Tôhoku Math. Journ.
32 (1980), 235-254.

EXISTENCE OF PERIODIC SOLUTIONS AT RESONANCE FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

MAX ASHKENAZI AND SHUI-NEE CHOW*

(Received June 6, 1979, revised October 18, 1979)

0. Introduction. The purpose of this paper is to study bifurcation of periodic orbits from the equilibrium for differential equation with time delays. One of the most important results for such problems is the Hopf bifurcation theorem. (See, for example, Chaffee [2], Hale [6], Chow and Mallet-Paret [4].) We are interested in similar problems in the case that there are time periodic perturbations. Such problems have been discussed in Perello [9], Hale [8, Chapter 9], Ashkenazi [1]. Similar problems have been encountered in the study of epidemic models [11]. Numerical studies indicate that instability may occur even if time periodic perturbation is small. On the other hand, if there are only autonomous perturbations, then a stable periodic solution will occur. We will give a partial answer to these phenomena by showing how the small parameter in Hopf's bifurcation theorem interacts with the periodic perturbation. Our main result (Theorem 4.1) shows how one can determine the regions in which one of the parameters is more dominant. We do not give a stability analysis for the periodic orbits bifurcating from the equilibrium.

Our approach to the above is in the spirit of [3]. In fact, Hale [6] called this the restricted unfolding approach. Here, we begin with a specific parametrized family of bifurcation equations, (a two parameter family of equations in this paper). Even though it may be possible to use theorems such as Malgrange-Weierstrass Transs preparation theorem to reduce the equations to a normal form, this may not be the best way for the problem. In our case, we have a two-parameter family of equations on $\boldsymbol{R}^d$ (Euclidean d-dimensional space). The normal form may envolve a large number of parameters which may be difficult to be identified with the original parameters. Thus, we use techniques such as scaling and the implicit function theorem to obtain quite precise information about the problem. The disadvantage in this approach

* Partially supported by NSF Grant MCS 76-06739

(restricted unfolding) is that there is no general approach.

In order to use such techniques, we will assume that the bifurcation equation satisfies certain generic or typical conditions. It will be clear that these conditions will be satisfied generically in the sense of Sard's theorem. In fact, one could make such statements by using theorems such as the Transversality theorem.

1. Preliminaries. Let $\boldsymbol{R}^n$ be the Euclidean n-space and $\tau > 0$ be finite. Let $C = C([-\tau, 0], \boldsymbol{R}^n)$ denote the Banach space of all continuous functions $\varphi: [-\tau, 0] \to \boldsymbol{R}^n$ with the usual sup norm

$$|\varphi| = \sup\{|\varphi(\theta)|: -\tau \leqq \theta \leqq 0\} .$$

Let $b > a$. If $x: [a - \tau, b) \to \boldsymbol{R}^n$ is continuous, then we let x_t, $a \leqq t < b$, denote the element of C defined by

$$x_t(\theta) = x(t + \theta) , \qquad -\tau \leqq \theta \leqq 0 .$$

The following may be found in Hale [7].

Consider the linear autonomous functional differential equation

$$x'(t) = L(x_t) , \qquad t \in \boldsymbol{R} \tag{1.1}$$

where $L: C \to \boldsymbol{R}^n$ is linear and continuous and "$'$" denotes d/dt. For each $\varphi \in C$, it is known that there exists a unique solution $x(\varphi)$ of (1.1) which satisfies the initial condition $x_0(\varphi) = \varphi$ and $x(\varphi)(t)$ is defined for $t \in [-\tau, \infty)$. This allows us to define for any $t \in [0, \infty)$ a bounded linear operator $T(t): C \to C$ by $T(t)\varphi = x_t(\varphi)$, $\varphi \in C$. Moreover, $\{T(t): t \geqq 0\}$ is a strongly continuous semigroup of operators and for $t \geqq \tau$, $T(t)$ is completely continuous.

By the Riesz Representation Theorem there exists a matrix valued function $\eta(\theta): [-\tau, 0] \to \boldsymbol{R}^{(n,n)}$, (the $n \times n$ matrices), whose elements are of bounded variation, such that

$$L(\varphi) = \int_{-\tau}^{0} [d\eta(\theta)]\eta(\theta) , \qquad \varphi \in C .$$

Then the characteristic equation of (1.1) is given by

$$\det\left(\lambda I_n - \int_{-\tau}^{0} [d\eta(\theta)]e^{\lambda\theta}\right) = 0 \tag{1.2}$$

where I_n denotes the $n \times n$ identity matrix. The solutions λ of (1.2) are called the characteristic roots or eigenvalues of (1.1). Each right half space of the complex plane contains at most a finite number of roots of Equation (1.2) and each of them has a finite multiplicity.

Let Λ be a finite set of eigenvalues of (1.1) and d be the sum of their multiplicities. Then it is possible to associate with Λ a unique d dimensional subspace $P = P_\Lambda$ of C, called the generalized eigenspace of (1.1) associated with Λ, a matrix valued function $\Phi = \Phi_\Lambda \colon [-\tau, 0] \to \boldsymbol{R}^{(n,d)}$ whose columns form a basis of P, and a $d \times d$ matrix $B = B_\Lambda$ the eigenvalues of which are exactly Λ, such that $T(t)\Phi = \Phi e^{Bt}$, $t \geqq 0$.

For any column vector $a \in \boldsymbol{R}^d$, the solution of (1.1) with initial condition $\varphi = \Phi a$ is $x_t(\varphi) = T(t)\Phi a = \Phi e^{Bt} a$. Furthermore, there exists a closed subspace $Q = Q_\Lambda$ of C such that

$$C = P \oplus Q \tag{1.3}$$

and both P and Q are invariant under $T(t)$, $t \geqq 0$.

We shall need an explicit characterization of the projection onto Q defined by the decomposition (1.3), which can be obtained by using the formal adjoint equation to (1.1)

$$y'(t) = -\int_{-\tau}^{0} y(t-\theta)d\eta(\theta) \tag{1.4}$$

where $y(t)$ is row n-vector. For any $\psi \in C^* \overset{\text{def}}{\equiv} C([0, \tau], \boldsymbol{R}^{n*})$, where $\boldsymbol{R}^{n*}$ are the row n-vectors, there exists a unique solution $y(\psi)$ of Equation (1.4) with initial condition ψ at $t = 0$ and which is defined on the interval $(-\infty, \tau]$. If we let Λ as before, there is a d-dimensional subspace $P^* = P_\Lambda^*$ of C^*, which is invariant with respect to the flow defined by (1.4), called the generalized eigenspace of (1.4) associated with Λ. Let $\Psi = \Psi_\Lambda \colon [0, \tau] \to \boldsymbol{R}^{(d,n)}$ be a function the rows of which form a basis of P^*. We introduce the bilinear form $(\ ,\ ) \colon C^* \times C \to \boldsymbol{R}$ defined by

$$(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} \psi(\xi-\theta)[d\eta(\theta)]\varphi(\xi)d\xi \qquad \psi \in C^*\,, \quad \varphi \in C\,.$$

Then (Ψ, Φ) is a nonsingular $d \times d$ matrix and hence by changing basis if necessary we can assume that $(\Psi, \Phi) = I_d$, the $d \times d$ identity matrix. The desired characterization of the decomposition (1.3) is given by the following:

THEOREM 1.1. *Let Λ be a finite set of eigenvalues of* (1.1), *Φ and Ψ bases of $P = P_\Lambda$ and $Q = Q_\Lambda$ respectively such that $(\Psi, \Phi) = I_d$. Then every $\varphi \in C$ has a unique decomposition*

$$\varphi = \varphi^P + \varphi^Q\,, \qquad \varphi^P = \Phi(\Psi, \varphi) \in P\,, \qquad \varphi^Q = \varphi - \varphi^P \in Q\,. \tag{1.5}$$

When this decomposition is used we say that C is decomposed by Λ.

Consider now the non homogeneous functional differential equation

$$x'(t) = L(x_t) + f(t) \tag{1.6}$$

where L is the same as before and $f\colon [0, \infty) \to \boldsymbol{R}^n$ is continuous. We denote by $x(\varphi, f)$ the solution of (1.6) with initial condition $x_0(\varphi, f) = \varphi$, and by $X(t)$ the $n \times n$ matrix valued function defined on $[-\tau, \infty)$ which is the solution of (1.1) for $t \geqq 0$ and satisfies the initial condition X_0 where

$$X_0(\theta) = \begin{cases} 0_{n\times n} & -\tau \leqq \theta < 0 \\ I_n & \theta = 0 \,. \end{cases}$$

With this notation the variation of parameters formula for (1.6) is

$$x_t(\varphi, f) = T(t)\varphi + \int_0^t T(t-s)X_0 f(s)ds \qquad t \geqq 0 \,. \tag{1.7}$$

Let E^P and E^Q denote the continuous projections of C onto P and Q respectively, defined by the decomposition (1.5). Then

$$x_t^P(\varphi, f) \equiv E^P x_t(\varphi, f) = T(t)\varphi^P + \int_0^t T(t-s)X_0^P f(s)ds$$

$$x_t^Q(\varphi, s) \equiv E^Q x_t(\varphi, f) = T(t)\varphi^Q + \int_0^t T(t-s)X_0^Q f(s)ds$$

where $X_0^P = \Phi(\Psi, X_0) = \Phi\Psi(0)$, $X_0^Q = X_0 - X_0^P$.

2. The bifurcation equation. Consider the functional differential equation

$$x'(t) = (1+\alpha)L(x_t) + \varepsilon f(t, x_t) + g(t, x_t) \tag{2.1}$$

where α and ε are assumed to be small parameters and the functions L, f and g satisfy the following conditions:

(H1) $L\colon C \to \boldsymbol{R}^n$ is linear and continuous.

(H2) The set Λ of eigenvalues of the linear problem

$$x'(t) = L(x_t) \tag{2.2}$$

of the form $2\pi in/\omega$, n integer, $\omega > 0$ fixed, is nonempty and are all simple.

(H3) f and g are continuously differentiable map from $\boldsymbol{R} \times C \to \boldsymbol{R}^n$.

(H4) f and g map bounded sets of $\boldsymbol{R} \times C$ into bounded sets of $\boldsymbol{R}^n$.

(H5) $f(t, \varphi)$ and $g(t, \varphi)$ are ω-periodic in t.

(H6) $g(t, \varphi) = O(|\varphi|^2)$ uniformly in t.

Let d be the number of elements of Λ. It is then known that Equation (2.2) has a d dimensional subspace of ω-periodic solutions of the form

$$x(t) = \Phi(0)e^{Bt}a \qquad a \in \boldsymbol{R}^d$$

where Φ and B are as Φ_Λ and B_Λ introduced in §1. Our purpose is to determine the ω-periodic solutions of (2.1) for non zero values of the parameters, which are 'close' to the zero solution when $\alpha = \varepsilon = 0$.

Our assumptions insure the existence and uniqueness of a solution $x = x(\varphi, \alpha, \varepsilon)$ of (2.1) which satisfies the initial condition $x_0 = \varphi$, $\varphi \in C$ and that such a solution is continuously differentiable in $(\varphi, \alpha, \varepsilon)$. See Hale [8]. We can state therefore

LEMMA 2.1. *Assume that conditions* (H1), (H3), (H4), (H6) *are satisfied. Then there exist* $\tilde{r} > 0$, $\tilde{\alpha} > 0$ *and* $\tilde{\varepsilon} > 0$ *such that if* $|\varphi| < \tilde{r}$, $|\alpha| < \tilde{\alpha}$ *and* $|\varepsilon| < \tilde{\varepsilon}$, *then the solution* $x_t(\varphi, \alpha, \varepsilon)$ *of* (2.1) *is defined at least for* t *in the interval* $[0, \omega]$.

Since the Equation (2.1) is ω-periodic in t, the existence of an ω-periodic solution of (2.1) is equivalent to the periodicity condition

$$x_\omega(\varphi, \alpha, \varepsilon) = \varphi . \tag{2.3}$$

The variation of parameters formula (1.7) for Equation (2.1) is

$$x_t = T(t)\varphi + \int_0^t T(t-s)X_0\{\alpha L(x_s) + \varepsilon f(s, x_s) + g(s, x_s)\}ds \tag{2.4}$$

where $x_t \equiv x_t(\varphi, \alpha, \varepsilon)$ and $T(t)$ is the solution operator for the linear equation (2.2). Combining (2.3) and (2.4) the periodicity condition becomes:

$$\begin{aligned}\{T(\omega) - I\}\varphi + \alpha\int_0^\omega T(\omega - s)X_0L(x_s)ds + \varepsilon\int_0^\omega T(\omega - s)X_0f(s, x_s)ds \\ + \int_0^\omega T(\omega - s)X_0g(s, x_s)ds = 0\end{aligned} \tag{2.5}$$

where $I: C \to C$ is the identity operator.

We view Equation (2.5) as a nonlinear equation in the Banach space C and we will carry out a Lyapunov-Schmidt reduction to derive a finite dimensional bifurcation equation.

We assume that C is decomposed by Λ, i.e.: $C = P \oplus Q$ where $P = P_\Lambda$ and $Q = Q_\Lambda$ are as in §1. From assumptions (H1), (H2) it follows that

$$P = N(T(\omega) - I), \qquad \dim P = d$$

where $N(T(\omega) - I)$ denotes the null space of $T(\omega) - I$. For a proof of this fact see Hale [7, Lemma 22.1, p. 112]. Therefore any $\varphi \in C$ can uniquely be represented as:

$$\varphi = \Phi a + \varphi^Q \qquad a \in \boldsymbol{R}^d, \qquad \varphi^Q \in Q .$$

Taking the projection of Equation (2.5) onto P and Q respectively we now obtain:

$$\alpha\int_0^\omega T(\omega-s)X_0^P L(x_s)ds+\varepsilon\int_0^\omega T(\omega-s)X_0^P f(s,x_s)ds$$
$$+\int_0^\omega T(\omega-s)X_0^P g(s,x_s)ds=0$$

$$\{T(\omega)-I\}\varphi^Q+\alpha\int_0^\omega T(\omega-s)X_0^Q L(x_s)ds+\varepsilon\int_0^\omega T(\omega-s)X_0^Q f(s,x_s)ds$$
$$+\int_0^\omega T(\omega-s)X_0^Q g(s,x_s)ds=0$$

where

$$x_s=x_s(\Phi a+\varphi^Q,\alpha,\varepsilon)\,.$$

Using the identity

$$T(t)X_0^P=\Phi e^{Bt}\Psi(0)\qquad t\geqq 0$$

and the fact that the restriction of $T(\omega)-I$ to Q is invertible ($\{T(\omega)-I\}|_Q$ is 1-1 and an iterate of $T(\omega)$ is compact), the above two equations yield:

$$\alpha\int_0^\omega e^{-Bs}\Psi(0)L(x_s)ds+\varepsilon\int_0^\omega e^{-Bs}\Psi(0)f(s,x_s)ds$$
$$+\int_0^\omega e^{-Bs}\Psi(0)g(s,x_s)ds=0 \tag{2.6}$$

$$\varphi^Q+K\int_0^\omega T(\omega-s)X_0^Q\{\alpha L(x_s)+\varepsilon f(s,x_s)+g(s,x_s)\}ds=0 \tag{2.7}$$

where

$$K=(\{T(\omega)-I\}|_Q)^{-1}\,.$$

Let $r_a>0$, $r_q>0$ be so that $r_a+r_q\leqq\tilde r$ and define

$B_{r_a}=\{a\in\boldsymbol{R}^d\colon|\Phi a|<r_a\}$, $\quad B_{r_q}=\{\varphi\in Q\colon|\varphi|<r_q\}$, $\quad I_\gamma=(-\gamma,\gamma)$ for $\gamma>0$.

Then by Lemma 2.1 we can view the left hand side of Equation (2.7) as a map:

$$F(a,\varphi^Q,\alpha,\varepsilon)\colon B_{r_a}\times B_{r_q}\times I_{\tilde\alpha}\times I_{\tilde\varepsilon}\to Q \tag{2.8}$$

so that (2.7) reads,

$$F(a,\varphi^Q,\alpha,\varepsilon)=0\,. \tag{2.9}$$

The mapping F satisfies:

$$F(0,0,0,0)=0\,,\qquad(\partial F/\partial\varphi^Q)(0,0,0,0)=\text{identity on } Q\,.$$

Therefore the Implicit Function Theorem applies and we obtain the following

LEMMA 2.2. *There exist* $\bar{r}_a > 0$, $\bar{r}_q > 0$, $\bar{\alpha} > 0$, $\bar{\varepsilon} > 0$ *and a unique continuously differentiable function*

$$\mathscr{H}(a, \alpha, \varepsilon): B_{\bar{r}_a} \times I_{\bar{\alpha}} \times I_{\varepsilon} \to B_{\bar{r}_q}$$

such that

$$\varphi^Q = \mathscr{H}(a, \alpha, \varepsilon) \tag{2.10}$$

is a solution of (2.9) *for all* $(a, \alpha, \varepsilon) \in B_{\bar{r}_a} \times I_{\bar{\alpha}} \times I_{\bar{\varepsilon}}$.

It is easily seen that $\mathscr{H}(0, \alpha, 0) = 0$ for all $\alpha \in I_{\bar{\alpha}}$.

By Lemma 2.2 we may substitute (2.10) in Equation (2.6), i.e., we set $x_s = x_s(\Phi a + \mathscr{H}(a, \alpha, \varepsilon), \alpha, \varepsilon)$. Then (2.6) becomes:

$$\alpha F_1(a, \alpha, \varepsilon) + \varepsilon F_2(a, \alpha, \varepsilon) + F_3(a, \alpha, \varepsilon) = 0 \tag{2.11}$$

where

$$F_1(a, \alpha, \varepsilon) = \int_0^{\omega} e^{-Bs}\Psi(0)L(x_s(\Phi a + \mathscr{H}(a, \alpha, \varepsilon), \alpha, \varepsilon))ds \tag{2.12}$$

$$F_2(a, \alpha, \varepsilon) = \int_0^{\omega} e^{-Bs}\Psi(0)f(s, x_s(\Phi a + \mathscr{H}(a, \alpha, \varepsilon), \alpha, \varepsilon))ds \tag{2.13}$$

$$F_3(a, \alpha, \varepsilon) = \int_0^{\omega} e^{-Bs}\Psi(0)g(s, x_s(\Phi a + \mathscr{H}(a, \alpha, \varepsilon), \alpha, \varepsilon))ds \tag{2.14}$$

and $F_i(a, \alpha, \varepsilon): B_{\bar{r}_a} \times I_{\bar{\alpha}} \times I_{\bar{\varepsilon}} \to \boldsymbol{R}^d$ $i = 1, 2, 3$.

(2.11) is a finite dimensional equation and is known in the literature as the determining equation or the bifurcation equation.

The above discussion can be summarized in the following

THEOREM 2.1. *Suppose that conditions* (H1)-(H6) *are satisfied. Then there exist* $\bar{r}_a > 0$, $\bar{r}_q > 0$, $\bar{\alpha} > 0$, $\bar{\varepsilon} > 0$ *such that if* $|\Phi a| < \bar{r}_a$, $|\alpha| < \bar{\alpha}$, $|\varepsilon| < \bar{\varepsilon}$ *and* (a, α, ε) *satisfies* (2.11), *then* $x(\Phi a + \mathscr{H}(a, \alpha, \varepsilon), \alpha, \varepsilon)$ *is an* ω-*periodic solution of* (2.1). *Conversely, any* ω-*periodic solution* $x(\Phi a + \varphi^Q, \alpha, \varepsilon)$ *of* (2.1) *with* $|\Phi a| < \bar{r}_a$, $|\varphi^Q| < \bar{r}_q$, $|\alpha| < \bar{\alpha}$ *and* $|\varepsilon| < \bar{\varepsilon}$ *is of the above form, i.e.,* $\varphi^Q = \mathscr{H}(a, \alpha, \varepsilon)$ *and* (a, α, ε) *satisfies* (2.11).

We show below that in the case $g(t, \varphi) = 0$, the bifurcation equation (2.11) is defined for a in an arbitrarily large ball, provided that the parameters α and ε are sufficiently small. Thus we consider the equation

$$x'(t) = (1 + \alpha)L(x_t) + \varepsilon f(t, x_t)\ . \tag{2.1a}$$

For this case $\tilde{r}$ of Lemma 2.1 can be chosen to be arbitrarily large and hence for any $r_a > 0$, $r_q > 0$, there exist $\tilde{\alpha} > 0$ and $\tilde{\varepsilon} > 0$ such that the mapping (2.8) is defined and it satisfies

$$F(a, 0, 0, 0) = 0 , \quad \frac{\partial F}{\partial \varphi^Q}(a, 0, 0, 0) = I|_Q \quad \text{for any} \quad a \in B_{r_a} .$$

An easy argument involving the Implicit Function Theorem shows that Lemma 2.2 can be modified to obtain

LEMMA 2.3. *For all $r_a > 0$, there exist $\bar{\alpha} > 0$, $\bar{\varepsilon} > 0$, a neighborhood V_q of 0 in Q and a unique continuously differentiable function*

$$\mathscr{H}(a, \alpha, \varepsilon): B_{r_a} \times I_{\bar{\alpha}} \times I_{\bar{\varepsilon}} \to V_q$$

such that $\varphi^Q = \mathscr{H}(a, \alpha, \varepsilon)$ is a solution of Equation (2.9) *for all* $(a, \alpha, \varepsilon) \in B_{r_a} \times I_{\bar{\alpha}} \times I_{\bar{\varepsilon}}$.

Clearly $\mathscr{H}(a, 0, 0) = 0$ for all $a \in B_{r_a}$.

As previously, Lemma 2.3 is used to obtain the bifurcation equation

$$\alpha F_1(a, \alpha, \varepsilon) + \varepsilon F_2(a, \alpha, \varepsilon) = 0 \tag{2.15}$$

where F_1 and F_2 are given by (2.12) and (2.13). The analog of Theorem 2.1 for this case is:

THEOREM 2.2. *Assume that conditions* (H1)-(H5) *are satisfied. Then for all $r_a > 0$ there exist $\bar{\alpha} > 0$, $\bar{\varepsilon} > 0$, a neighborhood V_q of 0 in Q and a unique continuously differentiable function $\mathscr{H}(a, \alpha, \varepsilon): B_{r_a} \times I_{\bar{\alpha}} \times I_{\bar{\varepsilon}} \to V_q$ such that if (a, α, ε) is a solution of the bifurcation equation* (2.15) *with $|\Phi a| < r_a$, $\varphi^Q \in V_q$, $|\alpha| < \bar{\alpha}$ and $|\varepsilon| < \bar{\varepsilon}$ then $x(\Phi a + \mathscr{H}(a, \alpha, \varepsilon), \alpha, \varepsilon)$ is an ω-periodic solution of* (2.1a). *Moreover any ω-periodic solution $x(\Phi a + \varphi^Q, \alpha, \varepsilon)$ of* (2.1a) *with $|\Phi a| < r_a$, $|\alpha| < \bar{\alpha}$ and $|\varepsilon| < \bar{\varepsilon}$ is of the above form, i.e., $\varphi^Q = \mathscr{H}(a, \alpha, \varepsilon)$ and (a, α, ε) satisfies* (2.15).

For the following section we need to consider also the case $\alpha = 0$ and $g(t, \varphi) = 0$, i.e., to look at the equation:

$$x'(t) = L(x_t) + \varepsilon f(t, x_t) \tag{2.1b}$$

For this case the bifurcation equation (2.15) after a division by ε becomes

$$F_2(a, 0, \varepsilon) = 0 . \tag{2.16}$$

where F_2 is given by Equation (2.13).
Similarly, we have for this case the following

THEOREM 2.3. *Assume that conditions* (H1)-(H5) *are satisfied. Then for all $r_a > 0$ there exist $\bar{\varepsilon} > 0$, a neighborhood V_q of 0 in Q and a unique continuously differentiable function $\mathscr{H}(a, \varepsilon): B_{r_a} \times I_{\bar{\varepsilon}} \to V_q$, with $\mathscr{H}(a, 0) = 0$ such that if (a, ε) is a solution of the bifurcation equation* (2.16) *with $|\Phi a| < r_a$ and $|\varepsilon| < \bar{\varepsilon}$ then $x(\Phi a + \mathscr{H}(a, \varepsilon), \varepsilon)$ is an ω-periodic* *solution of* (2.1b). *Moreover, any* ω*-periodic solution* $x(\Phi a + \varphi^Q, \varepsilon)$ *of* (2.1b) *with* $|\Phi a| < r_a$, $\varphi^Q \in V_q$, *and* $|\varepsilon| < \bar{\varepsilon}$ *is of the above form, i.e.,* $\varphi^Q = \mathscr{H}(a, \varepsilon)$ *and* (a, ε) *satisfies* (2.16).

3. The one parameter problem. We consider in this section the equation

$$x'(t) = L(x_t) + \varepsilon f(t, x_t) . \tag{3.1}$$

From the previous section we know that the bifurcation equation for this case is Equation (2.16) which reads:

$$M_\varepsilon(a) \equiv \int_0^\omega e^{-Bs}\Psi(0)f(s, x_s(\Phi a + \mathscr{H}(a, \varepsilon), \varepsilon))ds = 0 . \tag{3.2}$$

For $\varepsilon = 0$, Equation (3.2) becomes

$$M(a) = \int_0^\omega e^{-Bs}\Psi(0)f(s, x_s(\Phi a, 0))ds = 0$$

and since $x_s(\Phi a, 0) = \Phi e^{Bs}a$, we get

$$M(a) = \int_0^\omega e^{-Bs}\Psi(0)f(s, \Phi e^{Bs}a)ds = 0 . \tag{3.3}$$

THEOREM 3.1. *Assume that conditions* (H1)-(H5) *are satisfied. Assume also that* $M(\bar{a}) = 0$ *and* $\det(\partial M/\partial a)(\bar{a}) \neq 0$ *for some* $\bar{a} \in \boldsymbol{R}^d$. *Then for all* ε *with* $|\varepsilon|$ *sufficiently small, Equation* (3.1) *has an* ω*-periodic solution* $x(\varepsilon)$, *such that* $x(0)(t) = \Phi(0)e^{Bt}\bar{a}$.

PROOF. Let $r_a > 0$ be such that $|\Phi\bar{a}| < r_a$. Let $\bar{\varepsilon} > 0$, V_q and $\mathscr{H}(a, \varepsilon)$ be given as in Theorem 2.3. Then the mapping $M_\varepsilon(a)$ is continuously differentiable for $(a, \varepsilon) \in B_{r_a} \times I_{\bar{\varepsilon}}$. From the Implicit Function Theorem and our assumptions it follows that Equation (3.2) has a unique solution $a = a(\varepsilon)$ defined and continuously differentiable for all ε with $|\varepsilon|$ sufficiently small and such that $a(0) = \bar{a}$. From Theorem 2.3 it follows then, that $x(\varepsilon) = x(\Phi a(\varepsilon) + \mathscr{H}(a, \varepsilon), \varepsilon)$ is an ω-periodic solution of (3.1) and $x(0)(t) = x(\Phi\bar{a} + \mathscr{H}(\bar{a}, 0), 0)(t) = x(\Phi\bar{a}, 0)(t) = \Phi(0)e^{Bt}\bar{a}$.

We may also use topological degree to give existence theorems. Let $\Omega \subset \boldsymbol{R}^n$ be open and bounded. Suppose that $f\colon \bar{\Omega} \to \boldsymbol{R}^n$ is continuous and $0 \notin f(\partial\Omega)$ (where $\partial\Omega$ denotes the boundary of Ω). Then the topological degree of f with respect to Ω and $0 \in \boldsymbol{R}^n$, $\deg(f, \Omega, 0)$, is defined and integer valued. The reader is referred to Cronin [5] or Schwartz [10] for the details.

THEOREM 3.2. *Assume that conditions* (H1)-(H5) *are satisfied,* $r_a > 0$ *and* $\deg(M, B_{r_a}, 0)$ *is defined and different from* 0. *Then Equation* (3.1) *has at least one* ω*-periodic solution for all* ε *with* $|\varepsilon|$ *sufficiently small.*

PROOF. Similarly to the previous proof, $\exists\bar{\varepsilon} > 0$ such that $M_\varepsilon(a)$ is a continuously differentiable map for $(a, \varepsilon) \in \bar{B}_{r_a} \times \bar{I}_{\bar{\varepsilon}}$. The homotopy property of topological degree and our assumptions imply

$$\deg(M_\varepsilon, B_{r_a}, 0) = \deg(M, B_{r_a}, 0) \neq 0 .$$

Therefore Equation (3.2) has at least one solution in B_{r_a} for all ε with $|\varepsilon|$ sufficiently small. The conclusion of the theorem follows now from Theorem 2.3.

REMARK. Theorem 3.2 remains true if in our assumptions we replace $f(t, \varphi)$ by $f(t, \varphi) + f_1(t, \varphi, \varepsilon)$ where $f_1(t, \varphi, \varepsilon)$ is ω-periodic in t, continuously differentiable and $O(|\varepsilon|)$.

4. The two parameter problem. In this section we consider the equation

$$x'(t) = (1 + \alpha)L(x_t) + \varepsilon f(t, x_t) + g(t, x_t) \tag{4.1}$$

and we proceed by analyzing the corresponding bifurcation equation (2.11). Expressions (2.12), (2.13), (2.14) have the following form:

$$F_1(a, \alpha, \varepsilon) = l(a) + O(\|a\|^2 + |\varepsilon| + |\alpha\varepsilon| + \|\alpha a\| + \|\varepsilon a\|)$$
$$F_2(a, \alpha, \varepsilon) = p + O(\|a\| + |\varepsilon| + |\alpha\varepsilon| + \|\alpha a\| + \|\varepsilon a\|)$$
$$F_3(a, \alpha, \varepsilon) = Q(a) + O(\|a\|^3 + |\alpha|\|a\|^2 + \|\alpha\varepsilon a\| + \|\varepsilon a\| + |\varepsilon|^2)$$

where

$$l(a) = \int_0^\omega e^{-Bs}\Psi(0)L(\Phi e^{Bs}a)ds \tag{4.2a}$$

$$p = \int_0^\omega e^{-Bs}\Psi(0)f(s, 0)ds \tag{4.2b}$$

$$Q(a) = \int_0^\omega e^{-Bs}\Psi(0)\tilde{g}(s, \Phi e^{Bs}a)ds \tag{4.2c}$$

and $\tilde{g}(t, \varphi)$ contains the terms of order $O(|\varphi|^2)$ as $|\varphi| \to 0$ in the expansion of g. So $l(a)$ is linear in a, p is a constant d-vector and $Q(a)$ is quadratic in a. Therefore the bifurcation equation (2.11) can be written as

$$h(a, \alpha, \varepsilon) \equiv Q(a) + \alpha l(a) + \varepsilon p + \text{h.o.t.} = 0 \tag{4.3}$$

where h.o.t. designates higher order terms and refers to terms of order $O(\|a\|^3 + |\alpha|\|a\|^2 + \|\varepsilon a\| + |\alpha\varepsilon| + \|\alpha^2 a\| + |\varepsilon^2|)$.

In order to determine the ω-periodic solutions of (4.1) which are close to the origin, we have to analyze according to Theorem 2.1, the solutions of the bifurcation equation (4.3). This problem consists of studying the simultaneous solutions of

$$h_i(a, \alpha, \varepsilon) = 0 , \qquad i = 1, 2, \cdots, d$$

for (a, α, ε) near the origin, where $h = (h_1, h_2, \cdots, h_d) \in \boldsymbol{R}^d$. Let

$$M_i(\alpha, \varepsilon) = \{a \in \boldsymbol{R}^d : h_i(a, \alpha, \varepsilon) = 0\} , \qquad i = 1, 2, \cdots, d .$$

Then the problem is to study the intersection of the surfaces $M_i(\alpha, \varepsilon)$, $i = 1, \cdots, d$ in $\boldsymbol{R}^d$. If for some value of the parameters, say $(\alpha^*, \varepsilon^*)$ these surfaces intersect transversally at a point $a^* \in \boldsymbol{R}^d$, i.e., $h(a^*, \alpha^*, \varepsilon^*) = 0$ and $\det[(\partial h/\partial a)(a^*, \alpha^*, \varepsilon^*)] \neq 0$ then we expect the same situation to hold for (a, α, ε) near $(a^*, \alpha^*, \varepsilon^*)$, and hence the same number of solutions for (a, α, ε) near $(a^*, \alpha^*, \varepsilon^*)$. Thus, the condition

$$h(a, \alpha^*, \varepsilon^*) = 0 \Longrightarrow \det[(\partial h/\partial a)(a, \alpha^*, \varepsilon^*)] \neq 0 \tag{4.4}$$

implies that there is no bifurcation for values of the parameters near $(\alpha^*, \varepsilon^*)$. Therefore at a bifurcation point the following condition must hold:

$$h(a, \alpha, \varepsilon) = 0 , \qquad \det[(\partial h/\partial a)(a, \alpha, \varepsilon)] = 0 . \tag{4.5}$$

We shall later see that under certain conditions Equation (4.5) determines a finite number of curves in the parameter space, emanating from the origin, such that if the parameters cross one of these curves, then the number of solutions changes by two.

In order to obtain the bifurcation diagram, the variables (a, α, ε) must be scaled correctly. The scaling to be used is suggested by the following lemma:

LEMMA 4.1. *Assume that*

(E_1) *If* $Q(a) = 0$ *then* $a = 0$.

Then there exists a neighborhood V *of* $(a, \alpha, \varepsilon) = (0, 0, 0)$ *and a constant* $M > 0$ *such that any solution* $(a, \alpha, \varepsilon) \in V$ *of Equation* (4.3) *satisfies the estimate*:

$$\|a\| \leqq M(|\alpha| + |\varepsilon|^{1/2}) .$$

PROOF. Assume by contradiction that there exists a sequence of solutions $(a_n, \alpha_n, \varepsilon_n) \to (0, 0, 0)$ with $|\alpha_n|/\|a_n\| + |\varepsilon_n|^{1/2}/\|a_n\| \to 0$. From Equation (4.3) we obtain: $h(a_n, \alpha_n, \varepsilon_n)/\|a_n\|^2 = Q(a_n/\|a_n\|) + O(|\alpha_n|/\|a_n\| + |\varepsilon_n|/\|a_n\|^2 + |\alpha_n| + \|a_n\|)$. By taking a convergent subsequence we can assume that $a_n/\|a_n\| \to a_0 \neq 0$. We get $0 = Q(a_0)$, contradiction to (E_1).

Next we divide the (ε, α) plane into three regions. Let

$$R_1^+ = \{(\varepsilon, \alpha) : 0 \leqq \varepsilon \leqq c_1, |\alpha|^2 \leqq |\varepsilon|/c_2\}$$

$$R_1^- = \{(\varepsilon, \alpha) : -c_1 \leqq \varepsilon \leqq 0, |\alpha|^2 \leqq |\varepsilon|/c_2\}$$

$$R_2 = \{(\varepsilon, \alpha): |\alpha| \leqq c_1, |\varepsilon| \leqq c_2|\alpha|^2\}$$

where $c_1 \geqq 0$ and $c_2 \geqq 1$ are constants to be determined later. (See Figure 1.)

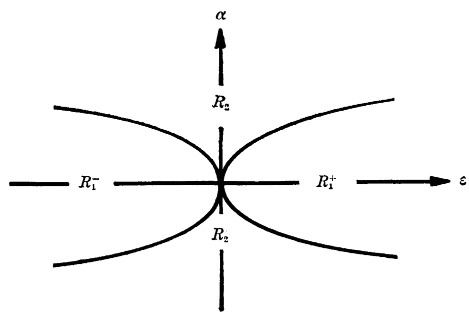


FIGURE 1

By considering separate scalings for the regions R_1^+, R_1^- and R_2 we obtain a complete description of the solutions of (4.3) in a neighborhood of the origin of the parameter space (ε, α).

We consider first the region R_1^+. By Lemma 4.1 we have the following estimate: $||a|| \leqq M(|\alpha| + |\varepsilon|^{1/2}) \leqq M(c_2^{-1/2} + 1)|\varepsilon|^{1/2} \leqq 2M|\varepsilon|^{1/2}$. Thus we may scale: $a = \lambda b$, $\varepsilon = \lambda^2$, $\alpha = \lambda\gamma$. With this scaling the bifurcation equation (4.3), after a division by λ^2, becomes

$$H(b, \lambda, \gamma) \equiv Q(b) + \gamma l(b) + p + O(|\lambda|) = 0 . \tag{4.6}$$

Here, λ has to be bounded near 0, and γ must satisfy the estimate $|\gamma| \leqq c_2^{-1/2}$ (follows from the definition of R_1^+ and γ). We assume now

(E_2) If $Q(b^*) \pm p = 0$ then $\det[(\partial Q/\partial b)(b^*)] \neq 0$.

Obviously, the equation

$$H(b, 0, 0) = Q(b) + p = 0$$

has a finite number of solutions. By condition (E_2), the Implicit Function Theorem is applicable at each such solution and therefore there exists a neighborhood of $(\gamma, \lambda) = (0, 0)$ in which the number of solutions of (4.6) is constant. This fixes the size of $|\lambda|$ and of $|\gamma|$ to be sufficiently small, which in turn fixes the constants c_1 and c_2, $c_1 \geqq 0$ and small, $c_2 \geqq 1$ and sufficiently large. Then, with this choice of c_1 and c_2 there is no bifurcation in the region R_1^+.

A similar analysis holds for the region R_1^- with the following changes: the scaling for ε should be now $\varepsilon = -\lambda^2$ and p should be replaced by $-p$ in the scaled Equation (4.6).

The above discussion can be summarized in the following:

LEMMA 4.2. *Assume that conditions* (E_1), (E_2) *are satisfied. Then there exist constants* $c_1 > 0$ *small,* $c_2 \gg 1$ *for which there is no bifurcation in the regions* R_1^+ *and* R_1^-.

To complete our picture, we have to carry out a bifurcation analysis for the region R_2. For this region Lemma 4.1 provides us with the following estimate: $\|a\| \leqq M(|\alpha| + |\varepsilon|^{1/2}) \leqq M(|\alpha| + c_2^{1/2}|\alpha|) = M(1+c_2^{1/2})|\alpha|$. This estimate suggests the following scaling for the region R_2: $a = \nu b$, $\varepsilon = \nu^2\delta$, $\alpha = \nu$.

Then the bifurcation (4.3), after a division by ν^2, becomes:

$$G(b, \nu, \delta) = Q(b) + l(b) + \delta p + O(|\nu|) = 0 \,. \tag{4.7}$$

Here $|\nu|$ must be bounded near 0 and δ has to satisfy the estimate $|\delta| \leqq c_2$ (follows from the definition of R_2 and δ) where $c_2 \gg 1$ is given by Lemma 4.2.

If for some b^* and δ^* we have $G(b^*,0,\delta^*)=0$ and $\det[(\partial G/\partial b)(b^*,0,\delta^*)]\neq 0$, then there is a unique zero b of (4.7) for every (ν, δ) near $(0, \delta^*)$.

It is now important to note that $G(b, 0, \delta)$ takes $\boldsymbol{R}^{d+1}$ into $\boldsymbol{R}^d$. Thus even if the inverse image of 0 for $G(b, 0, \delta)$ is a smooth curve, the above condition may not be verified. Hence we consider the case $G(b^*, 0, \delta^*) = 0$ and $\det[(\partial G/\partial b)(b^*, 0, \delta^*)] = 0$, i.e., b^* is a nonsimple zero of Equation (4.7) for $(\nu, \delta) = (0, \delta^*)$. We impose now the following condition on nonsimple zeros of Equation (4.7):

(E_3) If $Q(b^*) + l(b^*) + \delta^* p = 0$ and $\det[(\partial Q/\partial b)(b^*) + l] = 0$ then the $d \times d$ matrix $(\partial Q/\partial b)(b^*) + l$ has rank $d - 1$.

Assumption (E_3) implies that the matrix $(\partial G/\partial b)(b^*, 0, \delta^*)$ has a nonzero minor of order $d - 1$ and hence we can assume without loss of generality that $\det[(\partial(G_1, G_2, \cdots, G_{d-1})/\partial(b_1, b_2, \cdots, b_{d-1}))(b^*, 0, \delta^*)] \neq 0$. Let

$$\begin{aligned}
&G = (u, v) \quad \text{where} \quad u = (G_1, G_2, \cdots, G_{d-1}) \,, \qquad v = G_d \\
&b = (\eta, \zeta) \quad \text{where} \quad \eta = (b_1, b_2, \cdots, b_{d-1}) \,, \qquad \zeta = b_d \\
&b^* = (\eta^*, \zeta^*) \\
&\Delta_1 = \det[\partial(u, v)/\partial(\eta, \zeta)] \quad \text{and} \quad \Delta_2 = \det[\partial u/\partial\eta] \,.
\end{aligned}$$

With this notation we have

$$\begin{aligned}
&u(\eta^*, \zeta^*, 0, \delta^*) = 0 \,, \qquad v(\eta^*, \zeta^*, 0, \delta^*) = 0 \,, \\
&\Delta_2(\eta^*, \zeta^*, 0, \delta^*) \neq 0 \,, \qquad \Delta_1(\eta^*, \zeta^*, 0, \delta^*) = 0 \,.
\end{aligned}$$

Thus an application of the Implicit Function Theorem to the equation $u(\eta, \zeta, \nu, \delta) = 0$ shows that there exists a continuously differentiable function $\eta = e(\zeta, \nu, \delta)$ such that

$$e(\zeta^*, 0, \delta^*) = \eta^* , \qquad u(e(\zeta, \nu, \delta), \zeta, \nu, \delta) = 0 \tag{4.8}$$

for (ζ, ν, δ) in a neighborhood of $(\zeta^*, 0, \delta^*)$.

Thus, the bifurcation equation becomes:

$$w(\zeta, \nu, \delta) \equiv v(e(\zeta, \nu, \delta), \zeta, \nu, \delta) = 0 \tag{4.9}$$

which is a one dimensional equation and has to be solved in a neighborhood of the solution $(\zeta, \nu, \delta) = (\zeta^*, 0, \delta^*)$. Let

$$\Delta = \det [\partial(u, v, \Delta_1)/\partial(\eta, \zeta, \delta)]$$

and assume also that the following condition is satisfied:

(E_4) If $(b^*, 0, \delta^*)$ is a solution of the equations $G(b^*, 0, \delta^*) = 0$ and $\Delta_1(b^*, 0, \delta^*) = 0$, then $\Delta(b^*, 0, \delta^*) \neq 0$. Assumption ($E_4$) means that $G(b^*, 0, \delta^*) = 0$ and $\Delta_1(b^*, 0, \delta^*) = 0$ imply $\Delta(b^*, 0, \delta^*) \neq 0$.

LEMMA 4.3. *Assume that condition* (E_4) *is satisfied. Then at the point* $(\zeta, \nu, \delta) = (\zeta^*, 0, \delta^*)$ *the function* w *satisfies*:

$$w = \partial w/\partial\zeta = 0 \tag{4.10}$$

$$\partial^2 w/\partial\zeta^2 = \Delta_2^{-2} \det [\partial(u, \Delta_1)/\partial(\eta, \zeta)] \neq 0 \tag{4.11}$$

$$\partial w/\partial\delta = \Delta_2^{-1} \det [\partial(u, v)/\partial(\eta, \delta)] \neq 0 . \tag{4.12}$$

PROOF. We need the following identities:

$$\Delta_1 = [\partial v/\partial\zeta - (\partial v/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\zeta)]\Delta_2 \tag{4.13}$$

$$\det [\partial(u, \Delta_1)/\partial(\eta, \zeta)] = [\partial\Delta_1/\partial\zeta - (\partial\Delta_1/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\zeta)]\Delta_2 \tag{4.14}$$

$$\det [\partial(u, v)/\partial(\eta, \delta)] = [\partial v/\partial\delta - (\partial v/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\delta)]\Delta_2 . \tag{4.15}$$

Also

$$\begin{bmatrix} \partial u/\partial\delta \\ \partial v/\partial\delta \end{bmatrix} = p .$$

To prove (4.13) observe that by using row operations we have:

$$\begin{aligned} \Delta_1 &= \det \begin{bmatrix} \partial u/\partial\eta & \partial u/\partial\zeta \\ \partial v/\partial\eta & \partial v/\partial\zeta \end{bmatrix} = \det \begin{bmatrix} \partial u/\partial\eta & \partial u/\partial\zeta \\ 0 & \partial v/\partial\zeta - (\partial v/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\zeta) \end{bmatrix} \\ &= [\partial v/\partial\zeta - (\partial v/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\zeta)]\Delta_2 . \end{aligned}$$

This proves (4.13). (4.14) and (4.15) are obtained similarly. By implicit differentiation of Equation (4.9) we get: $\partial e/\partial\zeta = -(\partial u/\partial\eta)^{-1}(\partial u/\partial\zeta)$. Thus $\partial w/\partial\zeta = (\partial v/\partial\eta)(\partial e/\partial\zeta) + (\partial v/\partial\zeta) = \partial v/\partial\zeta - (\partial v/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\zeta) = \Delta_1/\Delta_2$ by

Equation (4.13). But, when $(\eta, \zeta, \nu, \delta) = (\eta^*, \zeta^*, 0, \delta^*)$ $\Delta_1 = 0$ and $\Delta_2 \neq 0$. This proves (4.10). Now $\partial^2 w/\partial\zeta^2 = \partial(\Delta_1/\Delta_2)/\partial\zeta = \Delta_2^{-2}[(\partial\Delta_1/\partial\zeta)\Delta_2 - \Delta_1(\partial\Delta_2/\partial\zeta)] = \Delta_2^{-1}(\partial\Delta_1/\partial\zeta)$ at $(\eta^*, \zeta^*, 0, \delta^*)$, $\partial(\Delta_1(e(\zeta, \nu, \delta), \zeta, \nu, \delta))/\partial\zeta = (\partial\Delta_1/\partial\eta)(\partial e/\partial\zeta) + \partial\Delta_1/\partial\zeta = \partial\Delta_1/\partial\zeta - (\partial\Delta_1/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u \partial\zeta) = \Delta_2^{-1}\det[\partial(u, \Delta_1)/\partial(\eta, \zeta)]$ by (4.14).

This proves the equality in (4.11). $\partial w/\partial\delta = (\partial v/\partial\eta)(\partial e/\partial\delta) + \partial v/\partial\delta = \partial v/\partial\delta - (\partial v/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\delta) = \Delta_2^{-1}\det[\partial(u, v)/\partial(\eta, \delta)]$ by (4.15), and this proves the equality in (4.12). The inequalities in (4.11) and (4.12) follow from condition (E_4) and the identity:

$$\Delta = -\Delta_2^2(\partial w/\partial\delta)(\partial^2 w/\partial\zeta^2) \quad \text{at} \quad (\eta^*, \zeta^*, 0, \delta^*) \tag{4.16}$$

which we prove now:

$$\begin{aligned}
\Delta &= \det\begin{bmatrix} \partial u/\partial\eta & \partial u/\partial\zeta & \partial u/\partial\delta \\ \partial v/\partial\eta & \partial v/\partial\zeta & \partial v/\partial\delta \\ \partial\Delta_1/\partial\eta & \partial\Delta_1/\partial\zeta & 0 \end{bmatrix} \\
&= \det\begin{bmatrix} \partial u/\partial\eta & \partial u/\partial\zeta & \partial u/\partial\delta \\ 0 & \partial v/\partial\zeta - (\partial v/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\zeta) & \partial v/\partial\delta - (\partial v/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\delta) \\ 0 & \partial\Delta_1/\partial\zeta - (\partial\Delta_1/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\zeta) & -(\partial\Delta_1/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\delta) \end{bmatrix} \\
&= \det\begin{bmatrix} \partial u/\partial\eta & \partial u/\partial\zeta & \partial u/\partial\delta \\ 0 & 0 & \Delta_2^{-1}\det[\partial(u, v)/\partial(\eta, \delta)] \\ 0 & \Delta_2^{-1}\det[\partial(u, \Delta_1)/\partial(\eta, \zeta)] & -(\partial\Delta_1/\partial\eta)(\partial u/\partial\eta)^{-1}(\partial u/\partial\delta) \end{bmatrix} \\
&= -\Delta_2^{-1}\det[\partial(u, v)/\partial(\eta, \delta)] \times \Delta_2 \times \Delta_2^{-1}\det[\partial(u, \Delta_1)/\partial(\eta, \zeta)] \\
&= -\Delta_2^{-1} \times \Delta_2(\partial w/\partial\delta) \times \Delta_2^2(\partial^2 w/\partial\zeta^2) = -\Delta_2^2(\partial w/\partial\delta)(\partial^2 w/\partial\zeta^2)\,.
\end{aligned}$$

REMARK. Lemma 4.3 shows essentially that the curve $u(\eta, \zeta, 0, \delta^*) = 0$ and the surface $v(\eta, \zeta, 0, \delta^*) = 0$ intersect transversally with respect to the parameter δ at the point (η^*, ζ^*). (See Figure 2.)

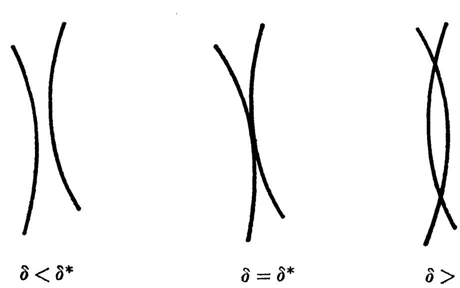


FIGURE 2

We are ready now for our main theorem:

THEOREM 4.1. *Assume that the two parameter bifurcation equation* (4.3) *satisfies conditions* (E_1)-(E_4). *Then there exist a finite number of curves of the form* $\varepsilon \sim \alpha^2\delta^*$ *as* $\alpha \to 0$ *on which bifurcation takes place and the bifurcating solution is given approximately by* $a = \nu b^*$ *where* (b^*, δ^*) *are simultaneous solutions of*

$$Q(b^*) + l(b^*) + \delta^* p = 0 \tag{4.17a}$$

$$\det[(\partial Q/\partial b)(b^*) + l] = 0 \,. \tag{4.17b}$$

Each such curve corresponds to a different value of δ^* *and as one crosses the curve from left to right two solutions appear if* $\Delta(b^*, 0, \delta^*) > 0$ *or they disappear if* $\Delta(b^*, 0, \delta^*) < 0$.

PROOF. We have seen that in order to have bifurcation at a point (b, ν, δ) near $(b^*, 0, \delta^*)$ it is necessary that equations (4.17a) and (4.17b) be satisfied simultaneously. Since $p \neq 0$ we can use one of the equations in (4.17a), to eliminate δ^* from the remaining $d-1$ ones. Together with (4.17b) we obtain then a system of d quadratic equations in $b^* = (b_1^*, b_2^*, \cdots, b_d^*)$. By a theorem of Bezout such a system can have at most 2^d solutions, hence only a finite number of δ^* have to be considered. The scaling of region R_2 implies that bifurcation can occur only on the curves $\varepsilon \sim \alpha^2\delta^*$.

Let $(b^*, 0, \delta^*) = (\eta^*, \zeta^*, 0, \delta^*)$ be a solution of (4.17a) and (4.17b). By Lemma 4.3 we can solve the system $(\partial w/\partial\zeta)(\zeta, \nu, \delta) = 0$ as $\zeta = \beta(\nu, \delta)$, $\beta(0, \delta^*) = \zeta^*$, $(\partial w/\partial\zeta)(\beta(\nu, \delta), \nu, \delta) = 0$. It follows that for all (ν, δ) close to $(0, \delta^*)$, $\zeta = \beta(\nu, \delta)$ is a critical point of w near ζ^*. This critical point is maximum if $\det[\partial(u, \Delta_1)/\partial(\eta, \zeta)] < 0$ and a minimum if $\det[\partial(u, \Delta_1)/\partial(\eta, \zeta)] > 0$. Let $Z(\nu, \delta) = w(\beta(\nu, \delta), \nu, \delta)$ be the value of w at this critical point. If the critical point is a minimum and $Z(\nu, \delta) < 0$ then the bifurcation equation has exactly two solutions near ζ^*, and if $Z(\nu, \delta) > 0$ there are no solution near ζ^*. Hence, in the parameter space (ν, δ), bifurcation occurs when

$$Z(\nu, \delta) = 0 \,. \tag{4.18}$$

Since $Z(0, \delta^*) = 0$, $(\partial Z/\partial\delta)(0, \delta^*) = (\partial w/\partial\delta)(0, \delta^*) \neq 0$ we can solve Equation (4.18) as $\delta = \gamma(\nu)$, $\gamma(0) = \delta^*$, $Z(\nu, \gamma(\nu)) = 0$, and for $\delta = \gamma(\nu)$ bifurcation occurs at the critical point $b = e(\beta(\nu, \delta), \nu, \delta), \beta(\nu, \delta), \nu, \delta) \equiv \psi(\delta)$, $\psi(0) = b^*$. The direction of bifurcation is determined as follows: If $\Delta(b^*, 0, \delta^*) > 0$ then $\partial^2 w/\partial\zeta^2$ and $\partial w/\partial\delta$ have opposite signs at $(\zeta^*, 0, \delta^*)$. Suppose that $\partial^2 w/\partial\zeta^2 > 0$ and $\partial w/\partial\delta < 0$. Then the critical point is a minimum and $Z < 0$ when $\delta - \gamma(\nu) > 0$. This means that bifurcation of two solutions from $(b^*, 0, \delta^*)$ occurs when $\nu > 0$ and this in turn means by our scaling, crossing the bifurcation curve $\varepsilon = \alpha^2\delta^*$ from left to right. The remaining possibilities are dealt with similarly.

5. Examples.

EXAMPLE 1. Consider the f.d.e.

$$x'(t) = -(\pi/2)x(t-1) + \varepsilon(x^2(t-1) - 1)\sin(\pi t/2)\ . \tag{5.1}$$

The various quantities associated with the linear equation

$$x'(t) = -(\pi/2)x(t-1) \tag{5.2}$$

were computed in Hale [7, p. 116] and they are: $\Lambda = \{\pi i/2, -\pi i/2\}$; thus $\omega = 4$ and $d = 2$; $\Phi = \{\varphi_1, \varphi_2\} = \{\sin(\pi\theta/2), \cos(\pi\theta/2)\}$, $-1 \leqq \theta \leqq 0$, is a basis of the periodic solutions of period 4 of (5.2);

$$B = \begin{bmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{bmatrix};$$

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 2\mu(\sin(\pi s/2) + (\pi/2)\cos(\pi s/2)) \\ 2\mu(-(\pi/2)\sin(\pi s/2) + \cos(\pi s/2)) \end{bmatrix} \qquad 0 \leqq s \leqq 1$$

where $\mu = 1/(1 + \pi^2/4)$.

Equation (5.1) is of the form (3.1) with $f(t, \varphi) = (\varphi^2(-1) - 1)\sin(\pi t/2)$. A series of easy but tedious computations, show that the bifurcation equation (3.3) for our example is:

$$M\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 2\mu\begin{bmatrix} -2 + a_1^2/2 - \pi a_1 a_2/2 + 3a_2^2/2 \\ \pi - \pi a_1^2/4 - a_1 a_2 - 3\pi a_2^2/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solutions of this equation are the intersection of the two ellipses $M_i(a) = 0$ $i = 1, 2$ and they are: $a^{(1)} = (2/\sqrt{3}, 0)$; $a^{(2)} = (0, 2)$; $a^{(3)} = (-2/\sqrt{3}, 0)$; $a^{(4)} = (0, -2)$. (See Figure 3.)

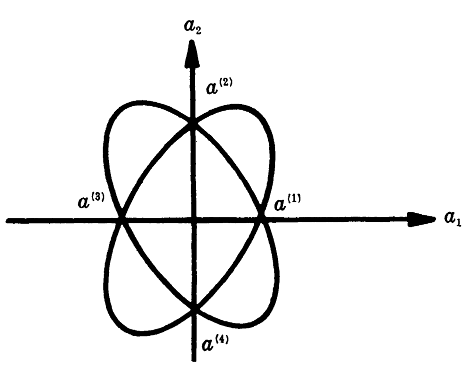


FIGURE 3

$$\det[\partial M(a)/\partial a]=2\mu \det\begin{bmatrix} a_1-\pi a_2/2 & -(\pi a_1/2)+3a_2 \\ -(\pi a_1/2)-a_2 & -a_1-3\pi a_2/2 \end{bmatrix} = 2\mu(1+\pi^2/4)(3a_2^2-a_1^2) .$$

Thus $\det[\partial M(a)/\partial a] \neq 0$ at the intersection points and so Theorem 3.1 applies. We obtain 4 periodic solutions of period 4, $x^{(i)}(\varepsilon)(t)$, $1 \leqq i \leqq 4$, for every ε with $|\varepsilon|$ sufficiently small such that $x^{(1)}(0)(t)=(2/\sqrt{3})\sin(\pi t/2)$, $x^{(2)}(0)(t)=2\cos(\pi t/2)$, $x^{(3)}(0)(t)=(-2/\sqrt{3})\sin(\pi t/2)$, $x^{(4)}(0)(t)=-2\cos(\pi t/2)$.

EXAMPLE 2. Consider the f.d.e.

$$x'(t) = -(\pi/2)x(t-1) + \varepsilon\{x^3(t)x(t-1)\cos(\pi t/2) + O(|\varepsilon|)\} . \tag{5.3}$$

The corresponding bifurcation equation is given by

$$M\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mu\begin{bmatrix} -(a_1^4 + \pi a_1^3 a_2 + \pi a_1 a_2^3 - a_2^4/2 \\ \pi a_1^4/4 - a_1^3 a_2 - a_1 a_2^3 - \pi a_2^4/4 \end{bmatrix} .$$

Consider the polynomials:

$$\alpha(x) = x^4 + \pi x^3 + \pi x - 1 , \qquad \beta(x) = \pi x^4/4 - x^3 - x - \pi/4 .$$

The roots of α are: $\alpha_1 \simeq -3.43$, $\alpha_2 \simeq 0.29$, $\alpha_3 = i$, $\alpha_4 = -i$. The roots of β are: $\beta_1 \simeq -0.55$, $\beta_2 \simeq 1.82$, $\beta_3 = i$, $\beta_4 = -i$. Thus $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$. Therefore it follows from Cronin [5, p. 40] that $\deg(M, B, O) = -2 \neq 0$ where B is a ball in $\boldsymbol{R}^2$ with center O. It follows from Theorem 3.2 that Equation (5.3) has at least one periodic solution of period 4 for all ε with $|\varepsilon|$ sufficiently small.

EXAMPLE 3. Consider the equation

$$x'(t) = (\alpha - \pi/2)x(t-1) + \varepsilon\sin(\pi t/2) + x^2(t-1)\sin(\pi t/2) + \text{h.o.t} = 0 . \tag{5.4}$$

The corresponding bifurcation equation is:

$$\begin{bmatrix} a_1^2/2 - a_1a_2/2 + 3a^2/2 \\ -\pi a_1^2/4 - a_1a_2 - 3\pi a_2^2/4 \end{bmatrix} + \varepsilon\begin{bmatrix} 2 \\ -\pi \end{bmatrix} + \alpha\begin{bmatrix} -\pi & 2 \\ -2 & -\pi \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \text{h.o.t.} = 0$$

which is of the form (4.3) with

$$Q(a) = \begin{bmatrix} a_1^2/2 - \pi a_1a_2/2 + 3a_2^2/2 \\ -\pi a_1^2/4 - a_1a_2 - 3\pi a_2^2/4 \end{bmatrix}; \qquad p = \begin{bmatrix} 2 \\ -\pi \end{bmatrix}; \qquad l = \begin{bmatrix} -\pi & 2 \\ -2 & -\pi \end{bmatrix} .$$

To apply Theorem 4.1 we have to check first that conditions (E_1)-(E_4) are satisfied. (E_1) is easy. To verify (E_2) we have to solve first the system $Q(b) \pm p = 0$. That is

$$b_1^2/2 - \pi b_1b_2/2 + 3b_2^2/2 \pm 2 = 0$$
$$-\pi b_1^2/4 - b_1b_2 - 3\pi b_2^2/4 \pm (-\pi) = 0 .$$

From Example 1 we know that this system with the + sign has no real solutions and with the $-$ sign it has exactly 4 solutions and $\det[\partial Q/\partial b] \neq 0$ at each solution. This shows that (E_2) is satisfied and furthermore it shows by the analysis done in §4 that in the region R_1^- there are 4 solutions, while in the region R_1^+ there are none. (See Figure 4.)

To verify conditions (E_3), (E_4) we have first to solve simultaneously the system $Q(b) + lb + \delta p = 0$, $\det[\partial Q/\partial b(b) + l] = 0$, that is

$$\begin{aligned} &b_1^2/2 - \pi b_1 b_2/2 + 3b_2^2/2 - \pi b_1 + 2b_2 + 2\delta = 0 \\ &-\pi b_1^2/4 - b_1 b_2 - 3\pi b_2^2/4 - 2b_1 - \pi b_2 - \pi\delta = 0 \\ &\Delta_1 = -b_1^2/4 + 3b_2^2/4 + 2b_2 + 1 = 0 \,. \end{aligned}$$

The solutions are

$$\begin{bmatrix} b_1 \\ b_2 \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} b_1 \\ b_2 \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \end{bmatrix} .$$

Thus, at first order the bifurcation curves are:

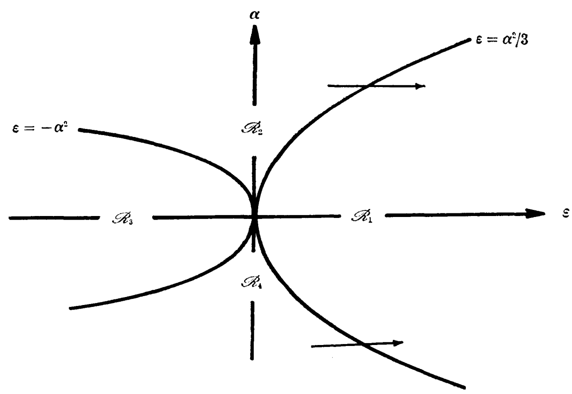
$$\varepsilon \sim -\alpha^2 \quad \text{as} \quad \alpha \to 0 \tag{5.5}$$

$$\varepsilon \sim \alpha^2/3 \quad \text{as} \quad \alpha \to 0 \,. \tag{5.6}$$

It is an easy matter to check that $[((\partial Q/\partial b) + l)]$ has rank 1 when evaluated at the solutions above. Thus (E_3) is satisfied. The direction of bifurcation is determined by Δ:

$$\begin{aligned} \Delta &= \det \begin{bmatrix} b_1 - \pi b_2/2 - \pi & -\pi b_1/2 + 3b_2 + 2 & 2 \\ -\pi/2 - b_2 - 2 & -b_1 - 3\pi/2 - \pi & -\pi \\ -b_1/2 & 3b_2/2 + 2 & 0 \end{bmatrix} \\ &= -(1 + \pi^2/4)(3b_2^2 + b_1^2) - (10 + 5\pi^2/2)b_2 - 8 - 2\pi^2 \,. \end{aligned}$$

On the curve (5.5) $\Delta = 0$ and on the curve (5.6) $\Delta < 0$. Thus by Theorem 4.1, one pair of solutions disappear as we cross the bifurcation curve (5.6) from left to right. (See Figure 4.) Theorem 4.1 does not give us information on the direction of bifurcation along the curve $\varepsilon = -\alpha^2$. However this can be obtained as follows: The number of solutions in each of the regions $\mathscr{R}_1, \cdots, \mathscr{R}_4$ is constant. Therefore in the region $\mathscr{R}_3$ there are 4 solutions, in the region $\mathscr{R}_1$ no solutions. Since there is a loss of two solutions when crossing the curve $\varepsilon = \alpha^2/3$ from left to right it follows that there are two solutions in the regions $\mathscr{R}_2$, $\mathscr{R}_4$. Since there are 4 solutions in $\mathscr{R}_3$ it follows that there is a loss of two solutions when crossing the curve $\varepsilon = -\alpha^2$ from left to right.



Bifurcation diagram
$\mathscr{R}_1$ = Region of no solutions
$\mathscr{R}_2, \mathscr{R}_4$ = Region of 2 solutions
$\mathscr{R}_3$ = Region of 4 solutions

FIGURE 4

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MATHEMATICS DEPARTMENT
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824
U.S.A.