(\eta')$ of the loxodromic element $U^M V U^{-M}$ are in D^*. Since the centers $(U^M V U^{-M})^{-N}(\infty)$ and $(U^M V U^{-M})^N(\infty)$ of isometric circles of $(U^M V U^{-M})^N$ and $(U^M V U^{-M})^{-N}$ tend to $U^M(\eta')$ and $U^M(\eta)$, respectively, as $N \to \infty$, and since by Lemma 4 radii of isometric circles of $(U^M V U^{-M})^{-N}$ and $(U^M V U^{-M})^N$ tend to zero as $N \to \infty$, we can find an integer N such that $I((U^M V U^{-M})^N)$ and $I((U^M V U^{-M})^{-N})$ are disjoint and are contained in D^*. Put $A_k = T'_k$ and $L = (U^M V U^{-M})^N$. Then it is immediate that the sequence of the Schottky groups $\{\langle L, A_k\rangle\}_{k=1}^{\infty}$ has the required property.

In the case (ii) we may assume that for all i

$$(1) \qquad |\text{trace } T_i| = |a_i + d_i| < 3\,, \qquad T_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

If $\overline{\lim}\,|a_i| < \infty$, we can find a subsequence $\{T_{i_j}\}_{j=1}^{\infty}$ of the given sequence $\{T_i\}_{i=1}^{\infty}$ such that the sequences $\{a_{i_j}\}_{j=1}^{\infty}$ and $\{d_{i_j}\}_{j=1}^{\infty}$ converge to complex numbers a and d, respectively, where $T_{i_j} = \begin{pmatrix} a_{i_j} & b_{i_j} \\ c_{i_j} & d_{i_j} \end{pmatrix}$. Hence we may assume that, for all j,

$$(2) \qquad |a_{i_j}| < 2|a| + 1\,, \qquad |d_{i_j}| < 2|d| + 1\,.$$

Here two cases can occur: the case where $b_{i_j} \neq 0$ for infinitely many j and the case otherwise.

If there are infinitely many T_{i_j} with $b_{i_j} \neq 0$ in $\{T_{i_j}\}_{j=1}^{\infty}$, then Lemma 4 and (2) imply the existence of a subsequence $\{T'_k\}_{k=1}^{\infty}$ of $\{T_{i_j}\}_{j=1}^{\infty}$ such that

$$(3) \qquad |b'_k| > |b'_{k+1}| > 0$$

and

$$(4) \qquad |b'_k c'_{k+1}| > 3 + 2(2|a| + 1)(2|d| + 1) + |b'_k c'_k| ,$$

where $T'_k = \begin{pmatrix} a'_k & b'_k \\ c'_k & d'_k \end{pmatrix}$. By (2), (3) and (4) we have

$$\begin{aligned} &|\operatorname{trace} T'_k T'^{-1}_{k+1}| \\ &\quad = |a'_k d'_{k+1} - b'_k c'_{k+1} - b'_{k+1} c'_k + a'_{k+1} d'_k| \\ &\quad \geqq |b'_k c'_{k+1}| - |a'_k d'_{k+1}| - |b'_{k+1} c'_k| - |a'_{k+1} d'_k| \\ &\quad \geqq 3 + 2(2|a| + 1)(2|d| + 1) + |b'_k c'_k| - 2(2|a| + 1)(2|d| + 1) - |b'_k c'_k| \\ &\quad = 3 . \end{aligned}$$

Thus the case has been reduced to the case (i) again.

In the remainder case, we may assume that b_{i_j} always vanishes. Since all loxodromic elements T_{i_j} have a common fixed point 0 and since G is discontinuous, the set $\{\xi, \xi'\}$ of fixed points of T_{i_1} is identical with of every T_{i_j} $(j > 1)$. Let $B \in PSL(2, C)$ satisfy $B(\xi) = 0$ and $B(\xi') = \infty$. We may assume that

$$BT_{i_j}B^{-1} = \begin{pmatrix} \rho_{i_j} & 0 \\ 0 & \rho_{i_j}^{-1} \end{pmatrix}, \qquad |\rho_{i_j}| > 1 .$$

Since $G^* = BGB^{-1}$ is discontinuous again, any $|\rho_{i_j}|$ must be greater than a real number $\rho > 1$. Hence we can find ring domains Δ_j such that Δ_j is a fundamental domain of the cyclic group $\langle BT_{i_j}B^{-1}\rangle$ and all Δ_j contain a ring domain Δ such that Δ contains a limit point of G^*. This last property of Δ can be easily verified from the fact that G^* is kleinian. As in the case (i) we can find two loxodromic elements BLB^{-1} and $BL^{-1}B^{-1}$ in $G^* = BGB^{-1}$ whose isometric circles are contained in Δ and are disjoint each other. Obviously Schottky subgroups $\{\langle L, T_{i_j}\rangle\}_{j=1}^{\infty}$ of G are desired.

If $\overline{\lim}_{i\to\infty} a_i = \infty$, there exists a subsequence $\{T_{i_j}\}_{j=1}^{\infty}$ of $\{T_i\}_{i=1}^{\infty}$ such that

$$(5) \qquad \lim_{j\to\infty} |a_{i_j}| = \lim_{j\to\infty} |d_{i_j}| = \infty .$$

First we assume $\lim_{j\to\infty} (d_{i_j}/c_{i_j}) = 0$. We can take a suitable subsequence $\{T'_k\}$ of $\{T_{i_j}\}_{j=1}^{\infty}$ such that

$$9\left|\frac{d'_{k+1}}{c'_{k+1}}\right| < \left|\frac{d'_k}{c'_k}\right|, \qquad T'_k = \begin{pmatrix} a'_k & b'_k \\ c'_k & d'_k \end{pmatrix}.$$

By an easy computation we have

$$\begin{aligned} \operatorname{trace} T'_k T'^{-1}_{k+1} &= a'_k d'_{k+1} - b'_k c'_{k+1} - b'_{k+1} c'_k + a'_{k+1} d'_k \\ &= a'_k d'_{k+1}\Big[1 - \Big(\frac{d'_k}{c'_k} - \frac{1}{a'_k c'_k}\Big)\frac{c'_{k+1}}{d'_{k+1}} \\ &\quad - \Big(\frac{a'_{k+1}}{c'_{k+1}} - \frac{1}{c'_{k+1} d'_{k+1}}\Big)\frac{c'_k}{a'_k} + \frac{a'_{k+1} d'_k}{a'_k d'_{k+1}}\Big]. \end{aligned}$$

From (1) and (5) we can conclude that, for a sufficiently large k,

$$\begin{aligned} &\frac{1}{2}\left|\frac{d'_k}{c'_k}\right| < \left|\frac{d'_k}{c'_k} - \frac{1}{a'_k c'_k}\right|, \\ &\left|\frac{a'_{k+1}}{c'_{k+1}} - \frac{1}{c'_{k+1} d'_{k+1}}\right| < 2\left|\frac{d'_{k+1}}{c'_{k+1}}\right|, \\ &\left|\frac{c'_k}{a'_k}\right| < 2\left|\frac{c'_k}{d'_k}\right| \quad \text{and} \quad \left|\frac{d'_k}{a'_k}\cdot\frac{a'_{k+1}}{d'_{k+1}}\right| < 2. \end{aligned}$$

By using these, we have

$$\begin{aligned} &|\operatorname{trace} T'_k T'^{-1}_{k+1}| \\ &\geqq |a'_k d'_{k+1}|\left(\frac{1}{2}\left|\frac{d'_k}{c'_k}\cdot\frac{c'_{k+1}}{d'_{k+1}}\right| - 1 - 4\left|\frac{d'_{k+1}}{c'_{k+1}}\right|\left|\frac{c'_k}{d'_k}\right| - 2\right) \\ &\geqq |a'_k d'_k|\left(\frac{1}{2}\cdot 9 - 1 - 4\cdot\frac{1}{9} - 2\right) \\ &\geqq |a'_k d'_k|. \end{aligned}$$

The condition (5) yields that $T'_k T'^{-1}_{k+1}$ is a loxodromic element in G and satisfies $|\operatorname{trace} T'_k T'^{-1}_{k+1}| > 3$ for a sufficiently large k. Thus our case can be reduced to the case (i).

When $\lim_{j\to\infty}(d_{i_j}/c_{i_j}) \neq 0$, we consider a suitable conjugate $WGW^{-1} = G'$ of G such that ∞ is also an ordinary point of G' and such that for $WT_{i_j}W^{-1} = \begin{pmatrix} a^*_{i_j} & b^*_{i_j} \\ c^*_{i_j} & d^*_{i_j} \end{pmatrix}$, it holds $\lim(d^*_{i_j}/c^*_{i_j}) = 0$. If $\overline{\lim}\,|a^*_{i_j}| = \infty$, then the above argument shows that our lemma holds for G', which establishes Lemma 5 itself. If $\overline{\lim}\,|a^*_{i_j}| < \infty$, then the proof in the case $\overline{\lim}\,|a_{i_j}| < \infty$ gives validity of Lemma 5 for G', so Lemma 5 also holds for G. Thus the proof of the lemma is complete.

3. In this section we prepare some results obtained under the assumption in our theorem. Let $G = \{S_1, S_2, \cdots\}$ and $G(n) = \{S_1(n), S_2(n), \cdots\}$ be kleinian groups. We restate the assumption of the theorem as follows: there exists a Möbius transformation Σ_m such that $\lim_{n\to\infty} S_m(n) = \Sigma_m$ for every m and there exists a type preserving isomorphism $\phi_n: S_m \mapsto S_m(n)$ of G onto $G(n)$ for every $n(m = 1, 2, \cdots)$.

Denote by Γ the group $\{\Sigma_1, \Sigma_2, \cdots\}$. First we prove the following.

LEMMA 6. *In addition to the assumption in our theorem, suppose that* $\infty \in \Omega(G)$ *and is not fixed by any element of* G. *Then* Γ *contains infinitely many loxodromic elements* $\{V_i\}_{i=1}^{\infty}$ *such that trace* V_i *is identical with trace* V_1 *for any* i *and such that* V_j *and* V_k *have no common fixed point for any* j *and* k.

PROOF. First, we shall show that Γ contains a loxodromic element V_1. Let U_1 be a loxodromic element in G. If $\phi(U_1)$ is loxodromic, we have nothing to prove. If $\phi(U_1)$ is not loxodromic, then by using Chuckrow's theorem we see $\phi(U_1)$ is parabolic. Let U_2 be a loxodromic element in G whose fixed points are different from the fixed points of U_1. Again we may assume $\phi(U_2)$ is parabolic. We observe that $\phi(U_1)$ and $\phi(U_2)$ have no common fixed point. In fact, if $\phi(U_1)$ and $\phi(U_2)$ have a common fixed point, then $\phi(U_1)$ and $\phi(U_2)$ are commutative. Hence U_1 and U_2 are commutative, which contradicts the fact that loxodromic elements U_1 and U_2 have no common fixed point. Therefore, we may assume that $\phi(U_1)$ and $\phi(U_2)$ are parabolic and of the form $\phi(U_1) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, $\lambda \neq 0$ and $\phi(U_2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c \neq 0$. Clearly we have trace $\phi(U_1)^N \phi(U_2) = a + N\lambda c + d$, which shows that for a sufficiently large integer N such that $V_1 = \phi(U_1)^N \phi(U_2)$ is a loxodromic element in Γ.

Next we shall show that Γ contains a transformation W which is not elliptic and has no common fixed point with V_1. It is of no loss of generarity to assume

$$V_1 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad |a| > 1 .$$

The fixed points of V_1 are 0 and ∞. We shall show the existence of a loxodromic element U in G such that $W = \phi(U)$ fixes neither 0 nor ∞. For the aim, suppose that for each loxodromic element U, $\phi(u)$ fixes either 0 or ∞. By our assumption we can find loxodromic elements U_1, U_2, U_3 and U_4 in G such that their fixed points are different from each other and such that $\phi(U_1)$, $\phi(U_2)$, $\phi(U_3)$ and $\phi(U_4)$ fix the point ∞, one of the fixed points of V_1. Since the centers $U_p^{-N}(\infty)$ and $U_p^N(\infty)$ $(p = 1, 2, 3, 4)$ of isometric circles of U_p^N and U_p^{-N} tend to the repelling and attracting fixed points of U_p, respectively, as $N \to \infty$, and since by Lemma 4 radii of the isometric circles of U_p^N and of U_p^{-N} tend to zero as $N \to \infty$, it is easy to see that for a sufficiently large integer N, these eight isometric circles of U_p^N and U_p^{-N} $(p = 1, 2, 3, 4)$ are mutually disjoint and bound an 8-ply connected domain containing the point ∞. Obviously $\langle U_1^N, U_2^N \rangle$ and $\langle U_3^N, U_4^N \rangle$ are Schottky subgroups of G and it is easily seen that one of the fixed points of the loxodromic element $U_1^N U_2^N U_1^{-N} U_2^{-N}$ is in the isometric circle of U_1^N and the other is in the isometric circle of U_2^N. For the loxodromic element $U_3^N U_4^N U_3^{-N} U_4^{-N}$, the situation is quite similar. Hence two loxodromic elements $U_1^N U_2^N U_1^{-N} U_2^{-N}$ and $U_3^N U_4^N U_3^{-N} U_4^{-N}$ have no common fixed point and they are not commutative. Therefore $\phi(U_1^N U_2^N U_1^{-N} U_2^{-N})$ and $\phi(U_3^N U_4^N U_3^{-N} U_4^{-N})$ must not be commutative. On the other hand, since $\phi(U_p^N)$ $(p = 1, 2, 3, 4)$ fix the point ∞, we can write as

$$\phi(U_1^N) = \begin{pmatrix} a_1 & b_1 \\ 0 & a_1^{-1} \end{pmatrix} \quad \text{and} \quad \phi(U_2^N) = \begin{pmatrix} a_2 & b_2 \\ 0 & a_2^{-1} \end{pmatrix}.$$

It is easy to see that

$$\begin{aligned} &\phi(U_1^N)\phi(U_2^N)\phi(U_1^{-N})\phi(U_2^{-N}) \\ &\quad = \begin{pmatrix} 1 & -a_2b_2 - a_1a_2^2b_1 + a_1^2a_2b_2 + a_1b_2 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence $\phi(U_1^N U_2^N U_1^{-N} U_2^{-N}) = \phi(U_1^N)\phi(U_2^N)\phi(U_1^{-N})\phi(U_2^{-N})$ is parabolic and fixes the point ∞. For the element $\phi(U_3^N U_4^N U_3^{-N} U_4^{-N})$, we have the same property. Therefore, they are commutative, which is absurd. Thus there exists a loxodromic element $U \in G$ such that $W = \phi(U) \in \Gamma$ has no fixed point common with V_1.

Put $V_{i+1} = W^i V_1 W^{-i}$, $i = 1, 2, \cdots$. Then obviously the set $\{V_i\}_{i=1}^\infty$ of loxodromic elements is the desired.

LEMMA 7. *In addition to the assumption in our theorem suppose that* $\infty \in \Omega(G)$ *and is not fixed by any element of* G. *If* Γ *is not discrete, then there exists a sequence* $\{V_k\}_{k=1}^\infty$ *of loxodromic elements in* Γ *such that* $\{V_k\}_{k=1}^\infty$ *converges to the identity.*

PROOF. Since Γ is not discrete, we can find a sequence $\{T_i\}_{i=1}^\infty$ in Γ which converges to the identity. If $\{T_i\}_{i=1}^\infty$ contains an infinite number of loxodromic elements, then we have nothing to prove more. So we may assume that $\{T_i\}_{i=1}^\infty$ contains no loxodromic elements. There are two cases: (i) the case when $\{T_i\}_{i=1}^\infty$ contains infinitely many elliptic elements and (ii) the case when $\{T_i\}_{i=1}^\infty$ contains at most a finite number of elliptic elements.

First we consider the case (i). By Lemma 6 there exist loxodromic elements L_p $(p = 1, 2, 3)$ which have no common fixed point. Hence Lemma

3 implies that there exist a loxodromic element L in Γ and a subsequence $\{T_{i_j}\}_{j=1}^{\infty}$ of $\{T_i\}_{i=1}^{\infty}$ such that L does not fix any fixed point T_{i_j} $(j = 1, 2, \cdots)$. We normalize T_{i_j} into the form

$$W_j T_{i_j} W_j^{-1} = \begin{pmatrix} e(\theta_{i_j}) & 0 \\ 0 & e(-\theta_{i_j}) \end{pmatrix}, \qquad e(\theta_{i_j}) \neq \pm 1 ,$$

where W_j is in $PSL(2, C)$, not necessary in Γ, and $e(\theta) = \exp(2\pi\sqrt{-1}\theta)$ and put

$$W_j L W_j^{-1} = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \qquad b_j c_j \neq 0 .$$

Then we can see that trace $X_j = 2 + 2b_j c_j(1 - \cos 2\theta_{i_j})$ for

$X_j = W_j T_{i_j} L T_{i_j}^{-1} L^{-1} W_j^{-1}$ and trace $\hat{X}_j = 2 + 2b_j c_j(a_j + d_j)^2(1 - \cos 2\theta_{i_j})$

for $\hat{X}_j = W_j T_{i_j} L^2 T_{i_j}^{-1} L^{-2} W_j^{-1}$.

Since both $\{W_j^{-1} X_j W_j\}_{j=1}^{\infty}$ and $\{W_j^{-1}\hat{X}_j W_j\}_{j=1}^{\infty}$ converge to the identity, it is sufficient to show that $W_j^{-1} X_j W_j$ or $W_j^{-1}\hat{X}_j W_j$ is loxodromic for every j. For the purpose we have only to prove that X_j or $\hat{X}_j$ is loxodromic for every j. If trace L is neither real nor pure imaginary, then trace $L =$ trace $W_j L W_j^{-1} = a_j + d_j$ is neither real nor pure imaginary, and at least one of trace X_j or trace $\hat{X}_j$ is not real, because $b_j c_j(1 - \cos 2_{i_j}) \neq 0$. If trace L is pure imaginary, we see easily that trace L^2 is real. Therefore as remainder we consider the case, where trace L is real. Then $W_j L W_j^{-1}$ is hyperbolic. If $W_j L W_j^{-1}$ transforms the disk $\{z; |z| \leqq \rho\}$ onto itself, then $W_j L W_j^{-1}$ is of the form

$$\begin{pmatrix} a_j & \rho b_j \\ \rho^{-1}\overline{b_j} & \overline{a_j} \end{pmatrix}, \qquad \rho b_j \neq 0 .$$

Hence trace $X_j = 2 + 2|b_j|^2(1 - \cos 2\theta_{i_j}) > 2$ and X_j is loxodromic. If for any $\rho > 0$ the disk $\{z; |z| \leqq \rho\}$ is not invariant under $W_j L W_j^{-1}$, then two elements L and T_{i_j} have no common invariant disk. Hence we may assume that L makes invariant the upper half plane and is of the form

$$L = \begin{pmatrix} 2 & \beta \\ 0 & 2^{-1} \end{pmatrix}, \qquad \beta > 0 ,$$

and that T_{i_j} does not make invariant the upper half plane and is of the form

$$T_{i_j} = \begin{pmatrix} a_{i_j} & b_{i_j} \\ c_{i_j} & d_{i_j} \end{pmatrix}, \qquad a_{i_j} d_{i_j} - b_{i_j} c_{i_j} = 1 ,$$

where at least one of a_{i_j}, b_{i_j}, c_{i_j} and d_{i_j} is not real and $c_{i_j} \neq 0$. Obviously trace $L^M T_{i_j}$ is equal to $2^M a_{i_j} + (2^{M-1} + 2^{M-3} + \cdots + 2^{-M+1})\beta c_{i_j} + 2^{-M} d_{i_j}$ and is not real for a sufficiently large integer M. In fact if trace $L^M T_{i_j}$ is real for any integer M, then a_{i_j}, c_{i_j} and d_{i_j} are clearly real, and $b_{i_j} = (a_{i_j} d_{i_j} - 1)/c_{i_j}$ is also real. This contradicts the assumption that at least one of a_{i_j}, b_{i_j}, c_{i_j} and d_{i_j} is not real. Further, if trace $L^M T_{i_j}$ is purely imaginary for infinitely many integers M and for any j, then a_{i_j}, c_{i_j} and d_{i_j} must be pure imaginary, for any j, which contradicts the fact that T_{i_j} tends to the identity as $j \to \infty$. Thus we have shown that Γ contains a loxodromic element $L^* = L^M T_{i_j}$ whose trace is neither real nor pure imaginary. Hence by Lemma 3 and Lemma 6 we see the existence of a subsequence $\{T'_k\}_{k=1}$ of $\{T_{i_j}\}_{j=1}$ and a loxodromic element L^{**} such that L^{**} does not fix any fixed point of an arbitrary T'_k and trace $L^{**} =$ trace L^*. Therefore, this case can be reduced to the previous case.

In the case (ii), we may assume that each T_i is parabolic. By the same way as in the case (i), we can find a loxodromic element L and a subsequence $\{T_{i_j}\}_{j=1}$ of $\{T_i\}_{i=1}$ in such that L does not fix the fixed points of any T_{i_j}. Since the sequence $\{T_i\}_{=1}$ converges to the identity, the sequence $\{T_{i_j} L T_{i_j}^{-1} L^{-1}\}$ also converges to the identity, so our final task is to show that $T_{i_j} L T_{i_j}^{-1} L^{-1}$ is loxodromic for each j. For the purpose, we normalize T_{i_j} into

$$W_j T_{i_j} W_j^{-1} = \begin{pmatrix} 1 & \lambda_j \\ 0 & 1 \end{pmatrix}, \qquad \lambda_j \neq 0\,,$$

where W_j is in $PSL(2, C)$. It is easily seen that $W_j(L T_{i_j} L^{-1}) W_j^{-1}$ is parabolic and does not fix the point ∞. Hence

$$W_j(L T_{i_j} L^{-1}) W_j^{-1} = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}, \qquad \gamma_j \neq 0$$

and, therefore, immediately we have

$$\begin{aligned} \text{trace } T_{i_j} L T_{i_j}^{-1} L^{-1} &= \text{trace } W_j(T_{i_j} L T_{i_j}^{-1} L^{-1}) W_j^{-1} \\ &= 2 + \gamma_j \lambda_j\,, \end{aligned}$$

which shows that $T_{i_j} L T_{i_j}^{-1} L^{-1}$ is not parabolic. If the sequence $\{T_{i_j} L T_{i_j}^{-1} L^{-1}\}_{j=1}^{\infty}$ contains an infinitely many elliptic elements, our case can be reduced to the case (i). If $\{T_{i_j} L T_{i_j}^{-1} L^{-1}\}_{j=1}^{\infty}$ contains at most a finite number of elliptic elements, this sequence contains infinitely many loxodromic elements. Thus our lemma is proved completely.

4. Now we can give the proof of our theorem.

First we note that we may restrict ourselves to the case where the set $\Omega(G)$ contains the point ∞ and any element of G does not fix ∞. Assume that Γ is not discrete. Then Lemma 7 implies the existence of a sequence $\{V_i\}_{i=1}^{\infty}$ in Γ such that every V_i is loxodromic and $\{V_i\}_{i=1}^{\infty}$ converges to the identity. Put $\phi^{-1}(V_i) = T_i$. Then for a sufficiently large n, $\phi_n(T_i)$ is loxodromic and hence T_i is also loxodromic. By Lemma 5 we can find a sequence of Schottky subgroups $\{\langle L, A_k\rangle\}_{k=1}^{\infty}$ of G such that A_k is some T_i or is of the form $T_i T_{i'}^{-1}$ where $i' > i$.

First we deal with the second case, that is, the case where A_k is of the form $T_i T_{i'}^{-1}$. For each k it holds that $\lim_{n\to\infty} \phi_n(A_k) = \lim_{n\to\infty} \phi_n(T_i T_{i'}^{-1}) = V_i V_{i'}^{-1}$. Consequently, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} \phi_{n_k}(A_k) = id$. On the other hand $\langle L, A_k\rangle$ is a free and purely loxodromic group. Since ϕ_{n_k} is a type preserving isomorphism of G onto $G(n_k)$, $\langle\phi_{n_k}(L), \phi_{n_k}(A_k)\rangle$ is also a free and purely loxodromic group. Moreover, since $\langle\phi_{n_k}(L), \phi_{n_k}(A_k)\rangle$ is a subgroup of a discontinuous group $G(n_k)$, $\langle\phi_{n_k}(L), \phi_{n_k}(A_k)\rangle$ is also a discontinuous group. By a theorem of Maskit [4], $\langle\phi_{n_k}(L), \phi_{n_k}(A_k)\rangle$ is a Schottky group, which contradicts Lemma 1 due to Chuckrow.

In the remainder case where A_k is some T_i, we arrive at the contradiction by the same reasoning as above. Thus we complete the proof of our theorem.

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