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A UNICITY THEOREM FOR MEROMORPHIC MAPS OF A COMPLETE KÄHLER MANIFOLD INTO $P^N(C)$

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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1. Introduction. In 1926, R. Nevanlinna proved the following unicity theorem for meromorphic functions on C ([12]).

THEOREM. *Let ϕ, ψ be nonconstant meromorphic functions on C. If there exist five distinct values $a_1, \cdots, a_5$ such that $\phi^{-1}(a_i) = \psi^{-1}(a_i)$ $(1 \leqq i \leqq 5)$, then $\phi \equiv \psi$.*

The author gave several types of generalizations of this to the case of meromorphic maps of C^n into $P^N(C)$ in his papers [3] $\sim$ [8]. In this paper, we study meromorphic maps of an n-dimensional complete Kähler manifold M into $P^N(C)$ and give a new type of unicity theorem in the case where the universal covering of M is biholomorphic to the ball in C^n and meromorphic maps satisfy a certain growth condition.

Let M be an n-dimensional connected Kähler manifold with Kähler form ω and f be a meromorphic map of M into $P^N(C)$. For $\rho \geqq 0$ we say that f satisfies the condition (C_ρ) if there exists a nonzero bounded continuous real-valued function h on M such that

$$\rho \Omega_f + dd^c \log h^2 \geqq \mathrm{Ric}\, \omega ,$$

where Ω_f denotes the pull-back of the Fubini-Study metric form on $P^N(C)$ by f and $d^c = (\sqrt{-1}/4\pi)(\bar\partial - \partial)$.

Take a point $p \in M$. We represent f as $f = (f_1 : \cdots : f_{N+1})$ on a neighborhood of p with holomorphic functions f_i, where $\boldsymbol{f} := (f_1, \cdots, f_{N+1}) \not\equiv (0, \cdots, 0)$. Let $\mathscr{M}_p$ denote the field of all germs of meromorphic functions at p. For each $k \geqq 0$ we consider the $\mathscr{M}_p$-submodule $\mathscr{F}_p^k$ of $\mathscr{M}_p^{N+1}$ generated by all elements $(\partial^{|\alpha|}/\partial z^\alpha)\boldsymbol{f}$ with $|\alpha| \leqq k$, where $z = (z_1, \cdots, z_n)$ is a system of holomorphic local coordinates around p and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \cdots, \alpha_n)$.

By definition, the k-th rank of f is given by

$$r_f(k) := \mathrm{rank}_{\mathscr{M}_p} \mathscr{F}_p^k - \mathrm{rank}_{\mathscr{M}_p} \mathscr{F}_p^{k-1} ,$$

which does not depend on the choices of a point $p \in M$, a reduced representation of f and holomorphic local coordinates z (cf. Section 2). Set

$$l_f := \sum_{k_l} k r_f(k) ,$$

$$m_f := \sum_{k,l} (k-l)^+ \min\left\{{}_{n-1}H_l, \left(r_f(k) - \sum_{\lambda=0}^{l-1} {}_{n-1}H_\lambda\right)^+\right\} ,$$

where $x^+ = \max\{x, 0\}$ for a real number x and ${}_{n-1}H_\lambda$ denotes the number of repeated combinations of λ elements among $n-1$ elements. We have always

$$0 \leqq m_f \leqq l_f \leqq \frac{N(N+1)}{2} .$$

The main result in this paper is stated as follows.

MAIN THEOREM. *Let M be a complete, connected Kähler manifold whose universal covering is biholomorphic to C^n or the unit ball in C^n, and let f and g be nondegenerate meromorphic maps of M into $P^N(C)$. If f and g satisfy the condition (C_ρ) and there exist $q(\geqq N+2)$ hyperplanes in $P^N(C)$ located in general position such that*

(i) $f = g$ *on* $\bigcup_{j=1}^q f^{-1}(H_j) \cup g^{-1}(H_j)$,

(ii) $q > N + 1 + \rho(l_f + l_g) + m_f + m_g$,

then $f \equiv g$.

If $n = N$ and f is of rank n, then $m_f = 1$ and $l_f = N$ (cf. Example 3.3). Therefore, we have:

COROLLARY 1. *In Main Theorem, if $n = N$ and f and g are of rank n, then the condition* (ii) *of Main Theorem can be replaced by*

(ii)′ $q > N + 2\rho N + 3$.

For the case $M = C^n$, we can take the flat metric whose Ricci form vanishes. Therefore, all meromorphic maps of C^n into $P^N(C)$ satisfy the condition (C_0). This gives:

COROLLARY 2. *Let $f, g: C^n \to P^N(C)$ be nondegenerate meromorphic maps. If there exist q hyperplanes $H_1, \cdots, H_q$ in general position such that*

(i) $f = g$ *on* $\bigcup_{j=1}^q f^{-1}(H_j) \cup g^{-1}(H_j)$,

(ii) $q > N + 1 + m_f + m_g$,

Then $f \equiv g$.

This yields the classical theorem of R. Nevanlinna for the case $n = N = 1$, and the result of S. J. Drouilhet for the case $n = N$ and f, g are of rank n (cf. [1]).

In Section 2 we shall recall some known facts which will be needed later and in Section 3 we shall furnish a lemma concerning the order of poles for a special type of meromorphic function. In Section 4 we shall prove Corollary 2 directly. Moreover, we shall study meromorphic maps f, g of the unit ball into $\boldsymbol{P}^N(\boldsymbol{C})$ satisfying the condition

$$\limsup_{r\to 1} \frac{\log(1/(1-r))}{T_f(r, r_0) + T_g(r, r_0)} < \infty$$

and give a unicity theorem for such maps. Main Theorem will be completely proved in Section 5.

2. Preliminaries. For later use, we recall some known results concerning meromorphic maps into $\boldsymbol{P}^N(\boldsymbol{C})$.

Let M be an n-dimensional complex manifold and $f: M \to \boldsymbol{P}^N(\boldsymbol{C})$ be a meromorphic map. We take a point $p \in M$ and denote by $\mathscr{M}_p$ the field of all germs of meromorphic functions at p. Let U be a holomorphic local coordinate neighborhood of p which is a Cousin-II domain. Then, f has a reduced representation on U, namely, a representation $f = (f_1: \cdots : f_{N+1})$ such that each f_i is a holomorphic function on U and $f(z) = (f_1(z): \cdots : f_{N+1}(z))$ outside the analytic set $\{z \in U: f_i(z) = 0, 1 \leqq i \leqq N+1\}$ of codimension $\geqq 2$. For a set $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers α_i, we set

$$D^\alpha \boldsymbol{f} = \left(\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} f_1, \cdots, \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} f_{N+1}\right) \in \mathscr{M}_p^{N+1},$$

where we mean $D^0 \boldsymbol{f} = \boldsymbol{f} := (f_1, \cdots, f_{N+1})$. For each $k \geqq 0$ we denote by $\mathscr{F}_p^k$ the $\mathscr{M}_p$-submodule of $\mathscr{M}_p^{N+1}$ generated by $\{D^\alpha \boldsymbol{f}: |\alpha| \leqq k\}$ and set $\mathscr{F}_p^{-1} = \{0\}$.

DEFINITION 2.1. We define the *k-th rank* of f by

$$r_f(k) := \operatorname{rank}_{\mathscr{M}_p} \mathscr{F}_p^k - \operatorname{rank}_{\mathscr{M}_p} \mathscr{F}_p^{k-1}.$$

(2.2) *The k-th rank $r_f(k)$ does not depend on the choices of a point p, a reduced representation of f and holomorphic local coordinates $(z_1, \cdots, z_n)$.*

For the proof, see [10, §4].

DEFINITION 2.3. We define the *total rank* of f by

$$r_f := \sum_{k \geqq 0} r_f(k) - 1$$

and the *total degree of the Jacobian matrix* of f by

$$l_f := \sum_{k \geqq 0} k r_f(k) .$$

(2.4) (i) *$l_f \leqq N(N+1)/2$ for all meromorphic maps into $\boldsymbol{P}^N(\boldsymbol{C})$.*

(ii) *A meromorphic map $f: M \to \boldsymbol{P}^N(\boldsymbol{C})$ is nondegenerate, namely, has the image not contained in any hyperplane, if and only if $r_f = N$.*

For the proof, see [10, §4].

We now consider a meromorphic map f of $B(R_0) := \{z \in \boldsymbol{C}^n : \|z\| < R_0\}$ $(0 < R_0 \leqq +\infty)$ into $\boldsymbol{P}^N(\boldsymbol{C})$, where $\|z\| = (\sum_{i=1}^n |z_i|^2)^{1/2}$ for $z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n$ and we mean $B(\infty) = \boldsymbol{C}^n$. Taking a reduced representation $f = (f_1 : \cdots : f_{N+1})$ on $B(R_0)$, we set

$$\|f\| = (|f_1|^2 + \cdots + |f_{N+1}|^2)^{1/2} .$$

By definition, the pull-back of the normalized Fubini-Study metric form by f is given by

$$\Omega_f := dd^c \log \|f\|^2 .$$

We set $v_l = (dd^c \|z\|^2)^l$, $\sigma_n = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1}$ and $S(r) = \{z \in \boldsymbol{C}^n : \|z\| = r\}$.

DEFINITION 2.5. The *characteristic function* of f is defined by

$$T_f(r, r_0) := \int_{r_0}^{r} \frac{dt}{t^{2n-1}} \int_{B(t)} \Omega_f \wedge v_{n-1} \quad (0 < r_0 < r < R_0) .$$

We then have

$$T_f(r, r_0) = \int_{S(r)} \log \|f\| \sigma_n - \int_{S(r_0)} \log \|f\| \sigma_n . \tag{2.6}$$

For the proof, see [15, pp. 251-255].

Let ϕ be a nonzero meromorphic function on $B(R_0)$. We may regard ϕ as a meromorphic map into $\boldsymbol{P}^1(\boldsymbol{C})$. For each $a \in \boldsymbol{P}^1(\boldsymbol{C})$ we denote the zero multiplicity of $\phi - a$ at a point $z \in B(R_0)$ by $\nu_\phi^a(z)$. Set

$$n_\phi^a(r) = \begin{cases} \dfrac{1}{r^{2n-2}} \displaystyle\int_{\{\phi = a\} \cap B(r)} \nu_\phi^a v_{n-1} & \text{if } \quad n > 1 \\ \displaystyle\sum_{z \in B(r)} \nu_\phi^a(z) & \text{if } \quad n = 1 , \end{cases}$$

and define the *valence function* for a by

$$N_\phi^a(r, r_0) = \int_{r_0}^{r} \frac{n_\phi^a(t)}{t} dt \quad (0 < r_0 < r < R_0) .$$

We then have the following Jensen formula:

$$\int_{S(r)} \log|\phi|\sigma_n - \int_{S(r_0)} \log|\phi|\sigma_n = N_\phi^0(r, r_0) - N_\phi^\infty(r, r_0) . \tag{2.7}$$

For the proof, see [15, p. 248].

Let $f: B(R_0) \to \boldsymbol{P}^N(\boldsymbol{C})$ be a nondegenerate meromorphic map with a reduced representation $f = (f_1 : \cdots : f_{N+1})$ and set $\boldsymbol{f} = (f_1, \cdots, f_{N+1})$.

DEFINITION 2.8. Let $\alpha^i = (\alpha_1^i, \cdots, \alpha_n^i)$ $(1 \leqq i \leqq N+1)$ be $N+1$ sets of nonnegative integers. The *generalized Wronskian* of f (or of $\boldsymbol{f}$) is defined by

$$W_{\alpha^1\cdots\alpha^{N+1}}(f) \equiv W_{\alpha^1\cdots\alpha^{N+1}}(\boldsymbol{f}) := \det(D^{\alpha^i}\boldsymbol{f}: 1 \leqq i \leqq N+1) .$$

DEFINITION 2.9. We say that a system $\{\alpha^1, \cdots, \alpha^{N+1}\}$ $(\alpha^i = (\alpha_1^i, \cdots, \alpha_{N+1}^i))$ is *admissible* for f (or for $\boldsymbol{f}$) if for each $k \geqq 0$ $\{D^{\alpha^1}\boldsymbol{f}, \cdots, D^{\alpha^{l(k)}}\boldsymbol{f}\}$ gives a basis for the $\mathscr{M}_p$-module $\mathscr{F}_p^k$, where p is an arbitrarily chosen point in M and $l(k) = \operatorname{rank}_{\mathscr{M}_p} \mathscr{F}_p^k$.

For an admissible system $\{\alpha^1, \cdots, \alpha^{N+1}\}$ for $\boldsymbol{f}$ and a holomorphic function g on $B(R_0)$, we see

$$W_{\alpha^1\cdots\alpha^{N+1}}(g\boldsymbol{f}) = g^{N+1} W_{\alpha^1\cdots\alpha^{N+1}}(\boldsymbol{f}) . \tag{2.10}$$

For the proof, see [10, Proposition 4.9].

Now, let us consider q $(\geqq N+2)$ hyperplanes

$$H_j: a_j^1 w_1 + \cdots + a_j^{N+1} w_{N+1} = 0 \quad (1 \leqq j \leqq q)$$

in $\boldsymbol{P}^N(\boldsymbol{C})$ located in general position and set

$$F_j = a_j^1 f_1 + \cdots + a_j^{N+1} f_{N+1} = 0 \quad (1 \leqq j \leqq q) .$$

Taking an admissible system $\{\alpha^1, \cdots, \alpha^{N+1}\}$ for f, we define

$$\phi := \frac{W_{\alpha^1\cdots\alpha^{N+1}}(f)}{F_1 F_2 \cdots F_q} , \tag{2.11}$$

which is a nonzero meromorphic function on $B(R_0)$. In this situation, we can prove:

PROPOSITION 2.12. *Let $0 < r_0 < R_0$ and $0 < l_f t < p' < 1$. Then, there xists a constant $K > 0$ such that for $r_0 < r < R < R_0$*

$$\int_{S(r)} |z^{\alpha^1 + \cdots + \alpha^{N+1}}\phi|^t \|f\|^{t(q-N-1)}\sigma_n \leqq K\left(\frac{R^{2n-1}}{R-r} T_f(R, r_0)\right)^{p'} ,$$

where $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for $z = (z_1, \cdots, z_n)$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$.

For the proof, see [10, Proposition 6.1].

For real-valued functions $f(r)$ and $g(r)$ on $[r_0, R_0)$ by notation $f(r) \leqq$

$g(r)\|$ we mean that $f(r) \leqq g(r)$ on $[r_0, R_0)$ outside a set E such that $\int_E dr < \infty$ in case $R_0 = \infty$ and $\int_E (R_0 - r)^{-1}dr < \infty$ in case $R_0 < \infty$. We can conclude from Proposition 2.12 the second main theorem in value distribution theory, which is stated as follows.

THEOREM 2.13. *Let $f: B(R_0) \to \boldsymbol{P}^N(\boldsymbol{C})$ be a nondegenerate meromorphic map and $H_1, \cdots, H_q$ be hyperplanes in general position. Then,*

$$(q - N - 1)T_f(r, r_0) \leqq N_\phi^\infty(r, r_0) + S_f(r) ,$$

where there exists a positive constant K such that

$$1^\circ \quad S_f(r) \leqq l_f \log \frac{1}{R_0 - r} + K \log^+ T_f(r, r_0)\| \quad if \quad R_0 < \infty$$

$$2^\circ \quad S_f(r) \leqq K(\log^+ T_f(r, r_0) + \log r)\| \quad if \quad R_0 = \infty .$$

The proof is given by the same argument as in the proof of [10, Proposition 6.2].

REMARK 2.14. In Theorem 2.13, if $R_0 = \infty$ and $\lim_{r\to\infty} T_f(r, r_0)/\log r < \infty$, or equivalently f is rational, then we can choose $S_f(r)$ to be bounded.

3. A lemma. Let $f: B(R_0) \to \boldsymbol{P}^N(\boldsymbol{C})$ be a nondegenerate meromorphic map with a reduced representation $f = (f_1: \cdots : f_{N+1})$.

DEFINITION 3.1. As stated in Section 1, we define

$$m_f := \sum_{k,l} (k - l)^+ \min\left\{{}_{n-1}H_l, \left(r_f(k) - \sum_{\lambda=0}^{l-1} {}_{n-1}H_\lambda\right)^+\right\} .$$

(3.2) It holds that $\qquad m_f \leqq l_f .$

Indeed, if we set $A(l) = \sum_{\lambda=0}^{l-1} {}_{n-1}H_\lambda$ and $l_0^k := \max\{l: A(l) \leqq r_f(k)\}$, then

$$\begin{aligned} m_f &\leqq \sum_{k\geqq 0} \left(\sum_{l=0}^{l_0^k - 1} (k - l)_{n-1}H_l + k(r_f(k) - A(l_0^k))\right) \\ &\leqq \sum_{k\geqq 0} k(A(l_0^k) + (r_f(k) - A(l_0^k))) = l_f . \end{aligned}$$

EXAMPLE 3.3. Suppose that $N = n$ and f is of rank N, namely, the Jacobian of f does not vanish at a point $p \notin I_f$, where I_f denotes the set of all indeterminate points of f. Then, we have $m_f = 1$ and $l_f = N$.

To see this, we take a point $p \notin I_f$ and a system of holomorphic local coordinates $z_1, \cdots, z_n$ around p. Changing indices if necessary, we may assume that $f_{N+1}(p) \neq 0$. Then, the Jacobian of f is given by

$$J_f := \det\left(\frac{\partial}{\partial z_i}\left(\frac{f_j}{f_{N+1}}\right): 1 \leqq i, j \leqq N\right) .$$

On the other hand, if we take

$$\alpha^1 = (0, \cdots, 0)\,, \qquad \alpha^2 = (1, 0, \cdots, 0)\,,$$
$$\alpha^3 = (0, 1, 0, \cdots, 0), \cdots, \qquad \alpha^{N+1} = (0, \cdots, 0, 1)\,,$$

then $W_{\alpha^1 \cdots \alpha^{N+1}}(f) = (-1)^N f_{N+1}^{N+1} J_f \not\equiv 0$. This shows that $m_f = 1$ and $l_f = N$.

Taking hyperplanes H_j $(1 \leqq j \leqq q)$ and an admissible system $\{\alpha^1, \cdots, \alpha^{N+1}\}$ for f, we consider as in the previous section holomorphic functions F_j and define the meromorphic functions ϕ by (2.11).

The purpose of this section is to prove:

LEMMA 3.4. *$\nu_\phi^\infty(p) \leqq m_f$ on $B(R_0)$ outside an analytic subset of codimension $\geqq 2$.*

For our purpose, we first note the following:

(3.5) *If $\{\alpha^i := (\alpha_1^i, \cdots, \alpha_n^i) : 1 \leqq i \leqq N+1\}$ is an admissible system for f, then*

$$\sum_{j=1}^{N+1} \alpha_i^j \leqq m_f \quad (1 \leqq i \leqq N+1)\,.$$

PROOF. Without loss of generality, we may assume $i = 1$. For each $k \geqq 0$, the number of j's with $|\alpha^j| = k$ is just $r_f(k)$. For each $l \leqq k$ the number of α's with $|\alpha| = k$ and $\alpha_1 = k - l$ is ${}_{n-1}H_l$ $(= {}_{n+l-2}C_l)$. If we choose α^j with $|\alpha^j| = k$ so that $\sum_{|\alpha^j|=k} \alpha_1^j$ attains the maxmum among all possible choices, we see

$$\sum_{|\alpha^j|=k} \alpha_1^j = \sum_l (k-l)^+ \min\{{}_{n-1}H_l, (r_f(k) - A(l))^+\}\,.$$

This concludes (3.5).

We next prove:

(3.6) *Let $\boldsymbol{F} = (F_1, \cdots, F_{N+1})$ be a system of holomorphic functions on $B(R_0)$ such that $F_1, \cdots, F_{N+1}$ are linearly independent over $\boldsymbol{C}$ and let $\{\alpha^1, \cdots, \alpha^{N+1}\}$ be an admissible system for $\boldsymbol{F}$. Take an arbitrary system of holomorphic local coordinates $u_1, \cdots, u_n$ around a point $p \in B(R_0)$. Then, there exist finitely many systems $\{\beta(\tau)^1, \cdots, \beta(\tau)^{N+1}\}$ $(1 \leqq \tau \leqq t)$ such that for each $k \geqq 0$*

$${}^uD^{\beta(\tau)^1}\boldsymbol{F}, \cdots, {}^uD^{\beta(\tau)^{l(k)}}\boldsymbol{F} \quad (l(k) = \operatorname{rank}_{\mathscr{M}_p}\mathscr{F}_p^k)$$

give a basis for the $\mathscr{M}_p$-module $\mathscr{F}_p^k$ and we can write

$$W_{\alpha^1 \cdots \alpha^{N+1}}(\boldsymbol{F}) = \sum_{\tau=1}^{t} h_\tau \det({}^uD^{\beta(\tau)^i} F_j : 1 \leqq i, j \leqq N+1)$$

with suitable holomorphic functions h_τ on a neighborhood of p, where we mean ${}^uD^\beta \boldsymbol{F} = (\partial^{|\beta|}/\partial u_1^{\beta_1} \cdots \partial u_n^{\beta_n})\boldsymbol{F}$ for $\beta = (\beta_1, \cdots, \beta_n)$.

PROOF. As is easily seen by induction on $|\alpha|$, each $D^{\alpha}\boldsymbol{F}$ can be written as

$$D^{\alpha}\boldsymbol{F}=\sum_{|\beta|\leqq|\alpha|} g_{\alpha\beta}{}^{u}D^{\beta}\boldsymbol{F}$$

with suitable holomorphic functions $g_{\alpha\beta}$ on a neighborhood of p. Since the determinant is linear as a function of each row vector, we get

$$W_{\alpha^1\cdots\alpha^{N+1}}(\boldsymbol{F})=\sum_{|\beta^i|\leqq|\alpha^i|} g_{\alpha^1\beta^1}\cdots g_{\alpha^{N+1}\beta^{N+1}}\det({}^{u}D^{\beta^i}\boldsymbol{F}\colon 1\leqq i\leqq N+1)\,.$$

Take a system $\{\beta^1,\cdots,\beta^{N+1}\}$ with $|\beta^i|\leqq|\alpha^i|$ $(1\leqq i\leqq N+1)$ and set $N_0:=\#\{i\colon|\beta^i|\leqq k\}$, where $\#A$ denotes the number of elements of a set A. We see easily $N_0\geqq l(k)$ because $|\beta^i|\leqq|\alpha^i|$ and $\#\{i\colon|\alpha^i|\leqq k\}=l(k)$. On the other hand, since $l(k)=\sum_{\kappa=0}^{k}r_f(\kappa)$ does not depend on the choice of holomorphic local coordinates by (2.2), we have necessarily $\det({}^{u}D^{\beta^i}F\colon 1\leqq i\leqq N+1)=0$ if $N_0>l(k)$. So we consider the only case where $\#\{i\colon|\beta^i|\leqq k\}=l(k)$. We denote by $\{\beta(\tau)^1,\cdots,\beta(\tau)^{N+1}\}$ $(1\leqq\tau\leqq t)$ all systems $\{\beta^1,\cdots,\beta^{N+1}\}$ such that

$$g_{\alpha^1\beta^1}\cdots g_{\alpha^{N+1}\beta^{N+1}}\det({}^{u}D^{\beta^i}\boldsymbol{F}\colon 1\leqq i\leqq N+1)\neq 0\,,$$

and set $h(\tau):=g_{\alpha^1\beta(\tau)^1}\cdots g_{\alpha^{N+1}\beta(\tau)^{N+1}}$. We then have the desired representation of $W_{\alpha^1\cdots\beta^{N+1}}(\boldsymbol{F})$.

PROOF OF LEMMA 3.4. Since $f=(f_1\colon\cdots\colon f_{N+1})$ is a reduced representation, the analytic set $I_f=\{f_1=\cdots=f_{N+1}=0\}$ is of codimension $\geqq 2$. On the other hand, if we set $Z:=\{z\in B(R_0);\,(F_1F_2\cdots F_q)(z)=0\}$, the set $S(Z)$ of all singularities of Z is an analytic set of codimension $\geqq 2$. We have only to show $\nu_{\phi}^{\infty}(p)\leqq m_f$ for each $p\in Z\setminus(I_f\cup S(Z))$.

Changing indices, we may assume that

$$|F_1(p)|\leqq|F_2(p)|\leqq\cdots\leqq|F_q(p)|\,.$$

By assumption, $f_1,\cdots,f_{N+1}$ can be written as a linear combination of $F_1,\cdots,F_{N+1}$ with constant coefficients. The assumption $p\notin I_f$ implies that $F_{N+1}(p)\cdots F_q(p)\neq 0$. Set

$$\tilde{\phi}:=\frac{W_{\alpha^1\cdots\alpha^{N+1}}((F_1,\cdots,F_{N+1}))}{F_1F_2\cdots F_{N+1}}\,.$$

Then, we can write

$$\phi=\frac{c}{F_{N+2}\cdots F_q}\tilde{\phi}$$

with a nonzero constant c and the function $c/(F_{N+2}\cdots F_q)$ is holomorphic and has no zero in a neighborhood of p. Therefore, we have $\nu_{\phi}^{\infty}(p)=\nu_{\tilde{\phi}}^{\infty}(p)$.

By the assumption $p \in Z \setminus S(Z)$, we can choose a system of holomorphic local coordinates $u_1, \cdots, u_n$ on a neighborhood U of p with $p = (0)$ such that $Z \cap U = \{u_1 = 0\}$. Then, we can write

$$F_i(u) = u_1^{p_i} g_i(u) \quad (1 \leqq i \leqq N+1)$$

with nowhere zero holomorphic functions $g_i(u)$ on U. By the help of (3.6), taking an arbitrary system $\beta^1, \cdots, \beta^{N+1}$ such that ${}^uD^{\beta^1}F, \cdots, {}^uD^{\beta^{l(k)}}F$ give a basis for the $\mathscr{M}_p$-module $\mathscr{F}_p^k$, we have only to show $\nu_{\phi^*}^{\infty}(p) \leqq m_f$ for the function

$$\phi^* := \frac{\det({}^uD^{\beta^i}F_j\colon 1 \leqq i, j \leqq N+1)}{F_1 F_2 \cdots F_{N+1}} .$$

For each $\beta = (\beta_1, \cdots, \beta_n)$, we have

$${}^uD^{\beta}F_i = \frac{\partial^{\beta_1}}{\partial u_1^{\beta_1}}\left(u_1^{p_i}\frac{\partial^{\beta_2+\cdots+\beta_n}}{\partial u_2^{\beta_2}\cdots\partial u_n^{\beta_n}}g_i(u)\right)$$

and hence we can easily show that ${}^uD^{\beta}F_i/F_i$ has a pole of order $\leqq \beta_1$ at p. Since

$$\phi^* = \sum_{\sigma = \binom{1\ 2\ \cdots N+1}{\sigma_1 \sigma_2 \cdots \sigma_{N+1}}} \operatorname{sgn}(\sigma) \frac{{}^uD^{\beta^1}F_{\sigma_1}}{F_{\sigma_1}} \cdots \frac{{}^uD^{\beta^{N+1}}F_{\sigma_{N+1}}}{F_{\sigma_{N+1}}} ,$$

we can conclude from (3.5)

$$\nu_{\phi^*}^{\infty}(p) \leqq \beta_1^1 + \beta_1^2 + \cdots + \beta_1^{N+1} \leqq m_f .$$

Therefore, we have Lemma 3.4.

4. The proof of Main Theorem for particular cases. The purpose of this section is to prove Corollary 2 stated in Section 1 and to give a unicity theorem for meromorphic maps of the unit ball into $\boldsymbol{P}^N(\boldsymbol{C})$ with suitable growth condition.

Let us consider two distinct nondegenerate meromorphic maps $f, g\colon B(R_0) \to \boldsymbol{P}^N(\boldsymbol{C})$ and assume that there exist q hyperplanes $H_1, \cdots, H_q$ in general position satisfying the condition (i) of Main Theorem. Take reduced representations $f = (f_1 \colon \cdots : f_{N+1})$ and $g = (g_1 \colon \cdots : g_{N+1})$ and set $\|f\| := (\sum_{i=1}^{N+1} |f_i|^2)^{1/2}$, $\|g\| = (\sum_{i=1}^{N+1} |g_i|^2)^{1/2}$. Let

$$H_j\colon a_j^1 w_1 + \cdots + a_j^{N+1} w_{N+1} = 0 \quad (1 \leqq j \leqq q) .$$

As in the previous sections, setting $F_j = \sum_{i=1}^{N+1} a_j^i f_i$ and $G_j = \sum_{i=1}^{N+1} a_j^i g_i$, we define the function ϕ by (2.11) and the function ψ by

$$\psi := \frac{W_{\beta 1 \ldots \beta N+1}(g)}{G_1 G_2 \cdots G_q} ,$$

where $\{\alpha^1, \cdots, \alpha^{N+1}\}$ and $\{\beta^1, \cdots, \beta^{N+1}\}$ are admissible systems for f and g, respectively. Then, according to Theorem 2.13 we get

$$(4.1) \qquad \begin{aligned}&(q-N-1)(T_f(r, r_0) + T_g(r, r_0)) \\ &\qquad \leqq N_\phi^\infty(r, r_0) + N_\psi^\infty(r, r_0) + S_f(r) + S_g(r)\ ,\end{aligned}$$

where $S_f(r)$ and $S_g(r)$ denote real-valued functions of r with the properties stated in Theorem 2.13 for maps f and g, respectively.

Now, we choose distinct indices i_0 and j_0 such that

$$\chi := f_{i_0}g_{j_0} - f_{j_0}g_{i_0} \not\equiv 0\ .$$

If $\nu_\phi^\infty(p) > 0$ for a point $p \in \mathrm{B}(R_0)$, then $F_j(p) = 0$ for some j $(1 \leqq j \leqq q)$ and so $p \in \cup_{j=1}^q f^{-1}(H_j)$. By assumption, we have $\chi(p) = 0$. Accordingly, we conclude from Lemma 3.4 that $\nu_\phi^\infty \leqq m_f \nu_\chi^0$ outside an analytic set of codimension $\geqq 2$, and hence

$$N_\phi^\infty(r, r_0) \leqq m_f N_\chi^0(r \cdot r_0) \quad (r_0 < r < R_0)\ .$$

Similarly, we have

$$N_\psi^\infty(r, r_0) \leqq m_g N_\chi^0(r, r_0) \quad (r_0 < r < R_0)\ .$$

On the other hand, since $|\chi| \leqq 2\|f\|\,\|g\|$, it follows from (2.6) and (2.7) that

$$\begin{aligned}N_\chi^0(r, r_0) &\leqq \int_{S(r)} \log|\chi|\sigma_n + O(1) \leqq \int_{S(r)} \log\|f\|\sigma_n + \int_{S(r)} \log\|g\|\sigma_n + O(1) \\ &\leqq T_f(r, r_0) + T_g(r, r_0) + O(1)\ ,\end{aligned}$$

where $O(1)$ denotes a bounded term. We thus conclude from (4.1)

$$(4.2) \qquad (q - N - 1 - m_f - m_g)(T_f(r, r_0) + T_g(r, r_0)) \leqq S_f(r) + S_g(r)\ .$$

Proof of Corollary 2. We now proceed to prove Corollary 2 stated in Section 1. We first consider the case where f and g are rational. Then, $S_f(r)$ and $S_g(r)$ can be taken to be bounded according to Remark 2.14. Then,

$$\lim_{r\to\infty} \frac{S_f(r) + S_g(r)}{T_f(r, r_0) + T_g(r, r_0)} = 0\ ,$$

and hence $q \leqq N + 1 + m_f + m_g$ as a result of (4.2). We next assume that f or g is transcendental. In this case, we have

$$\lim_{r\to\infty} \frac{\log r}{T_f(r, r_0) + T_g(r, r_0)} = 0\ .$$

On the other hand, by Theorem 2.13 we have

$$\frac{S_f(r) + S_g(r)}{T_f(r, r_0) + T_g(r, r_0)} \leqq \frac{K(\log^+(T_f(r, r_0) + T_g(r, r_0)) + \log r)}{T_f(r, r_0) + T_g(r, r_0)} \Big\| .$$

Therefore, we obtain

$$\liminf_{r\to\infty} \frac{S_f(r) + S_g(r)}{T_f(r, r_0) + T_g(r, r_0)} = 0 .$$

From this and (4.2) we conclude $q \leqq N + 1 + m_f + m_g$ in this case too. This completes the proof of Corollary 2.

Next, we consider meromorphic maps of the unit ball into $\boldsymbol{P}^N(\boldsymbol{C})$. We shall prove the following:

THEOREM 4.3. *Let f, g be nondegenerate meromorphic maps of the unit ball $B(1)$ into $\boldsymbol{P}^N(\boldsymbol{C})$. Suppose that*

$$\lambda := \limsup_{r\to\infty} \frac{\log(1/(1-r))}{T_f(r, r_0) + T_g(r, r_0)} < \infty .$$

If there exist q hyperplanes $H_1, \cdots, H_q$ in general position such that
(i) $f = g$ *on* $\bigcup_{j=1}^q f^{-1}(H_j) \cup g^{-1}(H_j)$,
(ii) $q > N + 1 + \lambda(l_f + l_g) + m_f + m_g$,
then $f \equiv g$.

PROOF. It suffices to show that

$$q \leqq N + 1 + \lambda(l_f + l_g) + m_f + m_g$$

under the assumption that $f \not\equiv g$ and they satisfy the condition (i) of Theorem 4.3. Theorem 2.13 implies that there exists a subset E of $[0, 1)$ such that $\int_E (1-r)^{-1} dr < \infty$ and, for every $r \notin E$,

$$\frac{S_f(r) + S_g(r)}{T_f(r, r_0) + T_g(r, r_0)} \leqq \frac{(l_f + l_g)\log(1/(1-r)) + K \log^+(T_f(r, r_0) + T_g(r, r_0))}{T_f(r, r_0) + T_g(r, r_0)} .$$

From this and (4.2), we can conclude

$$\begin{aligned} & q - N - 1 - m_f - m_g \\ & \quad \leqq (l_f + l_g) \liminf_{r\to 1, r\notin E} \frac{\log(1/(1-r)) + K\log^+(T_f(r, r_0) + T_g(r, r_0))}{T_f(r, r_0) + T_g(r, r_0)} \\ & \quad \leqq (l_f + l_g) \limsup_{r\to 1} \frac{\log(1/(1-r))}{T_f(r, r_0) + T_g(r, r_0)} \leqq \lambda(l_f + l_g) . \end{aligned}$$

REMARK 4.4. As is easily seen from the above proof, the quantity λ in the conclusion of Theorem 4.3 can be replaced by the least upper bound of the quantities $\tilde{\lambda}$ such that

$$\tilde{\lambda} = \liminf_{\lambda \to 1, r \notin E} \frac{\log(1/(1-r))}{T_f(r, r_0) + T_g(r, r_0)}$$

for some subset E of $[0, 1)$ with $\int_E (1-r)^{-1} dr < \infty$.

5. Proof of Main Theorem. We now proceed to prove Main Theorem. We first note:

(5.1) *For the proof of Main Theorem, we may assume that* $M = B(R_0)(\subset \boldsymbol{C}^n)$.

To see this, we consider the universal covering $\pi: \tilde{M} \to M$. For meromorphic maps f, g satisfying the assumptions of Main Theorem, if we set $\tilde{f} := f \circ \pi$ and $\tilde{g} := g \circ \pi$, they satisfy all assumptions of Main Theorem as meromorphic maps of the complete Kähler manifold $\tilde{M}$ with metric induced from M through π into $\boldsymbol{P}^N(\boldsymbol{C})$. Since $\tilde{f} = \tilde{g}$ on $\tilde{M}$ implies $f = g$ on M, we may assume $\tilde{M} = M$ for our purpose. Moreover, we may assume that $M = B(R_0)(0 < R_0 \leqq +\infty)$ by the assumption of Main Theorem.

Let $f, g: B(R_0) \to \boldsymbol{P}^N(\boldsymbol{C})$ be nondegenerate meromorphic maps satisfying all assumptions of Main Theorem. We shall show that they lead to a contradiction under the assumption $f \not\equiv g$. We use the same notations as in the previous section.

(5.2) *For the proof of Main Theorem we may assume that* $M = B(1)$ *and there exists a positive constant* K *such that*

$$T_f(r, r_0) + T_g(r, r_0) \leqq K \log \frac{1}{1-r} \quad (0 < r_0 \leqq r < 1). \tag{5.3}$$

In fact, the case $M = \boldsymbol{C}^n$ is nothing but Corollary 2 and so it suffices to study the case $M = B(1)$. Moreover, by virtue of Remark 4.4, Main Theorem is true unless there exist a subset E of $[0, 1)$ such that $\int_E (1-r)^{-1} dr < \infty$ and

$$\limsup_{r \to 1, r \notin E} \frac{T_f(r, r_0) + T_g(r, r_0)}{\log(1/(1-r))} < \infty. \tag{5.4}$$

On the other hand, by the same argument as in the proof of [9, Proposition 5.5] we can easily show that (5.4) implies (5.3).

Now, we represent the given Kähler metric form as

$$\omega = \sum_{i,j} h_{i\bar{j}} \frac{\sqrt{-1}}{2} dz_i \wedge d\bar{z}_j$$

on $B(1)$. By assumption we can take continuous plurisubharmonic functions u_1 and u_2 on $B(1)$ such that

$$e^{u_1}\det(h_{i\bar{j}})^{1/2} \leqq \|f\|^{\rho}\ ,$$
$$e^{u_2}\det(h_{i\bar{j}})^{1/2} \leqq \|g\|^{\rho}$$

(cf. Remark to [10, Definition 5.9]). Set $\tilde{\phi} := z^{\alpha^1+\cdots+\alpha^{N+1}}\phi$ and $\tilde{\psi} := z^{\beta^1+\cdots+\beta^{N+1}}\psi$. Since $\nu_{\tilde{\phi}}^{\infty} \leqq m_f\nu_{\chi}^0$ and $\nu_{\tilde{\psi}}^{\infty} \leqq m_g\nu_{\chi}^0$ outside an analytic set of codimension $\geqq 2$, the functions $\tilde{\phi}\chi^{m_f}$ and $\tilde{\psi}\chi^{m_g}$ are both holomorphic on $B(1)$. Therefore, if we set

$$t := \frac{\rho}{q-N-1-m_f-m_g}$$

and define

$$w := t\log|\tilde{\phi}\tilde{\psi}\chi^{m_f+m_g}|\ ,$$

then w is a plurisubharmonic function on $B(1)$. Since $t(m_f+m_g)+\rho = t(q-N-1)$ and $|\chi| \leqq 2\|f\|\,\|g\|$, we obtain

$$\begin{aligned}&\det(h_{i\bar{j}})e^{w+u_1+u_2}\\ &\quad\leqq |\tilde{\phi}|^t|\tilde{\psi}|^t|\chi|^{t(m_f+m_g)}\|f\|^{\rho}\|g\|^{\rho}\\ &\quad\leqq K_1|\tilde{\phi}|^t|\tilde{\psi}|^t\|f\|^{t(q-N-1)}\|g\|^{t(q-N-1)}\ ,\end{aligned}$$

where K_i are some positive constants. The volume form on M is given by

$$dV := c_n\det(h_{i\bar{j}})v_n\ ,$$

where c_n is a positive constant. Therefore, we have

$$\begin{aligned}I &:= \int_{B(1)} e^{w+u_1+u_2}dV\\ &\leqq K_2\int_{B(1)}|\tilde{\phi}|^t\|f\|^{t(q-N-1)}|\tilde{\psi}|^t\|g\|^{t(q-N-1)}v_n\ .\end{aligned}$$

Setting $p_1 = (l_f+l_g)/l_f$ and $p_2 = (l_f+l_g)/l_g$, we apply Hölder's inequality to obtain

$$I \leqq K_2\Big(\int_{B(1)}|\tilde{\phi}|^{tp_1}\|f\|^{tp_1(q-N-1)}v_n\Big)^{1/p_1}\Big(\int_{B(1)}|\tilde{\psi}|^{tp_2}\|g\|^{tp_2(q-N-1)}v_n\Big)^{1/p_2}\ .$$

Here, we see

$$(tp_1)l_f = (tp_2)l_g = t(l_f+l_g) = \frac{\rho(l_f+l_g)}{q-N-1-m_f-m_g} < 1\ .$$

Take some p' with $t(l_f+l_g) < p' < 1$. Then, by the help of Proposition 2.12, for $r_0 < r < R < R_0$

$$\int_{S(r)} |\tilde{\phi}|^{tp_1} \|f\|^{tp_1(q-N-1)} \sigma_n \leqq K_3 \Big(\frac{1}{R-r} T_f(R, r_0) \Big)^{p'} ,$$

$$\int_{S(r)} |\tilde{\psi}|^{tp_2} \|g\|^{tp_2(q-N-1)} \sigma_n \leqq K_4 \Big(\frac{1}{R-r} T_g(R, r_0) \Big)^{p'} .$$

By the same argument as in the proof of [10, Theorem 5.10], we can conclude

$$\int_{B(1)} e^{w+u_1+u_2} dV < \infty .$$

On the other hand, by the result of Yau ([17]) and Karp ([11]), we have necessarily

$$\int_{B(1)} e^{w+u_1+u_2} dV = \infty ,$$

because $w + u_1 + u_2$ is plurisubharmonic. This is a contradiction. Thus, Main Theorem is proved.

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