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OPERATOR-VALUED MARTINGALE TRANSFORMS

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Abstract. We develop a general theory of martingale transform operators with operator-valued multiplying sequences. Applications are given to classical operators such as Doob's maximal function and the square function. Some geometric properties of the underlying Banach spaces are also considered.

Introduction. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n\}_{n\geq 1}$ a nondecreasing sequence of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F} = V\mathcal{F}_n$. A martingale relative to $\{\mathcal{F}_n\}_{n\geq 1}$ is a sequence $f = \{f_n\}_{n\geq 1}$ of integrable variables such that each f_n is $\mathcal{F}_n$-measurable and $E(f_{n+1}|\mathcal{F}_n) = f_n$, in particular, $f_n = \sum_{k=1}^n d_k$, where d_k are the "increments" of the martingale f, i.e., $d_k = f_k - f_{k-1}$.

Given a uniformly bounded sequence of $\mathcal{F}_n$-predictable random variables $v = \{v_n\}_{n\geq 1}$ (i.e., v_n is $\mathcal{F}_{n-1}$-measurable), the martingale given by $(Tf)_n = \sum_{k=1}^n v_k d_k$ is called the martingale transform of f by the multiplying sequence v.

Martingale transform operators were introduced by Burkholder in [8]. Two objects were fundamental in this theory, namely "Doob's maximal function" defined by $f^*(\omega) = \sup_n |f_n(\omega)|$ and the "square function" $Sf = (\sum_{k=1}^\infty |d_k|^2)^{1/2}$. Their boundedness and the relation among them have been extensively studied, see [1], [8], [10], [14], [16], [19], [24].

The interplay between probability and harmonic analysis has been very successful, see for example [10], [11], [2], [4], [3] and [5]. There exists a large amount of objects in both fields that play a parallel role, namely "good λ" inequalities, maximal and square functions, Hardy and BMO spaces, etc. The well-known Calderón-Zygmund decomposition (see [12]), in Harmonic Analysis, of an integrable function has, in Probability, a counterpart due to Gundy, see [19].

In writing this paper we have been especially influenced by works of Benedek, Calderón and Panzone ([6]), Burkholder ([9]) and Rubio de Francia ([23]).

Our paper has two major aims. Firstly, we develop a general theory of martingale transform operators with operator-valued multiplying sequences, and secondly (we think this is the main contribution of this paper) we give several applications.

The first point is developed in Section 2, see Theorems 1 and 2. The philosophy behind these theorems is that the knowledge of the boundedness of the martingale transform operator in some fixed level, say, strong p with $p > 1$, weak $(1, 1)$, or even a convergence condition for martingales in L^1, is sufficient to assure the boundedness at the rest

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of the levels. Some related results in this direction are in a work by Burkholder, see [9]. He studied the class of Banach spaces B for which there exists a real number c_p such that $\|\varepsilon_1 d_1 + \cdots + \varepsilon_n d_n\|_{L^p_B} \leq c_p \|d_1 + \cdots + d_n\|_{L^p_B}$ for all B-valued martingale difference sequences $(d_1, d_2, \dots)$, all numbers $\varepsilon_1, \varepsilon_2, \dots$ in $\{-1, 1\}$, and all $n > 1$. Burkholder called this class "*UMD*" (unconditionality property for martingale differences). The technical proofs of these general results are given in Section 4. We follow some ideas developed by Burkholder, and also use a vector-valued version of Gundy's decomposition.

Our applications arise from considering Doob's maximal operator (Section 3.1) and the square function (Section 3.2) essentially as martingale transform operators given, respectively, by ℓ^∞-valued and ℓ^2-valued multiplying sequences. In particular, this allows us to get all known results about the square function as easy corollaries of the obvious L^2 boundedness. Particular relevance has an easy and straightforward proof of the equivalence, due to Davis (see [14]), $\|f^*\|_{L^1} \sim \|Sf\|_{L^1}$ (see (5) and (9) in Section 3.2). Moreover, some new ℓ^q-valued extensions are obtained, see Corollary 1. From the fact that the square function is essentially a martingale transform operator, we obtain some new characterizations of Hilbert spaces and UMD Banach lattices in terms of the existence of the square function; see Theorems 4, 5, 6. For example, we get the following results:

(1) *A Banach space B is isomorphic to a Hilbert space if and only if there exists a constant $c > 0$ such that $c^{-1}\|f^*\|_{L^1} \leq \|Sf\|_{L^1} \leq c\|f^*\|_{L^1}$, where Sf stands for the square function of f, i.e., $Sf = (\sum_{k=1}^{\infty} \|d_k\|_B^2)^{1/2}$.*

(2) *Let X be a Köthe lattice with Fatou property. Then X is UMD if and only if for every X-valued martingale f, it holds that, defining $S_X f = (\sum_{k=1}^{\infty} |d_k|^2)^{1/2}$,*

$$\|f\|_{L^1_X} < \infty \Rightarrow S_X f \in X \textit{ a.e., and } S_X f \in L^1_X \Rightarrow f \textit{ converges a.e.}$$

The organization of the paper is as follows. Notation and known results are collected in Section 1. Section 2 is devoted to a general setting of martingale transform operators with operator valued multiplying sequences. We give in Section 3 several applications. The technical proofs are collected in Section 4.

We would like to thank R. Gundy for many enlightening conversations and comments on some topics discussed in the paper.

1. Preliminaries. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n\}_{n\geq 1}$ a nondecreasing sequence of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F} = V\mathcal{F}_n$. Given a Banach space B, by a B-valued martingale relative to $\{\mathcal{F}_n\}_{n\geq 1}$ we mean a sequence $f = \{f_n\}_{n\geq 1}$ of B-valued variables such that f_n is $\mathcal{F}_n$-measurable, $E(\|f_n\|_B) < \infty$ and $E(f_{n+1}|\mathcal{F}_n) = f_n$ for every $n \geq 1$.

For any Banach space B and any B-valued random variable f defined on $(\Omega, \mathcal{F}, P)$, for p, $1 \leq p < \infty$, let

$$\|f\|_{L^p_B} = [E(\|f\|_B^p)]^{1/p}$$

be its L^p_B-norm, and in the case $p = \infty$, set

$$\|f\|_{L^\infty_B} = \operatorname{ess\,sup} \|f\|_B\,.$$

The space L_B^p is the space of functions with finite L_B^p-norm. When the Banach space B in the definitions above is the scalar field, the subindex B will be dropped.

For every martingale $f = \{f_n\}_{n\geq 1}$ we shall denote by $d_k f$, or simply d_k, the "increments" of the martingale f, defined by $d_k = d_k f = f_k - f_{k-1}$. In particular, a B-martingale relative to $\{\mathcal{F}_n\}_{n\geq 1}$ can be always expressed as $f_n = \sum_{k=1}^n d_k$, where d_k is $\mathcal{F}_k$-measurable, $\|d_k\|_{L_B^1}$ is finite and $E(d_{k+1}|\mathcal{F}_k) = 0$, $k \geq 1$. For a background on B-valued martingales, see [15].

Given a B-valued martingale f, we say that the martingale is L_B^p-bounded, $1 \leq p \leq \infty$, if $\|f\|_{L_B^p} = \sup_n \|f_n\|_{L_B^p}$ is finite. *Doob's maximal function* of f is defined by $f^*(\omega) = \sup_n \|f_n(\omega)\|_B$, $f_n^*(\omega) \sup_{1\leq k\leq n} \|f_k(\omega)\|_B$.

In what follows, C will denote an absolute constant. When C depends on some parameter, it will appear as a subindex. In both cases, the constants denoted by the same expression are not necessarily the same from one occurrence to another.

REMARK 1. For every Banach space B the sequence $\{\|f_n\|_B\}_{n\geq 1}$ is a real-valued submartingale. Thus, we have, see [16],

$$\lambda P(f_n^* > \lambda) \leq C \int_{\{f_n^* > \lambda\}} \|f_n\|_B dP,$$

which implies $\lambda P(f_n^* > \lambda) \leq C\|f_n\|_{L_B^1}$ and $\|f_n^*\|_{L^p} \leq C_p \|f_n\|_{L_B^p}$, for all p, $1 < p < \infty$.

REMARK 2. Given a B-valued martingale f such that $f^* \in L^p$, we can decompose it in two martingales g and h such that $f_n = g_n + h_n$ for all $n \geq 1$, and with the following properties:

(1) $h_n = \sum_{k=1}^n \alpha_k$ verifies $\left\| \sum_{k=1}^\infty \|\alpha_k\|_B \right\|_{L^p} \leq (4+4p)\|f^*\|_{L^p}$.

(2) For $g_n = \sum_{k=1}^n \beta_k$, there exists an adapted, positive, increasing process $\{\lambda_n\}$ such that $\|g_n\|_B \leq \lambda_{n-1}$ and $\|\lambda^*\|_{L^p} \leq (13+4p)\|f^*\|_{L^p}$.

This decomposition is due to Davis, see [14]. The proof of the decomposition for scalar-valued martingales also holds in the vector-valued case, if we define

$$\alpha_k = d_k \chi_{\{f_k^* > 2f_{k-1}^*\}} - E(d_k \chi_{\{f_k^* > 2f_{k-1}^*\}}|\mathcal{F}_{k-1}),$$
$$\beta_k = d_k \chi_{\{f_k^* \leq 2f_{k-1}^*\}} - E(d_k \chi_{\{f_k^* \leq 2f_{k-1}^*\}}|\mathcal{F}_{k-1}).$$

REMARK 3 (Gundy's decomposition [19]). Let $f = \{f_n\}_{n\geq 1}$ be a martingale bounded in L_B^1 and λ a strictly positive number. Then there exist martingales a, b and e such that

(1) $f_n = a_n + b_n + e_n$ for all $n \geq 1$.

(2) $a_n = \sum_{k=1}^n \alpha_k$ with $\|a\|_{L_B^1} \leq C\|f\|_{L_B^1}$, and $\lambda P\left(\left\{\sup_{k\geq 1} \|\alpha_k\|_B \neq 0\right\}\right) \leq C\|f\|_{L_B^1}$.

(3) $b_n = \sum_{k=1}^n \beta_k$ such that $\int_\Omega \sum_{k=1}^\infty \|\beta_k\|_B dP \leq C\|f\|_{L_B^1}$.

(4) $e_n = \sum_{k=1}^{n} \delta_k$ verifying $\sup_{n\geq 1} \left\| \sum_{k=1}^{n} \delta_k \right\|_B \leq C\lambda$, and $\|e\|_{L^1_B} \leq C\|f\|_{L^1_B}$.

2. Martingale transforms.

2.1. Main theorems and some observations.

DEFINITION 1 (Martingale transform operator). Let B_1 and B_2 be two Banach spaces, $\{\mathcal{F}_n\}_{n\geq 1}$ an increasing sequence of σ-algebras in a probability space $(\Omega, \mathcal{F}, P)$, $f = \{f_n\}_{n\geq 1}$ a B_1-valued martingale relative to $\{\mathcal{F}_n\}_{n\geq 1}$. Define $v = \{v_n\}_{n\geq 1}$ a sequence such that

(1) $\{v_n\}_{n\geq 1}$ is $\mathcal{F}_n$-predictable, i.e., v_n is $\mathcal{F}_{n-1}$-measurable, for $n \geq 2$ and v_1 is $\mathcal{F}_1$-measurable,

(2) each v_n is $\mathcal{L}(B_1, B_2)$-valued,

(3) v is a uniformly bounded sequence, with $\sup_{n\geq 1} \|v_n\|_{L^\infty_{\mathcal{L}(B_1,B_2)}} \leq 1$.

Such a sequence $v = \{v_n\}_{n\geq 1}$ will be called a *multiplying sequence*. The martingale given by

$$(Tf)_n = \sum_{k=1}^{n} v_k d_k$$

is called the *martingale transform* of f by the multiplying sequence v, where $d_k = f_k - f_{k-1}$ are the martingale differences of f. T will denote the martingale transform operator.

Observe that the following theorems will also hold for general uniformly bounded multiplying sequences, by just changing the operators $\{v_n\}_{n\geq 1}$ with $\{v_n/M\}_{n\geq 1}$, where $M = \sup_n \|v_n\|_{L^\infty_{\mathcal{L}(B_1,B_2)}}$.

THEOREM 1. *Let B_1 and B_2 be Banach spaces and T a martingale transform operator as above. Then the following statements are equivalent*:

(i) *There exists $C > 0$ such that $\lambda P\{(Tf)^* > \lambda\} \leq C\|f\|_{L^1_{B_1}}$ for any $\lambda > 0$.*

(ii) *There exists $C > 0$ such that $\lambda P\{(Tf)^* > \lambda\} \leq C\|f^*\|_{L^1}$ for any $\lambda > 0$.*

(iii) *Given any p, $1 \leq p < \infty$, there exists $C_p > 0$ such that $\|(Tf)^*\|_{L^p} \leq C_p\|f^*\|_{L^p}$.*

(iv) *Given any p, $1 < p < \infty$, there exists $C_p > 0$ such that $\|(Tf)^*\|_{L^p} \leq C_p\|f\|_{L^p_{B_1}}$.*

(v) *There exist p_0, $1 < p_0 < \infty$, and a constant C_0 such that $\|(Tf)^*\|_{L^{p_0}} \leq C_0\|f\|_{L^{p_0}_{B_1}}$.*

If B_2 has the Radon-Nikodym property, then any of these conditions implies

$$(*) \qquad \|f\|_{L^1_{B_1}} < \infty \Rightarrow Tf \text{ converges a.s.}$$

COROLLARY 1. *Let B_1 and B_2 be Banach spaces and T be a martingale transform operator satisfying any of the statements* (i)–(v) *of Theorem* 1 *and $F = \{F_n\}_{n\geq 1}$ a $\ell^q(B_1)$-valued martingale, $F_n = \{f_n^j\}_{j\geq 1}$, where $1 < q < \infty$. We define the operator $\tilde{T}$ such that $(\tilde{T}F)_n = \{(Tf^j)_n\}_{j=1}^\infty$. Then $\tilde{T}$ is a martingale transform operator and satisfies the following*:

(i) *There exists $C_q > 0$ such that $\lambda P\{(\tilde{T}F)^* > \lambda\} \leq C_q\|F\|_{L^1_{\ell^q(B_1)}}$ for any $\lambda > 0$.*

(ii) *There exists $C_q > 0$ such that $\lambda P\{(\tilde{T}F)^* > \lambda\} \leq C_q\|F^*\|_{L^1}$ for any $\lambda > 0$.*

(iii) *Given any* p, $1 \leq p < \infty$, *there exists* $C_{p,q} > 0$ *such that*

$$\|(\tilde{T}F)^*\|_{L^p} \leq C_{p,q}\|F^*\|_{L^p}\,.$$

(iv) *Given any* p, $1 < p < \infty$, *there exists* $C_{p,q} > 0$ *such that*

$$\|(\tilde{T}F)^*\|_{L^p} \leq C_{p,q}\|F\|_{L^p_{\ell^q(B_1)}}\,.$$

(v) *There exist* p_0, $1 < p_0 < \infty$, *and a constant* $C_{0,q}$ *such that*

$$\|(\tilde{T}F)^*\|_{L^{p_0}} \leq C_{0,q}\|F\|_{L^{p_0}_{\ell^q(B_1)}}\,.$$

DEFINITION 2. Let B_1, B_2 be Banach spaces and T be a martingale transform operator with multiplying sequence $\{v_k\}_{k\geq 1} \subset \mathcal{L}(B_1, B_2)$. We say that T is a *translation invariant* martingale transform operator if for any $k_0 \in N$, the sequence $\{u_k^{k_0}\}_{k\geq 1}$, $u_k^{k_0} = v_{k_0+k}$, defines an operator T_{k_0}, $(T_{k_0}f)_n = \sum_{k=1}^n v_{k+k_0}d_k$, such that for any martingale f bounded in $L^1_{B_1}$, $\|(Tf)_n\|_{B_2} = \|(T_{k_0}f)_n\|_{B_2}$ a.e.

REMARK 4 (Examples of translation invariant martingale transform operators).

(1) Given a scalar-valued martingale $f_n = \sum_{k=1}^n d_k$, we consider the martingale transform from scalar-valued martingales into ℓ^∞-valued martingales with multiplying sequence $\{w_k\}_{k\geq 1} \subset \mathcal{L}(\boldsymbol{R}, \ell^\infty) \cong \ell^\infty$ given by $w_k = (0, \overset{(k-1)}{\ldots}, 0, 1, 1, 1\ldots..)$. Then

$$(Tf)_n = \sum_{k=1}^n w_k d_k = \left(d_1, d_1 + d_2 \ldots, \sum_{k=1}^n d_k, \sum_{k=1}^n d_k, \ldots\right),$$

$$(T_{k_0}f)_n = \sum_{k=1}^n w_{k_0+k}d_k = \left(0, \overset{(k_0-1)}{\ldots}, 0, d_1, d_1 + d_2, \ldots, \sum_{k=1}^n d_k, \sum_{k=1}^n d_k, \ldots\right),$$

and we have

$$\|(Tf)_n(\omega)\|_{\ell^\infty} = \sup_{1\leq k\leq n}\left|\sum_{j=1}^k d_j(\omega)\right| = \|(T_{k_0}f)_n(\omega)\|_{\ell^\infty}\,.$$

(2) Given a scalar-valued martingale $f_n = \sum_{k=1}^n d_k$, we consider the martingale transform from scalar-valued martingales into ℓ^2-valued martingales with multiplying sequence $\{v_k\}_{k\geq 1} \subset \mathcal{L}(\boldsymbol{R}, \ell^2) \cong \ell^2$ given by $v_k = (0, \overset{(k-1)}{\ldots}, 0, 1, 0, 0\ldots..)$. Then

$$(Tf)_n = \sum_{k=1}^n v_k d_k = (d_1, d_2, \ldots, d_n, 0, \ldots)\,,$$

$$(T_{k_0}f)_n = \sum_{k=1}^n v_{k_0+k}d_k = (0, \overset{(k_0-1)}{\ldots}, 0, d_1, d_2, \ldots, d_n, 0, \ldots)\,,$$

and, as above, we have

$$\|(Tf)_n(\omega)\|_{\ell^2} = \left(\sum_{j=1}^n |d_j(\omega)|^2\right)^{1/2} = \|(T_{k_0}f)_n(\omega)\|_{\ell^2}\,.$$

Now we give a converse of Theorem 1.

THEOREM 2. *Let B_1 and B_2 be Banach spaces and T a translation invariant martingale transform operator such that each element v_k of the multiplying sequence is a constant operator from B_1 into B_2. Then the condition*

$$(*)\qquad \|f^*\|_{L^1} < \infty \Rightarrow Tf \text{ converges a.s.}$$

implies any of conditions (i) *to* (v) *of Theorem* 1.

3. Applications.

3.1. Maximal functions. Given a B-valued martingale $f_n = \sum_{k=1}^n d_k$, we have already defined its Doob's maximal function by $f^*(\omega) = \sup_n \|f_n(\omega)\|_B$.

Define a martingale transform operator T given by the multiplying sequence $\{w_k\}_{k\geq 1}$ of operators in $\mathcal{L}(B, \ell^\infty(B))$ with $w_k b = (0, \overset{(k-1)}{\ldots}, 0, b, b, b.....)$, $b \in B$, by

$$(Tf)_n = \sum_{k=1}^n w_k d_k = \left(d_1, d_1 + d_2, \ldots, \sum_{k=1}^n d_k, \sum_{k=1}^n d_k, \ldots\right).$$

Then

$$(1)\qquad \|(Tf)_n\|_{\ell^\infty(B)}(\omega) = \sup_{1\leq k\leq n} \|f_k(\omega)\|_B, \quad \text{and,} \quad (Tf)^*(\omega) = f^*(\omega).$$

It follows from Remark 1 that this martingale transform T satisfies (iv) and (i) in Theorem 1, with $B_1 = B$, $B_2 = \ell^\infty(B)$. In particular, $\tilde{T}$ defined as in Corollary 1 satisfies the hypothesis of this corollary. On the other hand, given $F = \{F_n\}_{n\geq 1}$ an $\ell^q(B)$-valued martingale, $F_n = \{f_n^j\}_{j\geq 1}$, by using (1) we have

$$(\tilde{T}F)^* = \sup_n \|(\tilde{T}F)_n\|_{\ell^q(\ell^\infty(B))} = \sup_n \left(\sum_j \|(Tf^j)_n\|^q_{\ell^\infty(B)}\right)^{1/q}$$
$$= \sup_n \left(\sum_j \left(\sup_{1\leq k\leq n} \|(f^j)_k\|_B\right)^q\right)^{1/q} = \left(\sum_j ((f^j)^*)^q\right)^{1/q}.$$

Therefore, given q, $1 < q < \infty$, there exists $C_q > 0$ such that for any $\lambda > 0$

$$P\left\{\sum_j ((f^j)^*)^q > \lambda^q\right\} \leq \frac{C_q}{\lambda}\left\|\left(\sum_j \|f^j\|_B^q\right)^{1/q}\right\|_{L^1},$$

and given any p, $1 < p < \infty$, there exists $C_{p,q} > 0$ such that

$$\left\|\left(\sum_j ((f^j)^*)^q\right)^{1/q}\right\|_{L^p} \leq C_{p,q}\left\|\left(\sum_j \|f^j\|_B^q\right)^{1/q}\right\|_{L^p}.$$

In other words we obtain (in the case $B = \boldsymbol{R}$) the martingale version of the well-known theorem of Fefferman and Stein (see [17]) for the Hardy-Littlewood maximal operator.

3.2. Square functions. Given B a Banach space and $f = \{f_n\}_{n\geq 1}$ a B-valued martingale, $f_n = \sum_{k=1}^n d_k$, the martingale square function of f is defined as

$$Sf = \left(\sum_{k=1}^{\infty} \|d_k\|_B^2\right)^{1/2}, \quad S_n f = \left(\sum_{k=1}^{n} \|d_k\|_B^2\right)^{1/2}.$$

In the case when f is a scalar-valued martingale, by orthogonality of the differences d_k it is clear that

$$\|S_n f\|_{L^2}^2 = \left\|\sum_{k=1}^{n} |d_k|^2\right\|_{L^1} = \left\|\left|\sum_{k=1}^{n} d_k\right|^2\right\|_{L^1} = \|f_n\|_{L^2}^2 . \tag{2}$$

Various results have been known for this function. The aim of this section is to show that these results can be obtained as easy corollaries of Theorem 1 and the straightforward L^2-bound (2).

Given the scalar-valued martingale f, we consider the martingale transform Q whose multiplying sequence is $\{v_k\}_{k\geq 1} \subset \mathcal{L}(\boldsymbol{R}, \ell^2) = \ell^2$ with $v_k = (0, \overset{(k-1)}{\ldots}, 0, 1, 0, \ldots)$. These functions are $\mathcal{F}_{k-1}$-measurable (since they are constants) and uniformly bounded by 1 (since $\|v_k\|_{\mathcal{L}(\boldsymbol{R},\ell^2)} = \|v_k\|_{\ell^2} = 1$). They define a martingale transform operator from scalar-valued martingales to ℓ^2-valued martingales by

$$(Qf)_n = \sum_{k=1}^{n} v_k d_k = (d_1, d_2, \ldots, d_n, 0, \ldots).$$

Observe that

$$(Qf)^* = \sup_n \|(Qf)_n\|_{\ell^2} = \sup_n \left(\sum_{k=1}^{n} |d_k|^2\right)^{1/2} = \left(\sum_{k=1}^{\infty} |d_k|^2\right)^{1/2} = Sf,$$

and

$$\left(\sum_{k=m+1}^{n} |d_k(\omega)|^2\right)^{1/2} = \|(Qf)_n(\omega) - (Qf)_m(\omega)\|_{\ell^2} . \tag{3}$$

Therefore, by using (2), we have that this martingale transform satisfies (v) in Theorem 1 with $p_0 = 2$, $B_1 = \boldsymbol{R}$, $B_2 = \ell^2$. Observe that ℓ^2 satisfies the Radon-Nikodym property. Then, applying Theorem 1, we have

$$P(Sf > \lambda) = P((Qf)^* > \lambda) \leq \frac{C}{\lambda}\|f\|_{L^1}, \tag{4}$$

$$\|Sf\|_{L^p} = \|(Qf)^*\|_{L^p} \leq C_p\|f^*\|_{L^p}, \quad 1 \leq p < \infty, \tag{5}$$

$$\|f\|_{L^1} < \infty \Rightarrow Qf \text{ converges a.s., i.e., } Sf < \infty \text{ a.e.} \tag{6}$$

Now we consider the ℓ^2-valued martingale $F = \{F_n\}_{n\geq 1}$, with $F_n = \sum_{k=1}^n D_k$, $D_k = \{D_k^j\}_{j\geq 1}$. Each D_k is $\mathcal{F}_k$-measurable and $E(D_k|\mathcal{F}_{k-1}) = 0$ (in ℓ^2). Therefore D_k and D_l are orthogonal (in $L^2_{\ell^2}$) for $k \neq l$ and then we have

$$\|S_n F\|^2_{L^2} = \int_\Omega \sum_{k=1}^n \|D_k\|^2_{\ell^2} dP = \int_\Omega \left\| \sum_{k=1}^n D_k \right\|^2_{\ell^2} dP = \|F_n\|^2_{L^2_{\ell^2}} . \tag{7}$$

Moreover, for each j the component D^j_k is $\mathcal{F}_k$-measurable and $E(D^j_k | \mathcal{F}_{k-1}) = 0$ (in $\boldsymbol{R}$), and hence D^j_k and D^j_ℓ are orthogonal (in L^2) for $k \neq l$. We consider the martingale transform operator R from ℓ^2-valued martingales into scalar-valued martingales given by the sequence $\{\tilde{v}_k\}_{k\geq 1} \subset \mathcal{L}(\ell^2, \boldsymbol{R}) \cong \ell^2$, $\tilde{v}_k x = \langle x, (0, \overset{(k-1)}{\ldots}, 0, 1, 0, \ldots)\rangle = x^k$, $x \in \ell^2$. Then

$$(RF)_n = \sum_{k=1}^n \tilde{v}_k D_k = D^1_1 + \cdots + D^n_n ,$$

and therefore applying (2) and (7), we have

$$\begin{aligned}\|(RF)_n\|^2_{L^2} = \|S_n(RF)\|^2_{L^2} &= \int_\Omega \sum_{k=1}^n |D^k_k|^2 dP \leq \int_\Omega \sum_{k=1}^n \left(\sum_j |D^j_k|^2 \right) dP \\ &= \int_\Omega \sum_{k=1}^n \|D_k\|^2_{\ell^2} dP = \|S_n F\|^2_{L^2} = \|F_n\|^2_{L^2_{\ell^2}} .\end{aligned}$$

In other words, R satisfies (v) of Theorem 1. Then

$$\begin{aligned}&P((RF)^* > \lambda) \leq \frac{C}{\lambda} \|F\|_{L^1_{\ell^2}} ,\\ &\|(RF)^*\|_{L^p} \leq C_p \|F^*\|_{L^p_{\ell^2}} , \quad 1 \leq p < \infty ,\\ &\|F\|_{L^1_{\ell^2}} < \infty \Rightarrow RF \text{ converges a.s.}\end{aligned}$$

Now, given the scalar-valued martingale $f = \{f_n\}_{n\geq 1}$, $f_n = \sum_{k=1}^n d_k$, we consider the ℓ^2-valued martingale $F = \{F_n\}_{n\geq 1}$, $F_n = \sum_{k=1}^n D_k$, where $D^k_k = d_k$ and $D^j_k = 0$, $j \neq k$. Then $\|F\|_{\ell^2} = Sf$, $(RF)_n = \sum_{k=1}^n \tilde{v}_k D_k = d_1 + \cdots + d_n = f_n$ and $(RF)^* = f^*$. Therefore we have

$$P(f^* > \lambda) \leq \frac{C}{\lambda} \|Sf\|_{L^1} , \tag{8}$$

$$\|f^*\|_{L^p} \leq C_p \|Sf\|_{L^p} , \quad 1 \leq p < \infty , \tag{9}$$

$$Sf \in L^1 \Rightarrow f \text{ converges a.s.} \tag{10}$$

Inequalities (4) and (8) are due to Burkholder, as well as (5) and (9) in the case $p > 1$ (see [8]). The case $p = 1$ in (5) and (9) was proved by Davis, see [14]. Result (6) is due to Austin [1].

3.3. Hilbert spaces. Let $\{r_n\}_{n\geq 1}$ be the Rademacher system defined by

$$r_n(t) = \operatorname{sign} \sin 2^n \pi t$$

for $t \in [0, 1]$. The following result is due to Kwapien, see [20]. Given a Banach space B, the following conditions are equivalent:

(1) B is isomorphic to a Hilbert space.

(2) There exists $C > 1$ such that for any sequence $\{x_j\}_{j=1}^n$, $n = 1, 2, \dots$ in B,

$$C^{-1}\sum_{j=1}^n \|x_j\|_B^2 \le \int_0^1 \left\|\sum_{j=1}^n x_j r_j(t)\right\|_B^2 dt \le C\sum_{j=1}^n \|x_j\|_B^2 .$$

Averaging over Ω, Kwapien's result can be formulated as follows.

THEOREM 3. *Given a Banach space B, the following conditions are equivalent*:

(1) *B is isomorphic to a Hilbert space.*

(2) *There exists $C > 1$ such that for any B-valued martingale sequence $f = \{f_n\}_{n\ge 1}$, we have*

$$C^{-1}\|f_n\|_{L_B^2} \le \|S_n f\|_{L^2} \le C\|f_n\|_{L_B^2}, \quad n = 1, 2, \dots .$$

Given a B-valued martingale f, we define, analogously as in the last section, the $\ell^2(B)$-valued martingale transform $Qf = \{(Qf)_n\}_{n\ge 1}$ with

$$(Qf)_n = \sum_{k=1}^n v_k d_k = (d_1, d_2, \dots, d_n, 0, \dots),$$

where $v_k = (0, \overset{(k-1)}{\dots}, 0, I_B, 0, \dots)$. Observe that

(11) $$\|(Qf)_n\|_{\ell^2(B)} = S_n f \quad \text{and} \quad (Qf)^* = Sf .$$

Given $F = \{F_n\}_{n\ge 1}$ a $\ell^2(B)$-valued martingale with $F_n = \sum_{k=1}^n D_k$, $D_k = \{D_k^j\}_{j\ge 1}$, we define the $\ell^2(\ell^2(B))$-valued martingale $\tilde{Q}F = \{(\tilde{Q}F)_n\}_{n\ge 1}$ by

$$(\tilde{Q}F)_n = \sum_{k=1}^n V_k D_k = (D_1, D_2, \dots, D_n, 0, \dots),$$

where $V_k = (0, \overset{(k-1)}{\dots}, 0, I_{\ell^2(B)}, 0, \dots)$, and the B-valued martingale $RF = \{(RF)_n\}_{n\ge 1}$ by

$$(RF)_n = \sum_{k=1}^n \tilde{v}_k D_k = \sum_{k=1}^n D_k^k ,$$

where $\tilde{v}_k x = x^k$ for all $x = \{x^j\}_{j\ge 1} \in \ell^2(B)$.

LEMMA 1. *Let B be a Banach space, and $F = \{F_n\}_{n\ge 1}$, $F_n = \sum_{k=1}^n D_k$, a $\ell^2(B)$-valued martingale. Then we have*

$$S_n(RF) \le S_n F .$$

PROOF. $S_n(RF)^2 = \sum_{k=1}^n \|D_k^k\|_B^2 \le \sum_{k=1}^n \sum_{j=1}^\infty \|D_k^j\|_B^2 = \sum_{k=1}^n \|D_k\|_{\ell^2(B)}^2 = S_n F^2$.

THEOREM 4. *Given a Banach space B, the following conditions are equivalent*:

(1) *B is isomorphic to a Hilbert space.*

(2) *There exists $C > 0$ such that for any $\lambda > 0$ we have*

$$P\{f^* > \lambda\} \le \frac{C}{\lambda}\|Sf\|_{L^1} \quad \textit{and} \quad P\{Sf > \lambda\} \le \frac{C}{\lambda}\|f\|_{L^1_B}.$$

(3) *There exists a constant $C > 0$ such that $C^{-1}\|f^*\|_{L^1} \le \|Sf\|_{L^1} \le C\|f^*\|_{L^1}$.*

(4) *There exist p, $1 < p < \infty$, and a constant C_p such that*

$$C_p^{-1}\|f\|_{L^p_B} \le \|Sf\|_{L^p} \le C_p\|f\|_{L^p_B}.$$

(5) *If $\|f\|_{L^1_B} < \infty$, then $Sf < \infty$ a.s., and if $Sf \in L^1$, then f converges a.s.*

PROOF. Assume (1). As B is isomorphic to a Hilbert space, we know that $\|S_n f\|_{L^2} \le C\|f_n\|_{L^2_B}$. Then by using (11) we have that the martingale transform operator Q satisfies (v) in Theorem 1 with $v_k = (0, \overset{(k-1)}{\ldots}, 0, I_B, 0\ldots) \in \mathcal{L}(B, \ell^2(B)) \cong \ell^2 \otimes B$, $B_1 = B$, $B_2 = \ell^2(B)$, $p_0 = 2$. In this way we obtain the inequalities appearing in (2), (3), (4) with S on the left. On the other hand, $\ell^2(B)$ is a Hilbert space and in particular satisfies the Radon-Nikodym property. Then by Theorem 1 we get, for martingales such that $\|f\|_{L^1_B} < \infty$, the a.s. convergence (in $\ell^2(B)$) of the martingale Qf, which in particular implies $Sf < \infty$, a.e.

On the other hand, again by Kwapien's Theorem, we have $\|f_n\|_{L^2_B} \le C\|S_n f\|_{L^2}$. Using Lemma 1 and the fact that $\ell^2(B)$ is also isomorphic to a Hilbert space, we have

$$\|(RF)_n\|^2_{L^2_B} \le C\|S_n(RF)\|^2_{L^2} \le C\|S_nF\|^2_{L^2} \le C'\|F_n\|^2_{L^2_{\ell^2(B)}}.$$

In other words, the martingale transform operator R satisfies (v) of Theorem 1 with $\tilde{v}_k \in \mathcal{L}(\ell^2(B), B) \cong \ell^2 \otimes B$, $\tilde{v}_k x = x^k$ for all $x = \{x^j\}_{j\ge1} \in \ell^2(B)$, $B_1 = \ell^2(B)$, $B_2 = B$, $p_0 = 2$. Now given the B-valued martingale $f = \{f_n\}_{n\ge1}$, $f_n = \sum_{k=1}^n d_k$, we can consider the $\ell^2(B)$-valued martingale $F_n = \sum_{k=1}^n D_k$, with $D_k^k = d_k$, $D_k^j = 0$ otherwise. Then

$$(RF)_n = f_n, \quad (RF)^* = f^*, \quad \|F_n\|_{\ell^2(B)} = S_n f, \quad (F)^* = Sf, \tag{12}$$

and we get the rest of (2), (3), (4) and (5).

Assume (2). By equation (11) we get that the martingale transform operator Q satisfies $\lambda P\{(Qf)^* > \lambda\} \le C\|f^*\|_{L^1_B}$, that is, (ii) in Theorem 1. Therefore we get $\|(Qf)^*\|_{L^p} = \|Sf\|_{L^p} \le C_p\|f^*\|_{L^p}$, $1 \le p < \infty$, and in particular $\|Sf\|_{L^2} \le C\|f\|_{L^2_B}$.

Applying Corollary 1, we have that the martingale transform $(\tilde{Q}F)_n = \{(Qf^j)_n\}_{j\ge1}$ satisfies (v) of this Corollary 1 with $B_1 = B$, $B_2 = \ell^2(B)$, $q = 2$, $p_0 = 2$. In particular, if $F = \{F_n\}_{n\ge1}$ is a $\ell^2(B)$-valued martingale with $F_n = \sum_{k=1}^n D_k$, $D_k = \{D_k^j\}_{j\ge1}$, for $1 \le p < \infty$, then

$$\|SF\|_{L^p} = \left\|\sup_n S_nF\right\|_{L^p} = \left\|\sup_n \|(\tilde{Q}F)_n\|_{\ell^2(\ell^2(B))}\right\|_{L^p} = \|(\tilde{Q}F)^*\|_{L^p} \le C\|F^*\|_{L^p}.$$

Then by our hypothesis (2) and Lemma 1, we get

$$P\{(RF)^* > \lambda\} \le \frac{C}{\lambda}\|S(RF)\|_{L^1} \le \frac{C}{\lambda}\|SF\|_{L^1} \le \frac{C}{\lambda}\|F^*\|_{L^1},$$

which shows that R satisfies (ii) in Theorem 1. Now given $f_n = \sum_{k=1}^n d_k$, we choose as before $F_n = \sum_{k=1}^n D_k$, with $D_k^k = d_k$, $D_k^j = 0$ otherwise. Then we get $\|f^*\|_{L^1} \le C\|Sf\|_{L^1}$, and also $\|f^*\|_{L_B^p} \le C_p\|Sf\|_{L^p}$, for $1 \le p < \infty$. So we have proved (2) $\Rightarrow$ (1), (3) and (4).

Using the operators Q and R, (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (2) can be proved in a similar way.

Finally we shall prove that (5) $\Rightarrow$ (1). By using (3) (in the B-valued setting) we have that $\|f\|_{L_B^1} < \infty \Rightarrow Sf < \infty$ implies that the martingale transform operator Q satisfies the hypothesis in Theorem 2. By (11) we get $\|Sf\|_{L^2} \le C\|f\|_{L^2}$.

Again, by using Corollary 1, we have $\|SF\|_{L^1} \le C\|F^*\|_{L^1}$. Given the operators $\tilde{v}_j \in \mathcal{L}(\ell^2(B), B) \cong \ell^2 \otimes B$, $\tilde{v}_j x = x^j$ for all $x = \{x^j\} \in \ell^2(B)$, consider an arbitrary sequence $v = \{\tilde{v}_{v(k)}\}_{k=1}^{\infty}$ of this operators and the corresponding martingale transform operator given by

$$(R_v F)_n = \sum_{k=1}^n \tilde{v}_{v(k)} D_k = D_1^{v(1)} + \cdots + D_n^{v(n)} .$$

As in Lemma 1, one can see that every sequence v satisfies $S_n(R_v F) \le S_n F$. Then by the part already proved, we get

$$\|S(R_v F)\|_{L^1} \le \|SF\|_{L^1} \le C\|F^*\|_{L^1} .$$

Using the hypothesis (5), we conclude that for any sequence v and any $\ell^2(B)$-valued martingale F such that $F^* \in L^1$, the martingale $R_v F$ converges a.s. (in B). Therefore, by Remark 7 after the proof of Theorem 2, we have that any martingale transform $R_v F$ satisfies the conditions in Theorem 1. Now given $f_n = \sum_{k=1}^n d_k$, we choose as before $F_n = \sum_{k=1}^n D_k$, with $D_k^k = d_k$, $D_k^j = 0$, and $\tilde{v}_{v(k)} = \tilde{v}_k$ and obtain $\|f\|_{L_B^p} \le C_p\|Sf\|_{L^p}$, $1 \le p < \infty$.

3.4. UMD Banach lattices.

DEFINITION 3. A Banach space X is said to be *UMD* (unconditionality property for martingale differences) if given p, $1 < p < \infty$ there exists a positive real number C_p such that

$$\|\varepsilon_1 d_1 + \cdots + \varepsilon_n d_n\|_{L_X^p} \le C_p\|d_1 + \cdots + d_n\|_{L_X^p}$$

for all X-valued martingale difference sequences $(d_1, d_2, \dots)$, all numbers $\varepsilon_1, \varepsilon_2, \dots$ in $\{-1, 1\}$, and all $n \ge 1$.

This definition is due to Burkholder, see [9]. It is known that the existence of one p_0 satisfying the inequality is enough to assure the existence of the rest of p's, $1 < p < \infty$, see [22].

By a Banach lattice we mean a Banach space X over the field of the real numbers, together with an order relation $\le$ on X, satisfying the following properties:

(i) $x \le y$ implies $x + z \le y + z$ for every $x, y, z \in X$.

(ii) $ax \ge 0$ for every $x \ge 0$ in X and every $a \ge 0$ in $\boldsymbol{R}$.

(iii) For every $x, y \in X$, there exists the least upper bound (say, $\sup\{x, y\}$) and also the greatest lower bound (say, $\inf\{x, y\}$).

(iv) If $|x|$ is defined as $|x| = \sup\{x, -x\}$, then the order relation $|x| \leq |y|$ implies the inequality between the norms $\|x\| \leq \|y\|$.

DEFINITION 4. Let X be a Banach lattice. X is said to be p-convex, $1 \leq p \leq \infty$, if the following inequality holds:

$$\left\| \left(\sum_{j=1}^{m} |x_j|^p \right)^{1/p} \right\|_X \leq C_p \left(\sum_{j=1}^{m} \|x_j\|_X^p \right)^{1/p},$$

and X is said to be q-concave, $1 \leq q \leq \infty$, if the following inequality holds:

$$\left(\sum_{j=1}^{m} \|x_j\|_X^q \right)^{1/q} \leq C_q \left\| \left(\sum_{j=1}^{m} |x_j|^q \right)^{1/q} \right\|_X,$$

where in both inequalities, the constants are independent of m.

Note that the Banach lattice L^p is p-convex and p-concave.

When X is a lattice of functions or, more generally, when X is order continuous, the concrete representation of the lattice allows us to define $(\sum_{j=1}^{m} |x_j|^p)^{1/p}$ in the obvious way. However, for a general lattice, these expressions need some technicalities to be defined, see [21].

REMARK 5. The following generalization of the classical inequality of Khintchine holds, see [21]. Let X be a q-concave Banach lattice for some $q < \infty$. Then there exists a constant $C < \infty$ such that, for every sequence $\{x_j\}_{j=1}^{m}$ of elements of X, we have

$$C^{-1} \left\| \left(\sum_{j=1}^{m} |x_j|^2 \right)^{1/2} \right\|_X \leq \int_{(0,1)} \left\| \sum_{j=1}^{m} r_j(t) x_j \right\|_X dt \leq C \left\| \left(\sum_{j=1}^{m} |x_j|^2 \right)^{1/2} \right\|_X.$$

DEFINITION 5. Let X be a Banach lattice and $f = \{f_n\}_{n\geq 1}$ a X-valued martingale, $f_n = \sum_{k=1}^{n} d_k$. For each positive integer N, we define the operators

$$S_{X,N} f(\omega) = \left(\sum_{k=1}^{N} |d_k(\omega)|^2 \right)^{1/2}.$$

Then $\|S_{X,N} f\|_X$ can be seen as the norm of the element $(d_1, \dots, d_N)$ in the Banach space

$$X(\ell_N^2) = \left\{ \{x_i\}_{i=1}^{N} \subset X;\ \left\| \left(\sum_{i=1}^{N} |x_i|^2 \right)^{1/2} \right\|_X < \infty \right\}.$$

$X(\ell_N^2)$ is also a Banach lattice with the coordinate-wise order when X is a Banach lattice. We can prove the following characterization of UMD Banach lattices.

THEOREM 5. *Given a Banach lattice X, the following statements are equivalent*:

(1) *X satisfies the UMD property.*

(2) *There exists $C > 0$ such that for any $\lambda > 0$ we have*

$$P\{f^* > \lambda\} \leq \frac{C}{\lambda} \sup_N \|S_{X,N} f\|_{L_X^1}, \quad \textit{and} \quad P\{\|S_{X,N} f\|_X > \lambda\} \leq \frac{C}{\lambda} \|f\|_{L_X^1}.$$

(3) *There exists a constant* $C > 0$ *such that* $C^{-1}\|f^*\|_{L^1} \leq \sup_N \|S_{X,N}f\|_{L^1_X} \leq C\|f^*\|_{L^1}$.

(4) *There exist* p, $1 < p < \infty$, *and a constant* C_p *such that*

$$C_p^{-1}\|f\|_{L^p_X} \leq \sup_N \|S_{X,N}f\|_{L^p_X} \leq C_p\|f\|_{L^p_X}.$$

PROOF. The proof follows the argument on the proof of Theorem 4 with some technical modifications.

(1) $\Leftrightarrow$ (4). This is due to Bourgain and Rubio de Francia, see [7] and [23]. It can be obtained by using Remark 5 and the fact that any lattice with the UMD property is q-concave for some $q < \infty$.

In order to see that (4) $\Rightarrow$ (2) and (3), we consider the martingale transform operator $Q_{X,N}f = \{(Q_{X,N}f)_n\}_{n\geq 1}$, defined by

$$(Q_{X,N}f)_n = \sum_{k=1}^{n} v_k d_k = (d_1, d_2, \dots, d_{n\wedge N}, 0, \dots, 0),$$

where $v_k = (0, \overset{(k-1)}{\dots}, 0, I_X, 0, \dots, 0) \in \mathcal{L}(X, X(\ell^2_N))$, $v_k x = (0, \overset{(k-1)}{\dots}, 0, x, 0, \dots, 0)$ for $x \in X$, when $k \leq N$ and $v_k = 0$ otherwise. Then $\|(Q_{X,N}f)_n\|_{X(\ell^2_N)} = \|S_{X,n\wedge N}f\|_X$ and, therefore, $Q_{X,N}$ satisfies (v) of Theorem 1 since for all n

$$\|(Q_{X,N}f)_n\|_{L^p_{X(\ell^2_N)}} \leq C\|f\|_{L^p_X}, \quad \text{for some } p,\ 1 < p < \infty.$$

Let us consider now the extension of $Q_{X,N}$, which we will call $\tilde{Q}_{X,N}$, defined for $X(\ell^2_N)$-valued martingales by

$$(\tilde{Q}_{X,N}F)_n = \sum_{k=1}^{n} V_k D_k = (D_1, \dots, D_n, 0, \dots, 0) \in X(\ell^2_N)(\ell^2_N)$$

where

$$X(\ell^2_N)(\ell^2_N) = \left\{ \{x_i\}_{i=1}^N \subset X(\ell^2_N);\ \left\| \left(\sum_{i=1}^{N} |x_i|^2 \right)^{1/2} \right\|_{X(\ell^2_N)} < \infty \right\},$$

and $V_k = (0, \overset{(k-1)}{\dots}, 0, I_{X(\ell^2_N)}, 0, \dots, 0)$ for $k \leq N$, $V_k = 0$ if $k > N$. This transform verifies for a $X(\ell^2_N)$-valued martingale F such that $F_n = (F_n^1, \dots, F_n^N)$:

$$\begin{aligned}
(\tilde{Q}_{X,N}F)^* &= \|(\tilde{Q}_{X,N}F)_N\|_{X(\ell^2_N)(\ell^2_N)} = \left\| \left(\sum_{k=1}^{N} |D_k|^2 \right)^{1/2} \right\|_{X(\ell^2_N)} \\
&= \left\| \left(\left(\sum_{k=1}^{N} |D_k^1|^2 \right)^{1/2}, \dots, \left(\sum_{k=1}^{N} |D_k^N|^2 \right)^{1/2} \right) \right\|_{X(\ell^2_N)} = \left\| \left(\sum_{j=1}^{N} \sum_{k=1}^{N} |D_k^j|^2 \right)^{1/2} \right\|_X \\
&= \left\| \left(\sum_{k=1}^{N} \sum_{j=1}^{N} |D_k^j|^2 \right)^{1/2} \right\|_X = \left\| \left(\sum_{j=1}^{N} |Q_{X,N}F_N^j|^2 \right)^{1/2} \right\|_{X(\ell^2_N)}.
\end{aligned}$$

Using Krivine's Theorem, see [21, page 93], we have that for $1 < p < \infty$

$$\|(\tilde{Q}_{X,N}F)^*\|_{L^p} = \|(\tilde{Q}_{X,N}F)_N\|_{L^p_{X(\ell^2_N)(\ell^2_N)}} = \left\| \left(\sum_{j=1}^{N} |Q_{X,N}F^j_N|^2 \right)^{1/2} \right\|_{L^p_{X(\ell^2_N)}}$$

$$\leq CK_G \left\| \left(\sum_{j=1}^{N} |F^j_N|^2 \right)^{1/2} \right\|_{L^p_X} = CK_G \|F_N\|_{L^p_{X(\ell^2_N)}} \leq CK_G \|F\|_{L^p_{X(\ell^2_N)}},$$

where K_G is the universal Grothendieck constant. This says that the martingale transform operator $\tilde{Q}_{X,N}$ also satisfies (iv) of Theorem 1.

On the other hand, given the $X(\ell^2_N)$-valued martingale $F = \{F_n\}_{n\geq 1}$, $F_n = \sum_{k=1}^n D_k$, $D_k = \{D^j_k\}_{j=1}^N$, we define a martingale transform operator $R_{X,N}$ as

$$(R_{X,N}F)_n = \sum_{k=1}^{n} \tilde{v}_k D_k = \sum_{k=1}^{n\wedge N} D^k_k,$$

where $\tilde{v}_j x = x^j$, $x = \{x^j\} \in X(\ell^2_N)$, for $j \leq N$ and $\tilde{v}_j = 0$ otherwise.

Then, using the hypothesis and the fact about $\tilde{Q}_{X,N}$, we have

$$\|(R_{X,N}F)_n\|_{L^p_X} \leq C \sup_M \|S_{X,M}(R_{X,N}F)\|_{L^p_X} \leq C \left\| \left(\sum_{k=1}^{N} |D^k_k|^2 \right)^{1/2} \right\|_{L^p_X}$$

$$\leq C \left\| \left(\sum_{j=1}^{N} \sum_{k=1}^{N} |D^j_k|^2 \right)^{1/2} \right\|_{L^p_X} = C \|(\tilde{Q}_{X,N}F)_N\|_{L^p_{X(\ell^2_N)(\ell^2_N)}}$$

$$\leq C \|F_N\|_{L^p_{X(\ell^2_N)}} \leq C \|F\|_{L^p_{X(\ell^2_N)}}.$$

This says that $R_{X,N}$ satisfies (iv) of Theorem 1. Now we have the same ingredients as in the proof of Theorem 4 and we leave the details of the remaining part of the proof to the reader.

3.5. UMD Köthe Banach lattices. For a general lattice X, size conditions over $S_{X,N}f$ state whether or not X is UMD. In the special case when the lattice is a Köthe function lattice with Fatou property, convergence conditions also characterize those that are UMD.

Let $(\Sigma, \mathcal{A}, \mu)$ be a σ-finite measure space. A Banach space X consisting of equivalence classes, modulo equality almost everywhere (a.e.), of real-valued, locally integrable functions defined on Σ, is called a Köthe function space if the following hold:

(1) If $|f(s)| \leq |g(s)|$ a.e. on Σ with f measurable and $g \in X$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.

(2) For every $E \in \mathcal{A}$ with $\mu(E) < \infty$, the characteristic function χ_E of E belongs to X.

Every Köthe function space is a Banach lattice with the obvious order ($f \geq 0$ if and only if $f(s) \geq 0$ a.e.).

A Köthe function space is said to have the Fatou property (see [21]) if for any sequence of functions $\{f_n\}$ in X such that $f_n \geq 0$ a.e., $f_n(s) \uparrow f(s)$ a.e. and $\sup_n \|f_n\|_X < \infty$, then $f \in X$ and $\|f\|_X = \sup_n \|f_n\|_X$.

For such a function space we have the following

THEOREM 6. *Let X be a Köthe lattice with the Fatou property. The following statements are equivalent*:

(1) *X is UMD.*

(2) *For every X-valued martingale f, it holds that, defining $S_X f = (\sum_{k=1}^{\infty} |d_k|^2)^{1/2}$,*

$$\|f\|_{L^1_X} < \infty \Rightarrow S_X f \in X \text{ a.e., and } S_X f \in L^1_X \Rightarrow f \text{ converges a.e.}$$

PROOF. Following the ideas in the proof of previous theorems, we try to regard $S_X f$ as the maximal of a martingale transform operator valued in certain Banach space. Define

$$X(\ell^2) = \left\{ \{x_i\}_{i=1}^{\infty} \subset X; \left\| \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2} \right\|_X < \infty \right\}.$$

If X has the Fatou property, then $X(\ell^2)$ is a Banach space. It is also a Banach lattice with the obvious order ($\{x_i\} \le \{y_i\}$ if and only if $x_i \le y_i$ for all i). Moreover, if X is UMD, $X(\ell^2)$ is also UMD (see [23]).

Suppose X is UMD. By Theorem 5, this is equivalent to the condition

$$\sup_N \|S_{X,N} f\|_{L^p_X} \sim \|f\|_{L^p_X}.$$

Since L^p_X has the Fatou property if X has, for any f bounded in L^p_X, $1 \le p < \infty$, we can define $S_X f = \sup_N S_{X,N} f$ in L^p_X. Also, we have $\|S_X f\|_{L^p_X} = \sup_N \|S_{X,N} f\|_{L^p_X}$.

Consider the martingale transform Q_X such that, for a X-valued martingale f, $f_n = \sum_{k=1}^n d_k$ gives the $X(\ell^2)$-valued martingale $(Q_X f)_n = \sum_{k=1}^n v_k d_k = (d_1, \dots, d_n, 0, \dots)$. Then, since X is UMD and $\|(Q_X f)_n\|_{X(\ell^2)} = \|S_{X,n} f\|_X$, Q_X verifies statement (iv) in Theorem 1 with $B_1 = X$ and $B_2 = X(\ell^2)$. Moreover, X is superreflexive (see [9]) and therefore it has the Radon-Nikodym property. Then, by $(*)$ in Theorem 1, we get that f being bounded in L^1_X implies that Qf converges a.e. in $X(\ell^2)$. But this is equivalent to $S_X f \in X$ a.e., and hence we obtain the first part of (2).

Define now the martingale transform R_X for $X(\ell^2)$-valued martingales $F_n = \sum_{k=1}^n D_k$ by $(R_X F)_n = \sum_{k=1}^n \tilde{v}_k D_k = \sum_{k=1}^n D_k^k$, where the operators $\tilde{v}_k$ are defined to be $\tilde{v}_j x = x^j$, for any $x = \{x^j\}_{j \ge 1} \in X(\ell^2)$. We have for $1 < p < \infty$.

$$\begin{aligned} \sup_n \|(R_X F)_n\|_{L^p_X} &\le C \sup_n \|S_{X,n}(R_X F)\|_{L^p_X} = C \sup_n \left\| \left(\sum_{k=1}^n |D_k^k|^2 \right)^{1/2} \right\|_{L^p_X} \\ &\le C \sup_n \left\| \left(\sum_{k=1}^n |D_k|^2 \right)^{1/2} \right\|_{L^p_{X(\ell^2)}} = C \sup_n \|S_{X(\ell^2),n} F\|_{L^p_{X(\ell^2)}} \\ &\le C \sup_n \|F_n\|_{L^p_{X(\ell^2)}}, \end{aligned}$$

where the first and last inequalities are due to the fact that $X(\ell^2)$ is a lattice and X and $X(\ell^2)$ are UMD.

Then, R_X verifies (iv) in Theorem 1 with $B_1 = X(\ell^2)$ and $B_2 = X$. For any X-valued martingale f, $f_n = \sum_{k=1}^n d_k$, taking F such that $D_k = (0, \overset{(k-1)}{\ldots}, 0, d_k, 0, \ldots)$, we obtain the second statement in (2).

To prove the converse, we do in the same way as in the proof of Theorem 5. It would suffice to conclude from (2) that transforms Q_X and R_X verify Theorem 1. Since Q_X is translation invariant, the first statement in (2) assures that it verifies the theorem, and we get $\sup_N \|S_{X,N} f\|_{L^p_X} \leq C\|f\|_{L^p_X}$.

R_X also verifies Theorem 1, since it can be seen, in the same way as in Theorem 5, that $F^* \in L^1$ implies that $S_X(R_{X,v}F)$ is in L^1_X for all transforms $R_{X,v}$. As it is usual, here v is a choice of operators $\tilde{v}_k$ in such a way that $(R_{X,v}F)_n = \sum_{k=1}^n D_k^{v(k)}$. The extension of the transform Q_X that we need this time, $\tilde{Q}_X$, is defined for $X(\ell^2)$-valued martingales F as the $X(\ell^2)(\ell^2)$-valued martingale $(\tilde{Q}_X F)_n = (D_1, \ldots, D_n, 0, \ldots)$. From Krivine's theorem we see that this transform verifies Theorem 1 if Q_X does, and this yields the desired result, with the same reasoning as in the proof of the former theorem.

3.6. The Hardy-Littlewood property. Given X a Banach lattice, J a finite subset of positive rational numbers and f a X-valued function defined in $\boldsymbol{R}^n$, consider the maximal operator

$$\mathcal{M}_J f(x) = \sup_{r\in J} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy\,.$$

X is said to have the Hardy-Littlewood property if there exists p_0, $1 < p_0 < \infty$ such that $\mathcal{M}_J$ are bounded in $L^{p_0}_X$ uniformly in J, see [18]. The definition depends neither on the dimension nor on p_0. Moreover, it can be seen, see [18], that X has the Hardy-Littlewood property if and only if $\mathcal{M}_J$ are uniformly bounded from L^1_X into weak $-L^1_X$.

For a X-valued martingale f, we can define the maximal operator

$$M_{X,N} f = \sup_{1\leq k\leq N} |f_k|\,,$$

and we say that X has the probabilistic Hardy-Littlewood property if there exist a constant C and a p_0, $1 < p_0 < \infty$, such that $\sup_N \|M_{X,N} f\|_{L^{p_0}_X} \leq C\|f\|_{L^{p_0}_X}$. Using our method we can prove the same kind of equivalent definitions of this property as in the Euclidean case. Moreover, we can go further and prove the following characterization.

THEOREM 7. *X has the probabilistic Hardy-Littlewood property if and only if*

$$\sup_N \|M_{X,N} f\|_{L^1_X} \leq C\|f^*\|_{L^1}\,.$$

PROOF. Observe that for each N

$$\|M_{X,N} f\|_X = \left\| \max_{1\leq k\leq N} |f_k| \right\|_X = \|\{f_k\}_{k=1}^N\|_{X(\ell^\infty_N)}\,.$$

Consider the martingale transform operator U_N such that it gives for a X-valued martingale f, the $X(\ell_N^\infty)$-valued martingale

$$(U_N f)_n = \sum_{k=1}^{n} w_k d_k = (d_1, d_1 + d_2, \dots, d_1 + \cdots + d_{n\wedge N}, \dots, d_1 + \cdots + d_{n\wedge N})$$
$$= (f_1, \dots, f_{n\wedge N}, \dots, f_{n\wedge N}) \in X(\ell_N^\infty),$$

where $w_k x = (0, \overset{(k-1)}{\dots}, 0, x, \dots, x) \in X(\ell_N^\infty)$ for all $x \in X$ and $k \leq N$, $w_k = 0$ otherwise. Then, if X has the probabilistic Hardy-Littlewood property, U_N verifies (iv) in Theorem 1 with $B_1 = X$, $B_2 = X(\ell_N^\infty)$, and the bounds are uniform in N. In particular, from statement (iii) in Theorem 1, we get the second statement, also uniformly in N. The converse can be obtained with the same reasoning.

REMARK 6. It is clear that the Hardy-Littlewood property implies the probabilistic Hardy-Littlewood for dyadic martingales. The converse is also true due to the fact that Hardy-Littlewood maximal operator is controled by an average of the dyadic maximal operator, see [17].

4. Proof of the general results.

PROOF OF THEOREM 1. We shall prove (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(i) and (iii)$\Rightarrow$($*$). That (iii)$\Rightarrow$(iv) follows from Remark 1. (i)$\Rightarrow$(ii) and (iv)$\Rightarrow$(v) are obvious.

We begin with proving (ii)$\Rightarrow$(iii). Consider a martingale f such that $f^* \in L^p$, $1 \leq p < \infty$ (in other case there is nothing to prove). We can decompose f, see Remark 2, as $f_n = g_n + h_n$, and then

$$\|(Tf)^*\|_{L^p} \leq \|(Tg)^*\|_{L^p} + \|(Th)^*\|_{L^p}.$$

By the fact that the sequence v_k is bounded in $\mathcal{L}(B_1, B_2)$, and the properties of the martingale h, see Remark 2, we get

$$\|(Th)^*\|_{L^p} = \left\| \sup_n \left\| \sum_{k=1}^{n} v_k \alpha_k \right\|_{B_2} \right\|_{L^p} \leq \left\| \sup_n \sum_{k=1}^{n} \|v_k\|_{\mathcal{L}(B_1,B_2)} \|\alpha_k\|_{B_1} \right\|_{L^p}$$
$$\leq \left\| \sup_n \sum_{k=1}^{n} \|\alpha_k\|_{B_1} \right\|_{L^p} = \left\| \sum_{k=1}^{\infty} \|\alpha_k\|_{B_1} \right\|_{L^p} \leq C_p \|f^*\|_{L^p}.$$

Define $\gamma_n = 2\lambda_n$. $\{\gamma_n\}$ is an adapted, positive, increasing process such that $\|g_n\|_{B_1} \leq \gamma_{n-1}$ and it also controls its martingale differences

$$\|\beta_n\|_{B_1} \leq \|g_n\|_{B_1} + \|g_{n-1}\|_{B_1} \leq \lambda_{n-1} + \lambda_{n-2} \leq 2\lambda_{n-1} = \gamma_{n-1}.$$

Let us fix $\lambda > 0$. For β, δ satisfying $\beta > \delta + 1$, define the following stopping times:

$$\mu = \inf\{n : \|(Tg)_n\|_{B_2} > \lambda\},$$
$$\nu = \inf\{n : \|(Tg)_n\|_{B_2} > \beta\lambda\},$$
$$\sigma = \inf\{n : \|g_n\|_{B_1} > \delta\lambda \text{ or } \gamma_n > \delta\lambda\}.$$

These are clearly stopping times, as g_n, $(Tg)_n$, and γ_n are $\mathcal{F}_n$-measurable and $\mu \leq \nu$. Define the functions $u_n = \chi_{\{\mu < n \leq \nu \wedge \sigma\}}$, which are uniformly bounded by 1 and $\mathcal{F}_n$-predictables, since $\{\mu < n \leq \nu \wedge \sigma\} = \{\mu < n\} \cap \{n \leq \nu \wedge \sigma\} = \{\mu \leq n-1\} \cap \{\sigma \leq n-1\}^c \cap \{\nu \leq n-1\}^c \in \mathcal{F}_{n-1}$.

Consider the martingales $a_n = \sum_{k=1}^n u_k \beta_k$ and $(Ta)_n = \sum_{k=1}^n v_k u_k \beta_k$. These martingales have the following properties:

(1) $a^* \leq 2\delta\lambda$ in the set $\{\mu < \infty\}$ and $a^* = 0$ in $\{\mu = \infty\}$. In particular, we have

$$\|a^*\|_{L^1} \leq 2\delta\lambda P(\mu < \infty) = 2\delta\lambda P((Tg)^* > \lambda).$$

(2) $P((Tg)^* > \beta\lambda, \gamma^* \leq \delta\lambda) \leq P((Ta)^* > (\beta - \delta - 1)\lambda)$.

From these properties of a and Ta and hypothesis (ii) we get

$$\begin{aligned} P((Tg)^* > \beta\lambda, \gamma^* \leq \delta\lambda) &\leq P((Ta)^* > (\beta - \delta - 1)\lambda) \\ &\leq \frac{C}{(\beta - \delta - 1)\lambda} \|a^*\|_{L^1} \\ &\leq \frac{C}{(\beta - \delta - 1)\lambda} 2\delta\lambda P((Tg)^* > \lambda) \\ &= \frac{C\delta}{\beta - \delta - 1} P((Tg)^* > \lambda). \end{aligned}$$

Also, this leads to the following inequality:

$$\begin{aligned} \|(Tg)^*\|_{L^p}^p &= p\int_0^\infty \lambda^{p-1} P((Tg)^* > \lambda) d\lambda = p\int_0^\infty (\beta\lambda)^{p-1} P((Tg)^* > \beta\lambda)\beta d\lambda \\ &= p\int_0^\infty (\beta\lambda)^{p-1} P((Tg)^* > \beta\lambda, \gamma^* \leq \delta\lambda)\beta d\lambda \\ &\quad + p\int_0^\infty (\beta\lambda)^{p-1} P((Tg)^* > \beta\lambda, \gamma^* > \delta\lambda)\beta d\lambda \\ &\leq p\int_0^\infty (\beta\lambda)^{p-1} \frac{C\delta}{\beta - \delta - 1} P((Tg)^* > \lambda)\beta d\lambda + p\int_0^\infty \frac{\beta^p}{\delta^p} t^{p-1} P(\gamma^* > t) dt \\ &= \beta^p \frac{C\delta}{\beta - \delta - 1} \|(Tg)^*\|_{L^p}^p + \frac{\beta^p}{\delta^p} \|\gamma^*\|_{L^p}^p. \end{aligned}$$

Taking β, δ such that $(C\delta\beta^p)/(\beta - \delta - 1) = 1/2$, we obtain

$$\|(Tg)^*\|_{L^p} \leq C\|\gamma^*\|_{L^p} = C\|\lambda^*\|_{L^p}.$$

With this the first part of the proof is complete, since for our martingale

$$\|(Tf)^*\|_{L^p} \leq \|(Tg)^*\|_{L^p} + \|(Th)^*\|_{L^p} \leq C(\|\lambda^*\|_{L^p} + \|f^*\|_{L^p}) \leq C\|f^*\|_{L^p}.$$

In order to prove (v)$\Rightarrow$(i), we shall use Gundy decomposition. Fix $\lambda > 0$ and decompose f, following Remark 3, as $f = a + b + e$. Then

$$P((Tf)^* > \lambda) \leq P((Ta)^* > \lambda/3) + P((Tb)^* > \lambda/3) + P((Te)^* > \lambda/3).$$

We will denote by α_k, β_k, δ_k the martingale difference sequences for a, b, e, respectively. By using the properties of the martingale a, we have

$$P\left((Ta)^* > \frac{\lambda}{3}\right) \leq P(Ta \neq 0) \leq P(a \neq 0) = P\left(\sup_k \|\alpha_k\|_{B_1} \neq 0\right) \leq \frac{C}{\lambda}\|f\|_{L^1_{B_1}} .$$

On the other hand, since the operators v_k are of norm 1 and β_k satisfy (3) in Remark 3, we have

$$\begin{aligned} P\left((Tb)^* > \frac{\lambda}{3}\right) &\leq \frac{3}{\lambda}\int (Tb)^* dP = \frac{3}{\lambda}\int \sup_n \|(Tb)_n\|_{B_2} dP \\ &= \frac{3}{\lambda}\int \sup_n \left\|\sum_{k=1}^n v_k\beta_k\right\|_{B_2} dP \leq \frac{3}{\lambda}\int \sum_{k=1}^\infty \|\beta_k\|_{B_1} dP \leq \frac{C}{\lambda}\|f\|_{L^1_{B_1}} . \end{aligned}$$

Hence, by using the hypothesis and the properties of e, we get

$$\begin{aligned} P\left((Te)^* > \frac{\lambda}{3}\right) &\leq \frac{C}{\lambda^{p_0}}\int (Te)^{*p_0} dP \leq \frac{C}{\lambda^{p_0}}\sup_n \int \|e_n\|_{B_1}^{p_0} dP \\ &\leq \frac{C}{\lambda}\sup_n \int \|e_n\|_{B_1} dP = \frac{C}{\lambda}\|e\|_{L^1_{B_1}} \leq \frac{C}{\lambda}\|f\|_{L^1_{B_1}} . \end{aligned}$$

Finally, we will prove that condition (iii) with $p = 1$ implies $(*)$.

Fix $\lambda > 0$ and define the stopping time $\mu = \inf\{n : \|f_n\|_{B_1} > \lambda\}$ and the scalar functions $u_n = \chi_{\{\mu \geq n\}} = \chi_{\{f^*_{n-1} \leq \lambda\}}$. These functions u_n are $\mathcal{F}_n$-predictable and uniformly bounded by 1. We consider the following martingales:

$$f_n = \sum_{k=1}^n d_k , \quad F_n = \sum_{k=1}^n u_k d_k , \quad (Tf)_n = \sum_{k=1}^n v_k d_k , \quad (TF)_n = \sum_{k=1}^n v_k u_k d_k .$$

Observe that if $n \geq \mu$, $F_n = \sum_{k=1}^n \chi_{\{\mu \geq k\}} d_k = f_\mu$, then $\|F_n\|_{B_1} = \|f_\mu\|_{B_1}$. On the other hand, if $n < \mu$, $F_n = f_n$. Therefore, if $n \geq \mu$,

$$\begin{aligned} F_n^* = \max_{1\leq k\leq n}\|F_k\|_{B_1} &\leq \max_{1\leq k\leq \mu-1}\|F_k\|_{B_1} + \max_{\mu\leq k\leq n}\|F_k\|_{B_1} \\ &= \max_{1\leq k\leq \mu-1}\|f_k\|_{B_1} + \max_{\mu\leq k\leq n}\|f_\mu\|_{B_1} \leq \lambda + \|f_\mu\|_{B_1} , \end{aligned}$$

and in case $n < \mu$, $F_n^* = \max_{1\leq k\leq n}\|F_k\|_{B_1} = \max_{1\leq k\leq n}\|f_k\|_{B_1} \leq \lambda$. Consequently,

$$\begin{aligned} \|F_n^*\|_{L^1_{B_1}} &\leq \lambda + \|f_{n\wedge\mu}\|_{L^1_{B_1}} = \lambda + \sum_{k=1}^\infty \int_{\{\mu=k\}} \|f_{n\wedge k}\|_{B_1} dP \\ &= \lambda + \sum_{k=1}^n \int_{\{\mu=k\}} \|f_k\|_{B_1} dP + \int_{\{\mu>n\}} \|f_n\|_{B_1} dP \\ &\leq \lambda + \sum_{k=1}^n \int_{\{\mu=k\}} \|f_n\|_{B_1} dP + \int_{\{\mu>n\}} \|f_n\|_{B_1} dP = \lambda + \|f_n\|_{L^1_{B_1}} , \end{aligned}$$

where in the last inequality we have used the martingale properties of f.

Applying (iii) to the martingale TF, we have that there exists a constant C such that $\|(TF)^*\|_{L^1} \leq C\|F^*\|_{L^1} \leq C(\lambda + \|f\|_{L^1})$. Then F^* and $(TF)^*$ are in L^1 and, since B_2

has the Radon-Nikodym property by assumption, $(TF)_n$ converges almost surely, see [13]. Now $Tf = TF$ in the set where $u_k = 1$ for all k, that is, in $\{f^* \le \lambda\}$. In other words, Tf converges almost surely in the set $\{f^* \le \lambda\}$, with λ any positive number. If we choose a sequence $\lambda_n \uparrow \infty$, Tf converges a.s. in $\{f^* \le \lambda_n\}$ for all n and therefore Tf converges a.s. in $\{f^* < \infty\} = \bigcup_{n=1}^{\infty}\{f^* \le \lambda_n\}$. But using Doob's inequality, we have

$$P\{f^* < \infty\} = \lim_{n\to\infty} P\{f^* \le \lambda_n\} \ge 1 - \lim_{n\to\infty} \frac{C}{\lambda_n}\|f\|_{L^1_{B_1}} = 1\,.$$

PROOF OF COROLLARY 1. Let $v_k \in \mathcal{L}(B_1, B_2)$ be the multiplying sequence of the martingale T. Given $F_n = \sum_{k=1}^n D_k = \{\sum_{k=1}^n d_k^j\}_{j=1}^\infty = \{f^j\}_{j=1}^\infty$, we have

$$(\tilde{T}F)_n = \{(Tf^j)_n\}_{j=1}^\infty = \left\{\sum_{k=1}^n v_k d_k^j\right\}_{j=1}^\infty = \sum_{k=1}^n V_k D_k\,,$$

where $V_k = \{v_k, v_k, v_k, \dots\} \in \ell^\infty(\mathcal{L}(B_1, B_2)) \subset \mathcal{L}(\ell^q(B_1), \ell^q(B_2))$, and $\|V_k\| \le \|v_k\|$. Therefore, $\tilde{T}$ is a martingale transform operator. On the other hand, given q, $1 < q < \infty$, since T satisfies Theorem 1, we have, $\|Tf^j\|_{L^q_{(B_2)}} \le \|(Tf^j)^*\|_q \le C_q\|f^j\|_{L^q_{(B_1)}}$ and therefore

$$\begin{aligned}\|\tilde{T}F\|_{L^q_{\ell^q(B_2)}} &= \sup_n \|(\tilde{T}F)_n\|_{L^q_{\ell^q(B_2)}} = \sup_n \left\|\left(\sum_j \|(Tf^j)_n\|_{B_2}^q\right)^{1/q}\right\|_{L^q} \\ &\le C_q \sup_n \left\|\left(\sum_j \|(f^j)_n\|_{B_1}^q\right)^{1/q}\right\|_{L^q} = C_q\|F\|_{L^q_{\ell^q(B_1)}}\,.\end{aligned}$$

By Remark 1 this means that $\tilde{T}$ satisfies (v) of Theorem 1 with B_1 replaced by $\ell^q(B_1)$ and B_2 by $\ell^q(B_2)$. Then the corollary follows.

PROOF OF THEOREM 2. We will prove (ii) of Theorem 1. The proof follows the argument for Theorem 1.1 in [9].

First, observe that we may assume $f_1 = d_1 = 0$. Given $f = \{f_n\}_{n\ge1}$ a martingale, with associated σ-algebra sequence $\{\mathcal{F}_n\}_{n\ge1}$ and other σ-algebra $\mathcal{G}$, construct the σ-algebras in the product space $\tilde{\mathcal{F}}_n = \mathcal{F}_n \times \Omega$, $\tilde{\mathcal{G}} = \Omega \times \mathcal{G}$. Consider a $\mathcal{G}$-measurable random variable r with $P(r = 1) = P(r = -1) = 1/2$, and define $\tilde{r}(\omega_1, \omega_2) = r(\omega_2)$. $\tilde{r}$ is independent with respect to the $\tilde{\mathcal{F}}_n$-martingale $\tilde{f}_n(\omega_1, \omega_2) = f_n(\omega_1)$ and the $\tilde{\mathcal{F}}_n$-multiplying sequence $\tilde{v}_n(\omega_1, \omega_2) = v_n(\omega_1)$.

We now define a martingale with the same behaviour as $\{f_n\}_{n\ge1}$ and such that $d_1 = f_1 = 0$. Let

$$\begin{aligned}&\mathcal{A}_1 = \{\emptyset, \Omega \times \Omega\}\,, \quad \mathcal{A}_2 = \sigma(\tilde{\mathcal{G}} \cup \tilde{\mathcal{F}}_1), \dots\,, \\ &\mathcal{A}_n = \sigma(\tilde{\mathcal{G}} \cup \tilde{F}_{n-1}) = \sigma\{A \times B : A \in \mathcal{F}_{n-1}, B \in \mathcal{G}\}\end{aligned}$$

be another σ-algebra sequence. The sequence $D = (D_1, D_2, D_3, \dots) = (0, \tilde{r}\tilde{d}_1, \tilde{r}\tilde{d}_2, \dots)$ is then a $\mathcal{A}_n$-martingale difference sequence. The martingale whose differences are D is

$F = \{F_n\}_{n\geq 1}$ defined by

$$F_1 = 0\,, \quad F_n = \sum_{k=1}^{n-1} \tilde{r}\tilde{d}_k = \tilde{r}\tilde{f}_{n-1}\,, \quad n \geq 2\,.$$

Consider now the $\mathcal{A}_n$-multiplying sequence $V = (l, \tilde{v}_1, \tilde{v}_2, \dots)$, where l is any operator in $\mathcal{L}(B_1, B_2)$. The martingale transform of F by V is $\tilde{T}F = \{(\tilde{T}F)_n\}$, with

$$(\tilde{T}F)_1 = 0\,, \quad (\tilde{T}F)_n = \sum_{k=1}^{n-1} \tilde{v}_k(\tilde{r}\tilde{d}_k) = \tilde{r}\sum_{k=1}^{n-1} \tilde{v}_k\tilde{d}_k = \tilde{r}(Tf)_{n-1}(\omega_1)\,, \quad n \geq 2\,.$$

Then we have $\|F_n\|_{B_1} = \|f_{n-1}\|_{B_1}$, $\|(\tilde{T}F)_n\|_{B_2} = \|(Tf)_{n-1}\|_{B_2}$ and that the set where $\tilde{T}F$ converges is the set $\{\omega_1 : Tf \text{ converges}\} \times \Omega$. Therefore the martingale f and its corresponding martingale transform Tf verify the theorem if and only if F and $\tilde{T}F$ do. The rest of the proof is for F and $\tilde{T}F$, but we will avoid unnecessary notation by keeping the names T, $(v_1, v_2, \dots)$ and f for the martingale transform operator, its multiplying sequence and the martingales that we will handle, respectively. As an intermediate step we shall prove

(∗∗) There exists an absolute constant $C > 0$ such that, given any $\lambda > 0$.

$$(Tf)^* > \lambda \text{ a.e.} \Rightarrow \|f^*\|_{L^1} \geq C\lambda\,.$$

It suffices to show (∗∗) for $\lambda = 1$. Suppose that (∗∗) is not true, i.e., for every $j > 0$ there exist a martingale f^j such that $(Tf^j)^* > 1$ a.s., but $\|(f^j)^*\|_{L^1} \leq 2^{-j}$. Then we are going to find a counterexample for (∗), that is, a martingale F such that $F^* \in L^1$ but TF does not converge almost surely.

We may assume that for each j there exists an index n_j such that

$$P((Tf^j)^*_{n_j} > 1) > \frac{1}{2}\,.$$

Otherwise, we would have $P((Tf^j)^*_m > 1) \leq 1/2$ for all m and then $P((Tf^j)^* > 1) \leq 1/2$, while our hypothesis is $(Tf^j)^* > 1$ a.s.

Each f^j is defined on a probability space $(\Omega^j, \mathcal{F}^j, P^j)$ and it is relative to an increasing sequence of σ-algebras, $\{\mathcal{F}^j_n\}_{n\geq 1}$. Transfer all the martingales f^j to the product space $(\Omega, \mathcal{F}, P) = \prod_{j=1}^\infty (\Omega^j, \mathcal{F}^j, P^j)$ to obtain independent martingales in the usual way, defining the σ-algebras

$$\tilde{\mathcal{F}}^j_n = \Omega^1 \times \cdots \times \Omega^{j-1} \times \mathcal{F}^j_n \times \Omega^{j+1} \times \cdots$$

and the sequences

$$\tilde{f}^j_n(\omega_1, \omega_2, \dots) = f^j_n(\omega_j)\,, \quad \tilde{d}^j_n(\omega_1, \omega_2, \dots) = d^j_n(\omega_j)\,.$$

Define a new sequence of σ-algebras by

$$\begin{aligned}\mathcal{A}_1 &= \tilde{\mathcal{F}}_1^1, \dots, \mathcal{A}_{n_1} = \tilde{\mathcal{F}}_{n_1}^1,\\ \mathcal{A}_{n_1+1} &= \sigma(\tilde{\mathcal{F}}_{n_1}^1 \cup \tilde{\mathcal{F}}_1^2), \dots, \mathcal{A}_{n_1+n_2} = \sigma(\tilde{\mathcal{F}}_{n_1}^1 \cup \tilde{\mathcal{F}}_{n_2}^2),\\ \mathcal{A}_{n_1+n_2+1} &= \sigma(\tilde{\mathcal{F}}_{n_1}^1 \cup \tilde{\mathcal{F}}_{n_2}^2 \cup \tilde{\mathcal{F}}_1^3), \dots,\\ \mathcal{A}_{n_1+n_2+\dots+n_k+h} &= \sigma(\tilde{\mathcal{F}}_{n_1}^1 \cup \tilde{\mathcal{F}}_{n_2}^2 \cup \dots \cup \tilde{\mathcal{F}}_{n_k}^k \cup \tilde{\mathcal{F}}_h^{k+1}),\end{aligned}$$

where $h = 1, \dots, n_{k+1}$. The sequence given by

$$D = (\tilde{d}_1^1, \tilde{d}_2^1, \dots, \tilde{d}_{n_1}^1, \tilde{d}_1^2, \dots, \tilde{d}_{n_2}^2, \tilde{d}_1^3, \dots) \tag{13}$$

is the $\mathcal{A}_n$-martingale difference sequence of the martingale F, that verifies

$$F_{n_1+\dots+n_k+h} = \tilde{f}_{n_1}^1 + \tilde{f}_{n_2}^2 + \dots + \tilde{f}_{n_k}^k + \tilde{f}_h^{k+1}, \quad h = 1, \dots, n_{k+1}.$$

Therefore, for $h = 1, \dots, n_{k+1}$, we have

$$\begin{aligned}(TF)_{n_1+\dots+n_k+h} &= \sum_{j=1}^{n_1+\dots+n_k+h} v_j D_j\\ &= v_1\tilde{d}_1^1 + v_2\tilde{d}_2^1 + \dots + v_{n_1}\tilde{d}_{n_1}^1 + v_{n_1+1}\tilde{d}_1^2 + \dots + v_{n_1+n_2}\tilde{d}_{n_2}^2\\ &\quad + v_{n_1+n_2+1}\tilde{d}_1^3 + \dots + v_{n_1+\dots+n_k+1}\tilde{d}_{n_k}^k + \dots + v_{n_1+\dots+n_k+h}\tilde{d}_h^{k+1}\\ &= (T\tilde{f}^1)_{n_1} + (T_{n_1}\tilde{f}^2)_{n_2} + \dots + (T_{n_1+\dots+n_{k-1}}\tilde{f}^k)_{n_k} + (T_{n_1+\dots+n_k}\tilde{f}^{k+1})_h.\end{aligned} \tag{14}$$

Observe that F^* is in L^1. Since, for $h = 1, \dots, n_{k+1}$, we have

$$\|F_{n_1+\dots+n_k+h}\|_{B_1} \le \sum_{i=1}^{k} \|\tilde{f}_{n_i}^i\|_{B_1} + \|\tilde{f}_h^{k+1}\|_{B_1} \le \sum_{i=1}^{k+1} (\tilde{f}^i)_{n_i}^*,$$

and then $\|F_{n_1+\dots+n_k+h}^*\|_{L^1} \le \sum_{i=1}^{k+1} 2^{-i} \le 1$. Thus, $\|F^*\|_{L^1} \le 1$.

Now we shall see that TF does not converge almost surely, that is, it does not verify $(*)$. It is clear that $\limsup_{n,m}\{\|(TF)_n - (TF)_m\|_{B_2} > 1\}$ is a subset of the set of points ω for which there does not exist $\lim_{n\to\infty}(TF)_n(\omega)$. Using the fact that $TF_{n_1+\dots+n_k+h} - TF_{n_1+\dots+n_k} = (T_{n_1+\dots+n_k}\tilde{f}^{k+1})_h$, the events

$$\begin{aligned}A_k &= \bigcup_{h=1}^{n_{k+1}} \{\|(TF)_{n_1+\dots+n_k+h} - (TF)_{n_1+\dots+n_k}\|_{B_2} > 1\}\\ &= \left\{\max_{1\le h\le n_{k+1}} \|(T_{n_1+\dots+n_k}\tilde{f}^{k+1})_h\|_{B_2} > 1\right\}\end{aligned}$$

are independent (since $(T_{n_1+\dots+n_k}\tilde{f}^{k+1})_h$ are independent in k) and verify, due to the choice of the n_k's and the fact that T is translation invariant,

$$P(A_k) = P\left(\max_{1\le h\le n_{k+1}} \|(T_{n_1+\dots+n_k}\tilde{f}^{k+1})_h\|_{B_2} > 1\right) = P((Tf^{k+1})_{n_{k+1}}^* > 1) > \frac{1}{2}.$$

Then Borel-Cantelli lemma tells us that $P(\limsup A_k) = 1$, and therefore that there does not exist $\lim_{n\to\infty}(TF)_n$ in a set of probability 1.

We shall prove now that $(*)$ and $(**)$ imply (ii) in Theorem 1. Choose a martingale f such that $\|f^*\|_{L^1} = 1$ and a fixed $\lambda > 0$. We can assume that there exists an index n_0 such that

$$P((Tf)^*_{n_0} > \lambda) = p > 0\,.$$

Otherwise, since $P((Tf)^*_n > \lambda) = 0$ for all n it leads to $P((Tf)^* > \lambda) = 0$, and there would be nothing to prove.

As in the preceding part of the proof, we can construct infinitely many independent copies of the sequences (f, Tf), say (f^j, Tf^j), moving the variables to the infinite product space, where f is now defined on a probability space $(\Omega, \mathcal{F}, P)$ and it is relative to an increasing sequence of σ-algebras, $\{\mathcal{F}_n\}_{n\geq 1}$. Define then the σ-algebra sequence $\tilde{\mathcal{F}}^j_n = \Omega \times \overset{(j-1)}{\cdots} \times \Omega \times \mathcal{F}_n \times \Omega \times \cdots$, and the sequences $\tilde{f}^j$, $T\tilde{f}^j$, as before.

Define now the functions $u_j = \chi_{\{(T\tilde{f}^j)^*_{n_0} \leq \lambda\}}$. Each u_j is $\tilde{\mathcal{F}}^j_{n_0}$-measurable, since

$$\begin{aligned}
(T\tilde{f}^j)^*_{n_0}(\omega_1, \omega_2, \dots) &= \max_{1\leq k\leq n_0} \|(T\tilde{f}^j)_k(\omega_1, \omega_2, \dots)\|_{B_2} \\
&= \max_{1\leq k\leq n_0} \|(Tf)_k(\omega_j)\|_{B_2} = (Tf)^*_{n_0}(\omega_j)\,,
\end{aligned}$$

and verifies for all j

$$E(u_j) = P((T\tilde{f}^j)^*_{n_0} \leq \lambda) = P((Tf)^*_{n_0} \leq \lambda) = 1 - p\,.$$

Define a new sequence of σ-algebras by

$$\begin{aligned}
\mathcal{A}_1 &= \tilde{\mathcal{F}}^1_1, \dots, \mathcal{A}_{n_0} = \tilde{\mathcal{F}}^1_{n_0}\,, \\
\mathcal{A}_{n_0+1} &= \sigma(\tilde{\mathcal{F}}^1_{n_0} \cup \tilde{\mathcal{F}}^2_1), \dots, \mathcal{A}_{2n_0} = \sigma(\tilde{\mathcal{F}}^1_{n_0} \cup \tilde{\mathcal{F}}^2_{n_0})\,, \\
\mathcal{A}_{2n_0+1} &= \sigma(\tilde{\mathcal{F}}^1_{n_0} \cup \tilde{\mathcal{F}}^2_{n_0} \cup \tilde{\mathcal{F}}^3_1), \dots, \\
\mathcal{A}_{kn_0+h} &= \sigma(\tilde{\mathcal{F}}^1_{n_0} \cup \tilde{\mathcal{F}}^2_{n_0} \cup \dots \cup \tilde{\mathcal{F}}^k_{n_0} \cup \tilde{\mathcal{F}}^{k+1}_h)\,,
\end{aligned}$$

where $h = 1, \dots, n_0$. Now, the sequence given by

$$\begin{aligned}
D &= (\tilde{d}^1_1, \tilde{d}^1_2, \dots, \tilde{d}^1_{n_0}, u_1\tilde{d}^2_1, \dots, u_1\tilde{d}^2_{n_0}, u_1u_2\tilde{d}^3_1, \dots)\,, \\
D_{kn_0+h} &= u_1 \dots u_k \tilde{d}^{k+1}_h\,, \quad h = 1, 2, \dots, n_0
\end{aligned}$$

is a $\mathcal{A}_n$-martingale difference sequence. The martingale whose differences are D is F, with

$$F_n = \sum_{k=1}^{n} D_k = \tilde{f}^1_{n_0} + u_1\tilde{f}^2_{n_0} + \dots + u_1 \dots u_k \tilde{f}^{k+1}_h\,, \quad n = kn_0 + h\,,\ h = 1, \dots, n_0\,. \tag{15}$$

Observe that for $n = kn_0 + h$, where $h = 1, \dots, n_0$ and $k = 0, 1, \dots$, it holds the following

$$\begin{aligned}
\|F_n\|_{B_1} &= \|\tilde{f}^1_{n_0} + u_1\tilde{f}^2_{n_0} + \dots + u_1 \dots u_k \tilde{f}^{k+1}_h\|_{B_1} \\
&\leq \|\tilde{f}^1_{n_0}\|_{B_1} + u_1\|\tilde{f}^2_{n_0}\|_{B_1} + \dots + u_1 \dots u_k\|\tilde{f}^{k+1}_h\|_{B_1}\,.
\end{aligned}$$

Then $F_n^* \le (\tilde f^1)^*_{n_0} + u_1(\tilde f^2)^*_{n_0} + \cdots + u_1 \dots u_k(\tilde f^{k+1})^*_h$. Using the fact that $\|f^*\|_{L^1} = 1$ and u_j is independent of $\tilde f^k_m$ and u_l for all $j \neq k, l$, we have that

$$\begin{aligned}\|F_n^*\|_{L^1} &\le \|(\tilde f^1)^*_{n_0}\|_{L^1} + E(u_1)\|(\tilde f^2)^*_{n_0}\|_{L^1} + \cdots + E(u_1)\dots E(u_k)\|(\tilde f^{k+1})^*_h\|_{L^1} \\ &\le 1 + (1-p) + \cdots + (1-p)^k \le \sum_{k=0}^{\infty}(1-p)^k = \frac{1}{p}.\end{aligned}$$

Therefore $\|F^*\|_{L^1} \le 1/p$.

Let us consider now the martingale TF, which is, for $n = kn_0 + h$, $h = 1, \dots, n_0$,

$$(TF)_n = \sum_{k=1}^{n} v_k D_k = (T\tilde f^1)_{n_0} + u_1(T_{n_0}\tilde f^2)_{n_0} + \cdots + u_1\dots u_k(T_{kn_0}\tilde f^{k+1})_h . \tag{16}$$

Since $(TF)_n = (T\tilde f^1)_n$ for $1 \le n \le n_0$, we have $\{(T\tilde f^1)^*_{n_0} > \lambda\} \subset \{(TF)^* > \lambda/2\}$. Also

$$\{(T\tilde f^1)^*_{n_0} \le \lambda\} \cap \{(T_{n_0}\tilde f^2)^*_{n_0} > \lambda\} \subset \left\{(TF)^* > \frac{\lambda}{2}\right\},$$

since in that set $u_1 = 1$, and $(TF)_{n_0} = (T\tilde f^1)_{n_0}$, $(TF)_{n_0+h} = (T\tilde f^1)_{n_0} + (T_{n_0}\tilde f^2)_h$, for $h = 1, \dots, n_0$. Therefore

$$\lambda < (T_{n_0}\tilde f^2)^*_{n_0} = \max_{1\le h\le n_0} \|(T_{n_0}\tilde f^2)_h\|_{B_2} = \max_{1\le h\le n_0} \|(TF)_{n_0+h} = (TF)_{n_0}\|_{B_2} \le 2(TF)^* .$$

By the same reasoning, all the sets of the form

$$\{(T\tilde f^1)^*_{n_0} \le \lambda\} \cap \cdots \cap \{(T_{(k-2)n_0}\tilde f^{k-1})^*_{n_0} \le \lambda\} \cap \{(T_{(k-1)n_0}\tilde f^k)^*_{n_0} > \lambda\}$$

are subsets of $\{(TF)^* > \lambda/2\}$. These sets are disjoint and $(T_{(j-1)n_0}\tilde f^j)^*_{n_0}$, $(T_{(k-1)n_0}\tilde f^k)^*_{n_0}$ are independent for $j \neq k$. Therefore, we have

$$\begin{aligned}&P\{(TF)^* > \lambda/2\} \\ &\quad\ge P\left\{\bigcup_{k=1}^{\infty}(\{(T\tilde f^1)^*_{n_0} \le \lambda\} \cap \cdots \cap \{(T_{(k-2)n_0}\tilde f^{k-1})^*_{n_0} \le \lambda\} \cap \{(T_{(k-1)n_0}\tilde f^k)^*_{n_0} > \lambda\})\right\} \\ &\quad= \sum_{k=1}^{\infty} P(\{(T\tilde f^1)^*_{n_0} \le \lambda\} \cap \cdots \cap \{(T_{(k-2)n_0}\tilde f^{k-1})^*_{n_0} \le \lambda\} \cap \{(T_{(k-1)n_0}\tilde f^k)^*_{n_0} > \lambda\}) \\ &\quad= \sum_{k=1}^{\infty} p(1-p)^{k-1} = 1 .\end{aligned}$$

Then, $(TF)^* > \lambda/2$ a.s. and applying $(**)$ we have for some absolute constant C

$$C\lambda \le \|F^*\|_{L^1} \le \frac{1}{p} \Rightarrow p = P\{(Tf)^*_{n_0} > \lambda\} \le \frac{C}{\lambda}.$$

Finally, we note that the index n_0 was chosen to verify $P((Tf)^*_{n_0} > \lambda) > 0$. As $(Tf)^*_{n_0+k} \ge (Tf)^*_{n_0}$ for all $k > 0$, we could have chosen any other index $n_0 + k$ and have reached for it the same conclusion $P((Tf)^*_{n_0+k} > \lambda) \le C/\lambda$ for the same constant C. Taking limits, we obtain $P((Tf)^* > \lambda) \le c/\lambda$ and the proof is finished.

REMARK 7. As far as we know, there does not exist a useful complete reciprocal to Theorem 1. Nevertheless, Theorem 2 also holds under other assumptions, for instance, the following condition

$$(*)' \qquad \|f^*\|_{L^1} < \infty \Rightarrow T^\alpha f \text{ converges a.s.}$$

when it is verified for any martingale f and any T^α in the following class of martingale transform operators. Consider the family $\{T^\alpha\}_{\alpha\in\Lambda}$ of martingale transform operators defined by multiplying sequences $\{v_k^\alpha\}$ such that each v_k^α is a constant operator and for any sequence $\{\alpha(k)\} \subset \Lambda$, the multiplying sequence $\{v_k^{\alpha(k)}\}$ defines a martingale transform operator $T^\beta \in \{T^\alpha\}_{\alpha\in\Lambda}$. This property of the family of martingale transform operators allows us to construct martingales F, TF in the same way we did in (13), (14), (15) and (16). Then, a slight modification of the proof above for Theorem 2 yields that any $T^\beta \in \{T^\alpha\}_{\alpha\in\Lambda}$ verifies statements (i)–(v) in Theorem 1.

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